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HOW TO SOLVE NONLINEAR EQUATIONS WHEN A THIRD ORDER METHOD IS NOT APPLICABLE *

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Abstract.

In this paper, we use a one-parametric family of second-order iterations to solve a nonlinear operator equation in a Banach space. A Kantorovich-type convergence theorem is proved, so that the first Fréchet derivative of the operator satisfies a Lipschitz condition. We also give an explicit error bound.

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1 Introduction.

Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ a nonlinear operator in an open convex nonvoid domain Ω. We consider the problem of solving the equation

$$F(x) = 0 \tag{1.1}$$

by means of iterative processes. Let us assume that $F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X.

Third order methods are not considered by many authors to solve (1.1) because of their high computational cost, mainly for the evaluation of the second Fréchet derivative. However, in some cases the increase in speed of convergence can justify their use. For instance, these methods have been successfully used in solving nonlinear integral equations [2, 5].

Our goal in this paper is to solve equation (1.1) when we cannot apply a third-order method. For that, from the third-order Super-Halley method, we construct a new iterative process of second-order with less operational cost and its convergence guaranteed under more relaxed conditions for the operator F. Moreover, the new iteration is faster than Newton's method.

In Section 2, the new method is defined and a semilocal convergence result is provided. Finally, in Section 3, we give some applications where the indicated properties of the new process are shown.

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2 A new iterative method.

It is known that the Super-Halley method or the Convex Acceleration of Newton's method [4, 7] is the iteration of order three:

$$x_{n+1} = x_n - \left[I + \frac{1}{2}L_F(x_n)H(x_n)\right]\Gamma_n F(x_n), \quad n \geq 0, \tag{2.1}$$

where I is the identity operator on X, $\Gamma_n = F'(x_n)^{-1}$,

$$L_F(x_n) = \Gamma_n F''(x_n)\Gamma_n F(x_n), \quad \text{and} \quad H(x_n) = [I - L_F(x_n)]^{-1}.$$

Generally necessary conditions for the convergence of third-order iterative processes have been established assuming that the second and the third Fréchet derivative of F are bounded in Ω [1]. Other conditions for the convergence of these methods have been established assuming that the second Fréchet derivative of F satisfies the Lipschitz condition

$$\|F''(x) - F''(y)\| \leq L\|x - y\|, \quad x, y \in \Omega. \tag{2.2}$$

See [2, 3, 12] for more information. The technique developed there is an extension of the one followed by Kantorovich and other authors [8, 10] for Newton's method.

As said in the introduction, our goal in this paper is to solve equation (1.1) when we cannot apply a third-order method. This is the case, for instance, if the third Fréchet derivative of F does not exist or if the second Fréchet derivative does not satisfy (2.2). For example, if we consider $G : \left(-\frac{1}{2}, \frac{3}{2}\right) \times \left(-\frac{1}{2}, \frac{3}{2}\right) \to \mathbb{R}^2$ where

$$G(x, y) = (x^3 \ln x^2 + 2y - 1/16,\, xy - 2x), \tag{2.3}$$

it is easy to prove that G''' does not exist and G'' exists but does not satisfy (2.2) in a neighbourhood of $(0, 0)$. Therefore we cannot apply a third order method to solve $G(x, y) = (0, 0)$. To solve this problem, the second Fréchet derivative is replaced by some fixed operator in (2.1). This is a similar technique to the one used in [6] for the Chebyshev method. So, from (2.1), we introduce a new family of iterations given by

$$x_{n+1} = x_n - \left[I + \frac{1}{2}L(x_n)\tilde{H}(x_n)\right]\Gamma_n F(x_n), \quad n \geq 0, \tag{2.4}$$

where $A : X \times X \to Y$ is a fixed bilinear operator satisfying $\|A\| = \alpha\, (\alpha \geq 0)$,

$$L(x_n) = \Gamma_n A \Gamma_n F(x_n) \quad \text{and} \quad \tilde{H}(x_n) = [I - L(x_n)]^{-1}.$$

We show that, if the linear operator F' satisfies a Lipschitz condition of type (2.2), we can approximate the solution of (1.1) by an iterative process of type (2.4). We use the majorant principle (see [8, 11, 12]) to prove the convergence of (2.4), where a majorizing sequence is obtained from the application of (2.4) to a second-degree polynomial. We finish this section, solving the equation $G(x,y) = (0,0)$, where G is given in (2.3) with an iterative process of the family (2.4).

In the sequel, we denote

$$\overline{B(x,r)} = \{y \in X; |y-x| \le r\} \quad \text{and} \quad B(x,r) = \{y \in X; |y-x| < r\}$$

and assume that F is a nonlinear once Fréchet differentiable operator in an open convex nonvoid domain Ω.

We assume throughout this section that

(I) There exists a continuous linear operator $\Gamma_0 = F'(x_0)^{-1}$, $x_0 \in \Omega$.

(II) $\|\Gamma_0(F'(x) - F'(y))\| \le K\|x-y\|$, $\quad x, y \in \Omega, \quad K \ge 0$.

(III) $\|A\| = \alpha, \qquad \|\Gamma_0 A\| \le \dfrac{\alpha}{b}, \qquad \|\Gamma_0 F(x_0)\| = \dfrac{a}{b}, \quad b > 0.$

(IV) $b - 2aK \ge 0$.

2.1 A convergence study for scalar quadratic polynomial equations.

Before establishing an existence-uniqueness result of equation (1.1) in Banach spaces, we study the convergence of (2.4) for scalar quadratic polynomial equations. The interest of this study is to construct a real function from which a scalar majorizing sequence for (2.4) is obtained to prove its convergence in Banach spaces.

LEMMA 2.1.

(i) *Let* α *satisfy* $0 \le \alpha < \dfrac{b}{8a}(b - 2aK)$. *Then*

$$\left[b + \frac{4\alpha}{K}, \frac{b^2}{2aK}\right] \ne \emptyset.$$

(ii) *Let* $N \ge 0$ *a parameter satisfying* $N \le \dfrac{b^2}{2aK}$. *Then the equation*

$$p(t) \equiv \frac{KN}{2}t^2 - bt + a = 0 \tag{2.5}$$

has two positive roots r_1 *and* r_2 $(r_1 \le r_2)$. *Moreover* $r_1 = r_2$ *iff* $N = \dfrac{b^2}{2aK}$.

Observe that we have introduced a modification in the usual "test" function p (see [3, 8, 11, 12]). The parameter N is considered in the polynomial p in order to prove the convergence of (2.4) under conditions (I)–(IV) and the hypotheses of Lemma 2.1.

Lemma 2.2. *Under the hypotheses of Lemma* 2.1*, let* p *be the polynomial defined in* (2.5)*. Then the sequence defined by* $t_0 = 0$*,*

(2.6)
$$t_{n+1} = t_n - \frac{p(t_n)}{2p'(t_n)}\left(1 + \frac{1}{1-\alpha D_p(t_n)}\right), \quad D_p(t_n) = \frac{p(t_n)}{p'(t_n)^2}, \quad n \geq 0,$$

is increasing and converges at least quadratically to r_1 *for all* $0 \leq \alpha \leq KN$*.*

Proof. Let
$$G(t) = t - \frac{p(t)}{2p'(t)}\left(1 + \frac{1}{1-\alpha D_p(t)}\right)$$
and $L_p(t) = KND_p(t)$. Then

$$\begin{aligned} G'(t) &= \frac{1}{2[1-\alpha D_p(t)]^2}\left[\alpha D_p(t)(\alpha D_p(t)-2) + (\alpha^2 D_p^2(t) - \alpha D_p(t) + 2)L_p(t)\right] \\ &= \frac{D_p(t)}{2[1-\alpha D_p(t)]^2}\left[KN\alpha^2 D_p^2(t) + (\alpha^2 - KN\alpha)D_p(t) + 2(KN-\alpha)\right]. \end{aligned}$$

Since $\alpha \leq KN$ implies
$$u(x) = KN\alpha^2 x^2 + (\alpha^2 - KN\alpha)x + 2(KN-\alpha) \geq 0,$$
we get $G'(t) \geq 0$. Notice that $t_0 \leq t_1 = G(t_0)$. From $G(t_0) - G(r_1) = G'(\xi)(t_0 - r_1) < 0$, we obtain $t_1 < r_1$. Then, by induction, $\{t_n\}$ is increasing and it is bounded from above by r_1. Therefore, $t^* := \lim_{n\to\infty} t_n \leq r_1$ exists, and $t^* = r_1$ follows from (2.6).

As $G(r_1) = r_1, G'(r_1) = 0$ we deduce that iteration (2.6) has at least quadratic convergence. □

In this case, it is easy to prove the asymptotic error constant of (2.4) is
$$C_{e,\alpha} = \left|\frac{G''(x^*)}{2!}\right| = \left|\frac{KN-\alpha}{2\sqrt{b^2 - 2KNa}}\right|.$$

Then, for $\alpha > 0$, (2.4) is a second order iterative processes faster than Newton's method ($\alpha = 0$). Therefore, as we use the majorant principle to prove the convergence of sequence (2.4) in Banach spaces, the real majorizing sequence obtained is faster than Newton's one. Moreover the error estimates are better than the ones arising from Newton's method.

Next, following Ostrowski (see [9]), we obtain the next error bounds for (2.4) when it is applied to polynomial (2.5).

Theorem 2.3. *Let* $\{t_n\}$ *be the sequence given in* (2.6)*. The following error bounds are obtained:*

(a) *If* $r_1 < r_2$*, let* $\theta = \dfrac{r_1}{r_2}$*,* $d_\alpha = \dfrac{\alpha}{KN}$*, and* $\Delta_\alpha = \theta(1-d_\alpha)$*. Then*
$$\frac{(r_2 - r_1)\Delta_\alpha^{2^n}}{1 - d_\alpha - \Delta_\alpha^{2^n}} < r_1 - t_n < \frac{(r_2 - r_1)\theta^{2^n}}{1-\theta^{2^n}}, \quad n \geq 0,$$
where $d_\alpha < 1, \theta < 1$ *and* $\Delta_\alpha < 1$*,*

(b) *If* $r_1 = r_2$, *we have*

$$r_1 - t_n = r_1 \left(\frac{1}{2} - \frac{d_\alpha}{4(2-d_\alpha)} \right)^n, \quad n \geq 0.$$

PROOF. To obtain the error estimates for the sequence $\{t_n\}$ defined by (2.6), we first set $a_n = r_1 - t_n$ and $b_n = r_2 - t_n$. Moreover

$$p(t_n) = \frac{KN}{2} a_n b_n \quad \text{and} \quad p'(t_n) = -\frac{KN}{2}(a_n + b_n).$$

Now, by (2.6), we have

$$a_n = \frac{a_{n-1}^2}{a_{n-1}+b_{n-1}} \frac{KN(a_{n-1}+b_{n-1})^2 - 2\alpha b_{n-1}(a_{n-1}+b_{n-1}) + \alpha b_{n-1}^2}{KN(a_{n-1}+b_{n-1})^2 - 2\alpha a_{n-1} b_{n-1}} \tag{2.7}$$

and

$$b_n = \frac{b_{n-1}^2}{a_{n-1}+b_{n-1}} \frac{KN(a_{n-1}+b_{n-1})^2 - 2\alpha a_{n-1}(a_{n-1}+b_{n-1}) + \alpha a_{n-1}^2}{KN(a_{n-1}+b_{n-1})^2 - 2\alpha a_{n-1} b_{n-1}}.$$

If $r_1 < r_2$, we denote the ratio of a_n and b_n by δ_n. So

(2.8)

$$\delta_n = \delta_{n-1}^2 \frac{KN(1+\delta_{n-1})^2 - \alpha(1+2\delta_{n-1})}{KN(1+\delta_{n-1})^2 - \alpha\delta_{n-1}(2+\delta_{n-1})} = \delta_{n-1}^2 W(\delta_{n-1}).$$

Taking into account that the function

$$W(x) = \frac{KN(1+x)^2 - \alpha(1+2x)}{KN(1+x)^2 - \alpha x(2+x)}$$

is nondecreasing for all $x \in (0,1)$, we obtain

$$\delta_n < \delta_{n-1}^2 < \cdots < \delta_0^{2^n}$$

and

$$\delta_n > (1-d_\alpha)\delta_{n-1}^2 > \cdots > (1-d_\alpha)^{2^n-1}\delta_0^{2^n}.$$

Then the first part holds.

If $r_1 = r_2$, then $a_n = b_n$ and, by (2.7), we get

$$a_n = \frac{a_{n-1}}{4} \frac{4KN - 3\alpha}{2KN - \alpha}$$

Now the second part holds by recurrence. □

2.2 A semilocal convergence result.

Next we obtain a semilocal convergence result under Kantorovich-type conditions.

THEOREM 2.4. *Let us assume that conditions* (I)–(IV) *hold, and that*

$$N \in \left[b + \frac{4\alpha}{K}, \frac{b^2}{2aK} \right], \qquad 0 \leq \alpha < \frac{b}{8a}(b - 2aK).$$

Then:

(i) *If* $\overline{B(x_0, r_1)} \subseteq \Omega$ *then the sequence* $\{x_n\}$ *defined by* (2.4) *converges to a solution* x^* *of equation* (1.1) *and* $x_n, x^* \in \overline{B(x_0, r_1)}$. *The limit* x^* *is the unique solution of* (1.1) *in* $B(x_0, r) \cap \Omega$, *where* $r = r_2 + 2(N - b)/(KN)$. *Moreover*

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \quad n \geq 0, \tag{2.9}$$

$$\|x^* - x_n\| \leq r_1 - t_n, \quad n \geq 0, \tag{2.10}$$

where $\{t_n\}$ *is defined by* (2.6).

(ii) *Following the notation used in Theorem* 2.3 *we have the following error bounds:*

(a) *If* $r_1 < r_2$,

$$\|x^* - x_n\| < \frac{(r_2 - r_1)\theta^{2^n}}{1 - \theta^{2^n}}, \quad n \geq 0,$$

where $d_\alpha < 1$, $\theta < 1$ *and* $\Delta_\alpha < 1$.

(b) *If* $r_1 = r_2$,

$$\|x^* - x_n\| \leq r_1 \left(\frac{1}{2} - \frac{d_\alpha}{4(2 - d_\alpha)} \right)^n, \quad n \geq 0.$$

Proof. To prove (i) we show the following items are true by mathematical induction on $n \geq 0$

[I_n] there exists $\Gamma_n = F'(x_n)^{-1}$.

[II_n] $\|\Gamma_n A\| \leq -\dfrac{\alpha}{p'(t_n)}$.

[III_n] $\|\Gamma_n F'(x_0)\| \leq \dfrac{p'(t_0)}{p'(t_n)}$.

[IV_n] $\|\Gamma_0 F(x_n)\| \leq -\dfrac{p(t_n)}{p'(t_0)}$.

[V_n] $\tilde{H}(x_n)$ exists and $\|\tilde{H}(x_n)\| \leq \dfrac{1}{1 - \alpha D_p(t_n)}$,

[VI_n] $\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \quad n \geq 0$.

Notice that [I_0]–[V_0] follow immediately from (I)–(IV). We prove [I_{n+1}]–[VI_{n+1}] by using mathematical induction. Following Altman [1] and Yamamoto [12], under our assumptions (I)–(IV), $\Gamma_{n+1} = F'(x_{n+1})^{-1}$ exists and so [I_{n+1}], [II_{n+1}] and [III_{n+1}] are true taking into account that $\|\Gamma_{n+1} A\| \leq \|\Gamma_{n+1} F'(x_0)\| \, \|\Gamma_0 A\|$. To prove [$\mathrm{IV}_{n+1}$], we infer by Taylor's formula (2.4) that

$$\begin{aligned} F(x_{n+1}) &= F(x_n) + F'(x_n)(x_{n+1} - x_n) + \int_{x_n}^{x_{n+1}} (F'(x) - F'(x_n))\, dx \\ &= -\frac{A}{2} \tilde{H}(x_n) \Gamma_n F(x_n) \Gamma_n F(x_n) + \int_{x_n}^{x_{n+1}} (F'(x) - F'(x_n))\, dx. \end{aligned}$$

Thus

$$\|\Gamma_0 F(x_{n+1})\| \leq \frac{\alpha D_p(t_n)p(t_n)}{2b(1-\alpha D_p(t_n))} + \frac{K}{2}(t_{n+1}-t_n)^2.$$

Repeating the same process for the polynomial p, we obtain

$$\begin{aligned} p(t_{n+1}) &= p(t_n) + p'(t_n)(t_{n+1}-t_n) + \int_{t_n}^{t_{n+1}} (p'(t) - p'(t_n))\,dt \\ &= -\frac{\alpha}{2}\frac{D_p(t_n)p(t_n)}{1-\alpha D_p(t_n)} + \frac{KN}{2}(t_{n+1}-t_n)^2. \end{aligned}$$

To show that

$$\|\Gamma_0 F(x_{n+1})\| \leq -\frac{p(t_{n+1})}{p'(t_0)}, \tag{2.11}$$

we note that

$$\begin{aligned} &\|\Gamma_0 F(x_{n+1})\| + \frac{p(t_{n+1})}{p'(t_0)} \\ &\quad \leq -\frac{\alpha}{p'(t_0)}\frac{D_p(t_n)p(t_n)}{1-\alpha D_p(t_n)} + \frac{K}{2}(t_{n+1}-t_n)^2\left[1+\frac{N}{p'(t_0)}\right] \\ &\quad = \left[\frac{p(t_n)}{p'(t_n)}\right]^2 \left[\frac{\alpha}{b}\frac{1}{1-\alpha D_p(t_n)} + \frac{K}{2}\left(1-\frac{N}{b}\right)\left[1+\frac{1}{1-\alpha D_p(t_n)}\right]^2\frac{1}{4}\right] \\ &\quad \leq \left(\frac{p(t_n)}{p'(t_n)}\right)^2\left(\frac{\alpha}{b}\frac{1}{1-\alpha D_p(t_n)} + \frac{K}{2}\left(1-\frac{N}{b}\right)\right) \end{aligned}$$

since $\frac{1}{4}\left[1+\frac{1}{1-\alpha D_p(t_n)}\right]^2 \geq 1$ and $N > b$.

In addition, as $D_p(t)$ is a decreasing function, then $0 \leq D_p(t_n) \leq D_p(t_0) = a/b^2$, and we have

$$1 \leq \frac{1}{1-\alpha D_p(t_n)} \leq 1 + \frac{\alpha a}{b^2-\alpha a} = \frac{b^2}{b^2-\alpha a}.$$

Therefore

$$\|\Gamma_0 F(x_{n+1})\| + \frac{p(t_{n+1})}{p'(t_0)} \leq \left(\frac{p(t_n)}{p'(t_n)}\right)^2\left(\frac{\alpha}{b}\frac{b^2}{b^2-\alpha a} + \frac{K}{2}\left(1-\frac{N}{b}\right)\right)$$

On the other hand, as $N \geq b + \frac{4\alpha}{K}$ implies $\alpha < KN$ and we have

$$\alpha a < KNa < Ka\frac{b^2}{2Ka} = \frac{b^2}{2},$$

it follows

$$\begin{aligned} \frac{\alpha}{b}\frac{b^2}{b^2-\alpha a} + \frac{K}{2}\left(1-\frac{N}{b}\right) &= \frac{\alpha}{b}\left[1+\frac{\alpha a}{b^2-\alpha a}\right] + \frac{K}{2}\left(1-\frac{N}{b}\right) \\ &\leq \frac{2\alpha}{b} + \frac{K}{2}\left(1-\frac{N}{b}\right) = \frac{4\alpha + Kb - NK}{2b} \leq 0 \end{aligned}$$

since $N \geq b + 4\alpha/K$. Then (2.11) holds.

Now, from the definition of $\tilde{H}(x)$, $[\mathrm{V}_{n+1}]$ follows immediately. Finally, it is easy to prove $[\mathrm{VI}_{n+1}]$:

$$\|x_{n+2} - x_{n+1}\| = \left\| \left(I + \frac{1}{2} L(x_{n+1}) \tilde{H}(x_{n+1}) \right) \Gamma_{n+1} F(x_{n+1}) \right\|$$
$$\leq - \left(1 + \frac{\alpha}{2} \frac{D_p(t_{n+1})}{1 - \alpha D_p(t_{n+1})} \right) \frac{p(t_{n+1})}{p'(t_{n+1})} = t_{n+2} - t_{n+1}.$$

Then the induction is complete and (2.9) is satisfied. The convergence of $\{t_n\}$, given by (2.6), implies the convergence of $\{x_n\}$ to a limit x^* (see [8]).To see that x^* is a solution of $F(x) = 0$, we have $\|\Gamma_n F(x_n)\| \to 0$ as $n \to \infty$. Taking into account $[\mathrm{V}_n]$, we infer that $\|F(x_n)\| \to 0$ as $n \to \infty$. Consequently, we obtain $F(x^*) = 0$ by the continuity of F.

Next, for $q \geq 0$,

$$\|x_{n+q} - x_n\| \leq t_{n+q} - t_n,$$

and letting $q \to \infty$ we get (2.10).

To show the uniqueness of the solution x^*, let us assume that y^* is another solution of (1.1) in $B(x_0, r)$, where $r = r_2 + \dfrac{2(N-b)}{KN}$. From the equality

$$0 = \Gamma_0[F(y^*) - F(x^*)] = \int_0^1 \Gamma_0 F'(x^* + t(y^* - x^*))\, dt\, (y^* - x^*),$$

we have to prove that the operator $\int_0^1 \Gamma_0 F'(x^* + t(y^* - x^*))\, dt = P$ is invertible, then $y^* = x^*$. Indeed, from

$$\|I - P\| \leq \int_0^1 \|\Gamma_0(F'(x^* + t(y^* - x^*)) - F'(x_0))\|\, dt \leq K \int_0^1 \|x^* + t(y^* - x^*) - x_0\|\, dt$$
$$\leq K \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)\, dt < K \left(\frac{r_1 + r}{2} \right) = 1,$$

it follows that $\left[\int_0^1 \Gamma_0 F'(x^* + t(y^* - x^*))\, dt \right]^{-1}$ exists by the Banach lemma.

Item (ii) follows from Theorem 2.3. □

Remark 2.1. In practice, notice that we can always consider $N = b + 4\alpha/K$, since for this value we get the smallest error bound for (2.6) Theorem 2.3. In fact, from (2.8), it is easy to check that, for $x \in (0, 1)$ fixed, the function W is nondecreasing for all $N \in [b + 4\alpha/K, b^2/(2aK)]$ and consequently for $N = b + 4\alpha/K$ we obtain the smallest error bound.

To illustrate Theorem 2.4, we consider the example cited in Section 2, in which a third-order iteration cannot be applied, but the convergence conditions for (2.4) hold.

EXAMPLE 2.1. Let us consider $\Omega = \left(-\frac{1}{2}, \frac{3}{2}\right) \times \left(-\frac{1}{2}, \frac{3}{2}\right)$ and the system of equations $G(x,y) = (0,0)$, where G is given in (2.3). Then we have

$$G'(x,y) = \begin{pmatrix} 3x^2 \ln x^2 + 2x^2 & 2 \\ y-2 & x \end{pmatrix}.$$

The second derivative is the bilinear operator given by

$$G''(x,y) = \left(\begin{array}{cc} 6x \ln x^2 + 10x & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 1 & 0 \end{array}\right).$$

We take the max-norm in $\mathbb{R}^2$ and the norm $\|C\| = \max\{|c_{11}|+|c_{12}|, |c_{21}|+|c_{22}|\}$ for

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

As in [10] we define the norm of a bilinear operator B on $\mathbb{R}^2$ by

$$\|B\| = \sup_{\|x\|=1} \max_i \sum_{j=1}^{2} \left| \sum_{k=1}^{2} b_i^{jk} x_k \right|.$$

where $x = (x_1, x_2)$ and

$$B = \left(\begin{array}{cc} b_1^{11} & b_1^{12} \\ b_1^{21} & b_1^{22} \\ \hline b_2^{11} & b_2^{12} \\ b_2^{21} & b_2^{22} \end{array}\right).$$

It is easy to prove that G''' does not exist and G'' does not satisfy a Lipschitz condition in a neighbourhood of $(0,0)$. So we cannot apply the classical theorems of convergence in Ω. On the other hand,

$$\|G'(x) - G'(y)\| \le 22.30\|x-y\|.$$

If we choose $\mathbf{x_0} = (0,0)$, we have $K = 11.15$, $b = 2$ and $a = 1/16$. Then $b - 2Ka = 0.60625 > 0$ and $b(b-2Ka)/(8a) = 2.425$. If we now consider, for instance, the bilinear operator A

$$A = \left(\begin{array}{cc} 2 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 1 & 0 \end{array}\right), \tag{2.12}$$

then $\|A\| = 2 = \alpha$ and $N = b + 4\alpha/K \approx 2.7174888$. So the conditions of Theorem 2.4 are satisfied. Thus polynomial (2.5) becomes

$$p(t) = 15.15t^2 - 2t + 1/16.$$

This has two positive roots $r_1 = 0.0507928$ and $r_2 = 0.0812203$. Therefore the process given by (2.4) where A is defined in (2.12) converges to $(x^*, y^*) = (0, 1/32)$. Moreover this solution is unique in $B(\mathbf{x_0}, 0.1160752)$ and the error bound expressions are for all $n \geq 0$:

$$\frac{0.0304275\,(0.5949908)^{2^n}}{0.9514209 - (0.5949908)^{2^n}} < 0.0507928 - t_n < \frac{0.0304275\,(0.6253707)^{2^n}}{1 - (0.6253707)^{2^n}}.$$

3 Applications.

In this section, we provide three examples, where the features of iteration (2.4) are shown.

Firstly we consider an integral equation, used by other authors as a test one (see [5]), where its solution is approximated by (2.4) without discretizing the integral equation. Observe that the discretization process is usually essential when third-order methods are used. However, if the second derivative operator is approximated by a bilinear operator as in (2.4), the approximations can be calculated directly.

Example 3.1. Let us consider

$$x(s) = s - \frac{1}{2}\int_0^1 s\cos(x(t))\,dt.$$

Then we can take the operator $F : C[0,1] \to C[0,1]$ defined by

$$F(x)(s) = x(s) - s + \frac{1}{2}\int_0^1 s\cos(x(t))\,dt, \tag{3.1}$$

where $C[0,1]$ is the space of all continuous functions defined on the interval $[0,1]$ with the sup norm $\|\cdot\| = \|\cdot\|_\infty$.

If we choose $x_0 = x_0(s) = s$, it is easy to prove

$$F(x_0)(s) = \frac{\sin 1}{2 - \sin 1 + \cos 1}\, s,$$

$$[F'(x_0)]^{-1}z(s) = z(s) + \frac{\int_0^1 z(s)\sin s\,ds}{2 - \sin 1 + \cos 1}\, s.$$

Now we consider the bilinear operator:

$$Ayz(s) = -\frac{7s}{100}\int_0^1 y(t)z(t)\,dt.$$

Therefore the parameters appearing in Theorem 2.4 are

$$\|\Gamma_0\| \leq \frac{3 - \sin 1}{2 - \sin 1 + \cos 1} = \beta = 1.2705964\ldots,$$

$$\|\Gamma_0(F'(x) - F'(y))\| \leq \frac{1}{2}\|\Gamma_0\|\|x - y\|, \quad \Longrightarrow \quad K = \frac{\beta}{2} = 0.635298\ldots,$$

Table 3.1: Slopes and errors.

n	k_n	$\|x_n - x^*\| = \|k_n - k^*\|$
0	1	0.4776533906006485
1	0.5079785015334464	0.01445810787330248
2	0.5224217509702833	0.00001485843646553686
3	0.5224366093836842	$2.30646612919827 \times 10^{-11}$
4	0.5224366093993514	$7.397416013077418 \times 10^{-12}$

$$\|A\| = \frac{7}{100} = 0.07, \quad b = \frac{1}{\beta} = 0.787031\ldots, \quad a = \frac{\sin 1}{3 - \sin 1} = 0.389835\ldots.$$

It follows that $b - 2aK = 0.291709\ldots$, and the conditions of Theorem 2.4 are satisfied since

$$0 \leq \alpha = 0.07 < \frac{b}{8a}(b - 2aK) = 0.0736157\ldots.$$

Therefore the function x_0 can be used for starting the iterations. If we apply (2.4) to x_0, we obtain $x_1(s) = k_1 s$, where $k_1 = 0.5079785015334464$. Following this process, we obtain $x_n(s) = k_n s$, where the slopes k_n of these lines are given in Table 3.1.

In Table 3.1, using 16 significant decimal figures, the iterations obtained by (2.4) and the error are shown, taking into account that the solution of (3.1) is $x^*(s) = k^* s$ with $k^* = 0.5224366093993515$.

Next, we consider a simple example where the speed of convergence is improved by applying (2.4) instead of Newton's method.

EXAMPLE 3.2. Let us consider the system of equations $F(x, y) = (0, 0)$ where $\Omega = B((0, 0), 1.4)$ and

$$F(x, y) = \left(x^2 - 2y + \tfrac{1}{3},\, y^2 - 4x + \tfrac{2}{3}\right).$$

Then we get

$$F'(x, y) = \begin{pmatrix} 2x & -2 \\ -4 & 2y \end{pmatrix} \quad \text{and} \quad F'(x, y)^{-1} = \frac{1}{2(xy - 2)} \begin{pmatrix} y & 1 \\ 2 & x \end{pmatrix}$$

if (x, y) does not belong to the hyperbole $xy = 2$.

By Theorem 2.4, if we take $\mathbf{x_0} = (0, 0)$, we have

$$\|\Gamma_0\| = 1/2, \qquad \|\Gamma_0 F(x_0)\| = 1/6, \qquad K = 1.$$

To apply Theorem 2.4 we take $b = 2$ and $a = 1/3$ to obtain $\alpha \in [0, 1)$. Next, we choose for instance the bilinear operator given by

$$A = \left(\begin{array}{cc} 0.9 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0.9 \end{array} \right) \tag{3.2}$$

Table 3.2: Errors for iteration (2.4) and Newton's method.

n	Iteration (2.4)	Newton′s method
0	0.1819695	0.1819695
1	0.011934	0.0153029
2	0.0000314731	0.0000622645
3	4.93886×10^{-10}	2.05831×10^{-9}

such that $\|A\| = 0.9 = \alpha$. Taking into account that

$$(x^*,\, y^*) = (0.1749448936348263,\, 0.181969524571117)$$

is the solution of the system $F(x, y) = (0, 0)$ with 16 significant decimal figures, we observe (see Table 3.2) that (2.4), where A is defined in (3.2), provides a better approximation to (x^*, y^*) than Newton's method.

To finish we provide an example where the operational cost of (2.4) and Newton's method are connected.

Example 3.3. To illustrate the application of (2.4) to nonlinear partial differential equations, we consider a numerical example. We use the equation cited by Rall [10]:

$$\Delta u = u^2, \tag{3.3}$$

which satisfies the boundary conditions

$$\begin{cases} u(x,0) = 2x^2 - x + 1, & 0 \le x \le 1, \\ u(1,y) = 2, & 0 \le y \le 1, \\ u(x,1) = 2, & 0 \le x \le 1, \\ u(0,y) = 2y^2 - y + 1, & 0 \le y \le 1. \end{cases}$$

Rall uses Newton's method to solve (3.3). We then show that (2.4) approximates the solution of (3.3) faster than the Newton iteration. Moreover the operational cost of (2.4) is slightly higher than the corresponding one of Newton's method.

Now, this equation is discretized to replace it by a finite dimensional problem. So, we consider

$$P_{i,j} = (ih, jh), \quad h = \frac{1}{n+1}, \quad i, j = 0, 1, \ldots, n+1,$$

and the nonlinear system

$$-x_{i+1,j} - x_{i-1,j} - x_{i,j+1} - x_{i,j-1} + 4x_{i,j} = -h^2 x_{i,j}^2, \quad i, j = 1, \ldots, n, \tag{3.4}$$

where $x_{i,j}$, is the approximation to $u(P_{i,j})$, $i, j = 0, 1, \ldots, n+1$, which results from the discretization. If we denote

$$x_1 = x_{1,1}, \ldots, x_n = x_{n,1}, \quad x_{n+1} = x_{1,2}, \ldots, \quad x_m = x_{n,n}, \quad (m = n^2)$$

the system (3.4) is given by $B\overline{x} + \Phi(\overline{x}) = \overline{b}$, where

$$B = \begin{pmatrix} C & -I & 0 & \cdots & 0 \\ -I & C & -I & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -I \\ 0 & \cdots & 0 & -I & C \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 4 \end{pmatrix},$$

$B \in M(m\times m)$, $C \in M(n\times n)$, I is the identity matrix in $\mathbb{R}^n$, $\overline{x} = (x_1 \dots, x_m)^T$, $\Phi(\overline{x}) = h^2(x_1^2 \dots, x_m^2)^T$ and $\overline{b}$ is the vector given by the initial conditions.

If we consider $n = 3$ and $m = 9$, then $\overline{b}$ is

$$\overline{b} = (7/4, 1, 27/8, 1, 0, 2, 27/8, 2, 4)^T,$$

and (3.4) is given by the operator

$$F(\overline{x}) = B\overline{x} + \Phi(\overline{x}) - \overline{b}.$$

Hence

$$F'(\overline{x}) = B + \frac{1}{8}\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_9 \end{pmatrix}.$$

We now choose A as the bilinear operator given by

$$A\overline{u}\overline{v} = \frac{1}{8}\overline{u}\overline{v}, \quad \forall \overline{u}, \overline{v} \in \mathbb{R}^9.$$

To obtain Newton's sequence, given $\overline{x}_k \in \mathbb{R}^9$, we do the following steps:

1. Solve the linear system $F'(\overline{x}_k)\overline{c}_k = -F(\overline{x}_k)$.
2. Define $\overline{x}_{k+1} = \overline{x}_k + \overline{c}_k$.

Using (2.4), we have

$$L(x)(I - L(x))^{-1} = (I - L(x))^{-1}L(x)$$

and do the following steps:

1. Solve the linear system $F'(\overline{x}_k)\overline{c}_k = -F(\overline{x}_k)$.
2. Solve the linear system

$$[F'(\overline{x}_k) + A\overline{c}_k]\overline{d}_k = -F(\overline{x}_k) + \frac{1}{2}A\overline{c}_k^2.$$

3. Define $\overline{x}_{k+1} = \overline{x}_k + \overline{d}_k$.

Table 3.3: The solution $\overline{x} = (x_1, \dots, x_9)$ of (3.4).

x_1	1.02591171169
x_2	1.20971388714
x_3	1.51670303096
x_4	1.20971388714
x_5	1.38770378644
x_6	1.62587249196
x_7	1.51670303096
x_8	1.62587249196
x_9	1.76429948544

Table 3.4: Errors $\max_{0\le i\le m} |x_i^{(k)} - x_i|$ with intial approximation $x_i^{(0)} = 90$, $i = 1, \dots, 9$.

k	Newton's method	Iteration (2.4)
0	88.9740882883100000	88.9740882883100000
1	42.6345115411600000	18.6066802709600000
2	18.6066802709600000	1.0635641385100000
3	6.2177246388800000	$4.66358099999 \cdot 10^{-4}$
4	1.0635641385100000	0.0

In this case, given $\overline{x}$, $\overline{y} \in \mathbb{R}^m$, $A\overline{y}$ is a linear application with a diagonal associated matrix. If $\overline{x} = (x_1, \dots, x_9)$, $\overline{y} = (y_1, \dots, y_9)$, the associated matrix is $2h^2\text{diag}\{y_1, \dots, y_9\}$. Then Step 2 for (2.4) is very simple. Moreover (2.4) reaches the solution faster than Newton's method (see Table 3.4).

The solution $\overline{x} = (x_1, \dots, x_9)$ of (3.4) is given in Table 3.3 (see [10]). We denote the k-th iteration by $\overline{x}_k = \left(x_1^{(k)}, \dots, x_9^{(k)}\right)^T$. If we choose $x_i^{(0)} = 90$, $i = 1, 2, \dots, 9$, we obtain the results of Table 3.4 using 16 significant decimal digits. Table 3.4 contains the errors

$$\max_{0\le i\le m} \left|x_i^{(k)} - x_i\right|$$

for the iterates generated by Newton's method and iteration (2.4).

Remark 3.1. We indicate that the choice of the matrix A is an open study for future papers. In this paper, we only choose those with a little operational cost taking into account the influence of $\|A\|$ in the speed of convergence of iteration (2.4).

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