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Letter to the Editor

Remark on the convergence of the midpoint method under mild differentiability conditions[1]

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Abstract

We establish a convergence theorem for the Midpoint method using a new system of recurrence relations. The purpose of this note is to relax its convergence conditions. We also give an example where our convergence theorem can be applied but other ones cannot. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The purpose of this paper is the study of the Midpoint method for solving nonlinear operator equations. Chen and Argyros studied in [1] the convergence of this iterative process of order three, and defined, for all $n \geqslant 0$, by

$$y_n = x_n - [F'(x_n)]^{-1}F(x_n),$$

$$z_n = x_n + \tfrac{1}{2}(y_n - x_n),$$

$$x_{n+1} = x_n - F'(z_n)^{-1}F(x_n), \tag{1}$$

where F is a nonlinear operator defined on an open convex domain Ω of a Banach space X with values in another Banach space Y. Assume that F has a first-order continuous Fréchet derivative on Ω.

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Chen and Argyros give in [1] a Kantorovich-like convergence theorem, where the convergence conditions

$$\|F''(x)\| \leqslant L_1, \qquad \|F''(x) - F''(y)\| \leqslant L_2\|x-y\|, \quad x, y \in \Omega \tag{2}$$

are required.

In this note, we give another convergence theorem for nonlinear operator equations using a new type of recurrence relations for this method. In order to provide the convergence of (1), it is assumed

$$\|F'(x) - F'(y)\| \leqslant K\|x-y\|, \quad x, y \in \Omega, \tag{3}$$

instead of assumption (2). Observe that we can apply method (1) under the same condition (3) as for Newton's method (see [4]). Finally, we provide an example where condition (2) fails but (3) does not.

We denote $\overline{B(x,r)} = \{y \in X;\ \|y-x\| \leqslant r\}$ and $B(x,r) = \{y \in X;\ \|y-x\| < r\}$.

2. A convergence theorem

Let $x_0 \in \Omega$ and suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathscr{L}(Y,X)$ exists, where $\mathscr{L}(Y,X)$ is the set of bounded linear operators from Y into X.

Let us assume that

(c_1) $\|\Gamma_0\| \leqslant \beta$,

(c_2) $\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta$,

(c_3) $\|F'(x) - F'(y)\| \leqslant K\|x-y\|, \quad x, y \in \Omega,\ K \geqslant 0$.

We denote $a_0 = K\beta\eta$ and define the sequence

$$a_{n+1} = a_n f(a_n)^2 g(a_n),$$

where

$$f(x) = \frac{2-x}{2-3x} \quad \text{and} \quad g(x) = \frac{x(4-x)}{(2-x)^2}.$$

We study the convergence of the sequence $\{x_n\}$ given by (1) to a solution x^* of an equation $F(x) = 0$. Assuming $x_0, y_0 \in \Omega$, $a_0 < s = 0.310102\ldots$ (s is the smallest positive root of polynomial $q(x) = 5x^2 - 8x + 2$) and initial hypotheses (c_1)–(c_3), as $z_0 \in \Omega$, we have

$$\|I - \Gamma_0 F'(z_0)\| \leqslant \|\Gamma_0\| \|F'(x_0) - F'(z_0)\| \leqslant \frac{K}{2}\|\Gamma_0\|\|y_0 - x_0\| \leqslant \frac{a_0}{2} < 1$$

and, by the Banach lemma, $\overline{\Gamma}_0\Gamma_0^{-1} = F'(z_0)^{-1}F'(x_0)$ exists and $\|\overline{\Gamma}_0\Gamma_0^{-1}\| \leqslant 2/(2-a_0)$. Then, x_1 is well defined and

$$\|x_1 - x_0\| \leqslant \|\overline{\Gamma}_0 F(x_0)\| \leqslant \|\overline{\Gamma}_0 F'(x_0)\|\|\Gamma_0 F(x_0)\| \leqslant \frac{2\eta}{2-a_0}.$$

Under the hypotheses mentioned above, $\{a_n\}$ is a decreasing sequence. Applying mathematical induction on n and assuming $x_n, y_n \in \Omega$ we can prove the following statements for all $n \geqslant 1$:

[I_n] $\|\Gamma_n\Gamma_{n-1}^{-1}\| = \|F'(x_n)^{-1}F'(x_{n-1})\| \leqslant f(a_{n-1})$,
[II_n] $\|y_n - x_n\| = \|\Gamma_n F(x_n)\| \leqslant f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\|$,
[III_n] $K\|\Gamma_n\|\|y_n - x_n\| \leqslant a_n$,
[IV_n] $\|\overline{\Gamma}_n\Gamma_n^{-1}\| = \|F'(z_n)^{-1}F'(x_n)\| \leqslant \frac{2}{2-a_n}$,
[V_n] $\|x_{n+1} - x_n\| \leqslant \frac{2}{2-a_n}\|y_n - x_n\|$.

The existence of $\Gamma_1\Gamma_0^{-1}$ and item [I_1] follows from the Banach lemma. Taking into account the Taylor's formula and (1), we have

$$\Gamma_0 F(x_1) = \Gamma_0 F(x_0) + (x_1 - x_0) + \int_{x_0}^{x_1} \Gamma_0[F'(x) - F'(x_0)]\,\mathrm{d}x$$

$$= \overline{\Gamma}_0[F'(z_0) - F'(x_0)]\Gamma_0 F(x_0) + \int_0^1 \Gamma_0[F'(x_0 + t(x_1 - x_0)) - F'(x_0)](x_1 - x_0)\,\mathrm{d}t.$$

In consequence [II_1]–[V_1] hold.

Now assuming that [I_n]–[V_n] are true for a fixed $n \geqslant 1$ we can easily prove [I_{n+1}]–[V_{n+1}] and the induction is complete.

Then the next convergence theorem is established.

Theorem 2.1. *Let X, Y be Banach spaces and $F: \Omega \subseteq X \to Y$ be a nonlinear once Fréchet differentiable operator in an open convex domain Ω. Let us assume that $\Gamma_0 = F'(x_0)^{-1} \in \mathscr{L}(Y,X)$ exists at some $x_0 \in \Omega$ and* (c$_1$)–(c$_3$) *are satisfied. Let us denote $a_0 = K\beta\eta$. Suppose that $a_0 \in (0,s)$. Then, if $\overline{B(x_0, \eta/a_0)} \subseteq \Omega$, sequence $\{x_n\}$ defined in* (1) *and starting at x_0 converges at least R-quadratically to a solution x^* of $F(x) = 0$. In that case, the solution x^* and the iterates x_n, y_n and z_n belong to $\overline{B(x_0, \eta/a_0)}$. Moreover, the solution x^* is unique in $B(x_0, 2/(K\beta) - \eta/a_0) \cap \Omega$.*

Proof. We prove that $\{x_n\}$ is a Cauchy sequence. For that we denote

$$M_n = \frac{a_n(4 - a_n)}{(2 - a_n)(2 - 3a_n)} = f(a_n)g(a_n).$$

It is easy to prove that

$$a_n \leqslant \gamma^{2^{n-1}} a_{n-1} \leqslant \cdots \leqslant \gamma^{2^n - 1} a_0 \quad \text{where } \gamma = a_1/a_0 \in (0,1).$$

We now obtain $M_n \leqslant \gamma^{2^n - 1} M_0 = \gamma^{2^n}\Delta$ with $\Delta = 1/f(a_0) < 1$, and consequently

$$\|x_{n+m} - x_n\| \leqslant \|x_{n+m} - x_{n+m-1}\| + \cdots + \|x_{n+1} - x_n\|$$

$$\leqslant \frac{2}{2 - a_0}\left(\prod_{j=0}^{n+m-2} M_j + \cdots + \prod_{j=0}^{n-1} M_j\right)\|y_0 - x_0\| \leqslant \frac{2\eta\Delta^n(1 - \Delta^m)}{(2 - a_0)\gamma(1 - \Delta)}\gamma^{2^n}.$$

Therefore, $\{x_n\}$ given by (1), is a Cauchy sequence. Moreover, from a similar reasoning, it immediately follows that $x_n, y_n, z_n \in \overline{B(x_0, \eta/a_0)} \subseteq \Omega$. As a result, the sequence $\{x_n\}$ converges to $x^* \in \overline{B(x_0, \eta/a_0)}$. We now deduce that $F(x^*)=0$ from the continuity of F and the fact that $\|y_n - x_n\| \leqslant M_0^n \|y_0 - x_0\| \to 0$ when $n \to \infty$.

On the other hand, we have

$$\|x^* - x_n\| \leqslant \frac{2\eta}{(2-a_0)\gamma(1-\Delta)} \gamma^{2^n},$$

so this method converges at least R-quadratically [3] to x^*.

Finally, to show the uniqueness, let us assume that $y^* \in B(x_0, 2/(K\beta) - \eta/a_0) \cap \Omega$ is another solution of $F(x)=0$. From the approximation

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\,dt\,(y^* - x^*),$$

in a similar way than in [2], we prove that the operator $[\int_0^1 F'(x^* + t(y^* - x^*))\,dt]^{-1}$ exists and therefore $y^* = x^*$. □

3. Example

We provide an example where the assumptions given in (2) fail but the conditions of Theorem 2.1 are fulfilled.

Let us consider the system of equations $F(x,y)=0$ where $F : (-\frac{1}{2}, \frac{3}{2}) \times (-\frac{1}{2}, \frac{3}{2}) \to \mathbb{R}^2$ such that

$$F(x,y) = (x^3 \ln x^2 + 4y - 1/8,\ x(y-4)).$$

If we choose $\mathbf{x}_0 = (0,0)$, then F does not satisfy the Lipschitz condition for F'' given in (2).

On the other hand, we have

$$F'(x,y) = \begin{pmatrix} 3x^2 \ln x^2 + 2x^2 & 2 \\ y-2 & x \end{pmatrix}.$$

Taking into account the max-norm in $\mathbb{R}^2$ and the norm $\|C\| = \max\{|c_{11}| + |c_{12}|, |c_{21}| + |c_{22}|\}$ for

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

we can apply Theorem 2.1, since

$$\beta = \|\Gamma_0\| = 1/4, \quad \eta = \|y_0 - x_0\| = 1/32,$$

$$\|F'(x,y) - F'(u,v)\| \leqslant 22.30 \|(x,y) - (u,v)\|$$

and, consequently, $a_0 = K\beta\eta = 0.1742187 \leqslant s = 0.310102\ldots$. As a result, we can study the convergence of this system of equations by Theorem 2.1.

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