Journal of Number Theory 130 (2010) 1876–1887

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Moments of combinatorial and Catalan numbers ☆

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ARTICLE INFO

Article history:
Received 20 August 2009
Revised 10 December 2009
Available online 9 April 2010
Communicated by Pilar Bayer

Keywords:
Catalan numbers
Combinatorial identities
Binomial coefficients

ABSTRACT

In this paper we obtain the moments $\{\Phi_m\}_{m\geqslant 0}$ defined by

$$\Phi_m(n) := \sum_{p=1}^{n+1} (2p-1)^m \binom{2n+1}{n+1-p}^2,$$

$$n \in \mathbb{N},\ m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where $\binom{m}{n}$ is the usual combinatorial number. We also provide the moments in the Catalan triangle whose (n, p) entry is defined by

$$A_{n,p} := \frac{2p-1}{2n+1}\binom{2n+1}{n+1-p}, \quad n, p \in \mathbb{N},\ p \leqslant n+1,$$

and, in particular, new identities involving the well-known Catalan numbers.

0. Introduction

Although there exist several triangles known as the “Catalan triangle”, the following one is one of the most-standing form

☆ The first author has been partly supported by the DGI-FEDER Project MTM 2007-61446 and the DGA project E-64. The second author has been partly supported by the Ministry of Science and Innovation, Project MTM 2008-01952.

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2010.01.018

$k \setminus m$	0	1	2	3	4	5	6	...
0	1							
1	1	1						
2	1	2	2					
3	1	3	5	5				
4	1	4	9	14	14			
5	1	5	14	28	42	42		
6	1	6	20	48	90	132	132	
...	...	...	...	...	...	...	...	

see for example [10]. Each entry $C_{k,m}$ is defined by

$$C_{k,m} := \frac{(k+m)!(k-m+1)}{m!(k+1)!}, \quad 0 \leqslant m \leqslant k.$$

Notice that $C_{k,k}$ is the well-known Catalan number C_k, given by the formula

$$C_k = \frac{1}{k+1}\binom{2k}{k}, \quad k \geqslant 1.$$

The Catalan numbers may be defined recursively by $C_0 = 1$ and $C_k = \sum_{i=0}^{k-1} C_i C_{k-1-i}$ for $k \geqslant 1$. These numbers appear in a wide range of problems, see [11]. For instance, the Catalan number C_k counts the number of ways to triangulate a regular polygon with $k+2$ sides; or, let $2k$ people seat around a circular table, the Catalan number C_k gives the number of ways that all of them are simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other.

In the Catalan triangle, we now consider numbers $C_{k,m}$ in the same diagonal such that $k+m$ is odd. We write $k+m = 2n-1$ and $p = n-m$ to get Shapiro's triangle introduced in [8],

$n \setminus p$	1	2	3	4	5	6	...
1	1						
2	2	1					
3	5	4	1				
4	14	14	6	1			
5	42	48	27	8	1		
6	132	165	110	44	10	1	
...	...	...	...	...	...	...	...

(1)

whose entries are given by

$$B_{n,p} := \frac{p}{n}\binom{2n}{n-p}, \quad n, p \in \mathbb{N}, \; p \leqslant n.$$

On the other hand, when $k+m = 2n$ and $p = n-m+1$, we recover the following triangle

$n \setminus p$	1	2	3	4	5	6	...
1	1	1					
2	2	3	1				
3	5	9	5	1			
4	14	28	20	7	1		
5	42	90	75	35	9	1	
6	132	297	275	154	54	11	1
...	...	...	...	...	...	...	...

(2)

whose entries are defined by

$$A_{n,p} := \frac{2p-1}{2n+1}\binom{2n+1}{n+1-p}, \quad n, p \in \mathbb{N},\ p \leqslant n+1. \tag{3}$$

Observe that the numbers $A_{n,p}$ satisfy the following recurrence relation

$$A_{n,p} = A_{n-1,p-1} + 2A_{n-1,p} + A_{n-1,p+1}, \quad p \geqslant 2.$$

In [5] other generalized Catalan numbers are considered.

Recently, Catalan triangle (1) has been studied in detail. The formula

$$\sum_{p=1}^{i} B_{n,p} B_{n,n+p-i}(n+2p-i) = (n+1)C_n\binom{2(n-1)}{i-1}, \quad i \leqslant n, \tag{4}$$

which appears in a problem related with the dynamical behavior of a family of iterative processes has been proved in [4, Theorem 5]. Using the WZ-theory (see [7,12]), the moments Ω_m defined by

$$\Omega_m(n) := \sum_{p=1}^{n} p^m B_{n,p}^2, \quad n \in \mathbb{N},\ m \in \mathbb{N}_0,$$

were given for $1 \leqslant m \leqslant 7$ in [6, Theorems 2.1 and 2.2]. New techniques based on the symmetric functions were used in [1] to give explicit expressions of $\{\Theta_m\}_{m \geqslant 0}$ defined by

$$\Theta_m(n) := \sum_{p=1}^{n} p^m \binom{2n}{n-p}^2, \quad n \in \mathbb{N},\ m \in \mathbb{N}_0,$$

and consequently for $\{\Omega_m\}_{m \geqslant 0}$, since $n^2\Omega_m(n) = \Theta_{m+2}(n)$, for $n \in \mathbb{N}, m \in \mathbb{N}_0$. Moreover, in [9] equivalent expressions of Ω_m are shown using some recurrence relations. More recently, divisibility properties of sums of products of binomial coefficients are obtained in [3] using the Newton interpolation formula.

In the first section of this paper, we improve some results presented in [1] to apply them to some questions posed in [6]. In the second and third sections, we consider the moments $\{\Phi_m\}_{m \geqslant 0}$ defined by

$$\Phi_m(n) := \sum_{p=1}^{n+1} (2p-1)^m \binom{2n+1}{n+1-p}^2, \quad n \in \mathbb{N}.$$

Using symmetric functions, we obtain explicit results for $\{\Phi_m\}_{m\geqslant 0}$ which are similar to the expressions of $\{\Theta_m\}_{m\geqslant 0}$. We apply these results to give explicit formulae of $\{\Psi_m\}_{m\geqslant 0}$, where

$$\Psi_m(n) := \sum_{p=1}^{n+1}(2p-1)^m A_{n,p}^2, \quad n\in\mathbb{N}.$$

Note that

$$\Psi_m(n) = \frac{1}{(2n+1)^2}\Phi_{m+2}(n), \quad m,n\in\mathbb{N}_0. \tag{5}$$

Finally, we remark that the expressions of $\{\Omega_m\}_{m\geqslant 0}$ and $\{\Psi_m\}_{m\geqslant 0}$ involve the Catalan numbers and rational functions on the discrete variable n, see Theorems 2, 6 and 10. However, checking particular values of m, we conjecture some improvements of these results which are posed in Remarks 3, 7 and 11, respectively.

1. Symmetric functions and combinatorial numbers

In what follows, we denote by $n!$ and $\langle x\rangle_m$ the usual factorial number and the Pochhammer function given by

$$\langle x\rangle_0 := 1, \qquad \langle x\rangle_m := x(x-1)\cdots(x-m+1), \quad x\in\mathbb{R},\ m\in\mathbb{N}.$$

We use the usual combinatorial notation $\binom{x}{m}$ to represent the polynomial of degree m in the variable x given by $\binom{x}{m} := \frac{\langle x\rangle_m}{m!}$.

Using the theory of symmetric functions, we may write

$$x^{2m} = \sum_{k=0}^{m}(-1)^k\langle y+x\rangle_k\langle y-x\rangle_k\sigma_{k,m-k}(y), \quad x,y\in\mathbb{R},\ m\in\mathbb{N}_0, \tag{6}$$

where the polynomial $\sigma_{k,m-k}$ is defined by

$$\sigma_{k,m-k}(y) := \sum_{0\leqslant l_1\leqslant l_2\leqslant\cdots\leqslant l_{m-k}\leqslant k}(y-l_1)^2(y-l_2)^2\cdots(y-l_{m-k})^2, \quad y\in\mathbb{R},$$

with $l_i\in\mathbb{N}_0$ for $i\in\{1,\ldots,m-k\}$. Observe that the degree of $\sigma_{k,m-k}$ is $2(m-k)$ and

$$\sigma_{k,m-k}(y) = \frac{2(-1)^k}{\langle 2y\rangle_{2k+1}}\sum_{i=0}^{k}\binom{2y}{i}\binom{2k-2y}{k-i}(y-i)^{2m+1}, \quad y\in\mathbb{R}. \tag{7}$$

We denote by $\lambda_k(m,y)$ the polynomial in the variable y given by

$$\lambda_k(m,y) := \sum_{i=0}^{k}\binom{2y}{i}\binom{2k-2y}{k-i}(y-i)^{2m+1}, \quad y\in\mathbb{R}. \tag{8}$$

The first polynomials $\lambda_k(m,y)$ for $k=0,1,2$ are the following

$$\lambda_0(m, y) = y^{1+2m},$$
$$\lambda_1(m, y) = 2y(1-y)\big(y^{2m} - (y-1)^{2m}\big),$$
$$\lambda_2(m, y) = y(y-2)\big((2y-3)y^{2m} - 4(y-1)^{2m+1} + (2y-1)(y-2)^{2m}\big),$$

for $m \geqslant 2$ and $y \in \mathbb{R}$, see more details in [1,2]. In the following lemma we give interesting properties of these polynomials $\lambda_k(m, y)$.

Lemma 1. *Let $m \in \mathbb{N}_0$ be and $0 \leqslant k \leqslant m$. The polynomial $\lambda_k(m, y)$ defined in (8) satisfies the following properties.*

(i) *The degree of $\lambda_k(m, y)$ is at most $2m+1$ and 2^k divides to $\lambda_k(m, y)$.*
(ii) *The equality $\lambda_k(m, \frac{k}{2} - y) = -\lambda_k(m, \frac{k}{2} + y)$ holds for all $y \in \mathbb{R}$.*
(iii) *The values of $\frac{j}{2}$ with $0 \leqslant j \leqslant 2k$ are roots of $\lambda_k(m, y)$.*

Proof. To show (i), note that $2(-1)^k\lambda_k(m, y) = \sigma_{k,m-k}(y)\langle 2y\rangle_{2k+1}$. On the other hand, $\sigma_{k,m-k}$ is a polynomial of degree $2(m-k)$ and $\langle 2y\rangle_{2k+1} = 2^{k+1}P(y)$ where P is a polynomial with integer coefficients.

We take $y \in \mathbb{R}$ and change the index i by $k - j$ in (8) to get

$$\begin{aligned}
\lambda_k\left(m, \frac{k}{2} - y\right) &= -\sum_{j=0}^{k} \binom{k-2y}{k-j}\binom{k+2y}{j}\left(\frac{k}{2} + y - j\right)^{2m+1} \\
&= -\sum_{j=0}^{k} \binom{2(\frac{k}{2}+y)}{j}\binom{2k - 2(\frac{k}{2}+y)}{k-j}\left(\frac{k}{2} + y - j\right)^{2m+1} \\
&= -\lambda_k\left(m, \frac{k}{2} + y\right).
\end{aligned}$$

The proof of item (ii) is concluded. From the equality $2(-1)^k\lambda_k(m, y) = \sigma_{k,m-k}(y)\langle 2y\rangle_{2k+1}$, we deduce that the values of $\frac{j}{2}$ with $0 \leqslant j \leqslant 2k$ are roots of $\lambda_k(m, y)$ and the proof of the lemma is finished. □

Next result answers the first and the second questions posed in [6, Section 3]. The main aim of these problems is to write $\Omega_m(n)$ in terms of the Catalan numbers and rational functions on the variable n.

Theorem 2. *There exist P_{3m+1}, Q_{2m+2}, R_{3m-1} polynomials of integer coefficients and degree at most $3m+1$, $2m+2$ and $3m-1$ respectively, such that*

$$\begin{aligned}
\Omega_{2m}(n) &= \frac{P_{3m+1}(n)}{\prod_{l=1}^{m}(4n - (2l+1))}C_{2n-1}, \quad m \geqslant 0, \\
\Omega_{2m+1}(n) &= Q_{2m+2}(n+1)C_nC_{n-2}, \quad 3 \geqslant m \geqslant 0, \\
\Omega_{2m+1}(n) &= \frac{R_{3m-1}(n)}{\prod_{l=1}^{m-3}(2n - (2l+3))}(n+1)C_nC_{n-2}, \quad m \geqslant 4,
\end{aligned}$$

for $n \in \mathbb{N}$.

Proof. It is direct to check that $C_{2n} = \frac{2(4n-1)}{2n+1} C_{2n-1}$ and

$$C_{2n-k} = \frac{(2n)\cdots(2n-k+2)}{2^{k-1}(4n-3)(4n-5)\cdots(4n-(2k-1))} C_{2n-1}, \quad k \geqslant 2.$$

Applying [1, Proposition 5], we obtain

$$\begin{aligned}\Omega_{2m}(n) &= C_{2n-1}(4n-1)n^{2m} + 2C_{2n-1}\big((n-1)^{2m+2} - n^{2m+2}\big) \\ &\quad + C_{2n-1} \sum_{k=2}^{m+1} \frac{\lambda_k(1+m,n)(2n-1)\cdots(2n-k+1)}{2^{k-1}n(n-k)(4n-3)(4n-5)\cdots(4n-(2k-1))}.\end{aligned}$$

From items (i) and (iii) of Lemma 1, $2^{k-1}n(n-k)$ divides to $\lambda_k(1+m,n)$ and

$$\Omega_{2m}(n) = \frac{P_{3m+1}(n)}{\prod_{l=1}^{m}(4n-(2l+1))} C_{2n-1}$$

where P_{3m+1} is a polynomial of integer coefficients and degree at most $3m+1$.

Now we write $C_n = \frac{4(2n-1)(2n-3)}{(n+1)n} C_{n-2}$, $C_{n-1} = \frac{2(2n-3)}{n} C_{n-2}$ and

$$C_{n-k} = \frac{(n-1)(n-2)\cdots(n+2-k)}{2^{k-2}(2n-5)\cdots(2n+1-2k)} C_{n-2}, \quad k \geqslant 3.$$

And from [1, Proposition 5], it follows

$$\begin{aligned}\frac{1}{(n+1)C_nC_{n-2}}\Omega_{2m+1}(n) &= (2n-1)(2n-3)n^{2m} + 2(1-n)\frac{(2n-3)}{2n-1}\big(n^{2m+2} - (n-1)^{2m+2}\big) \\ &\quad + \frac{\lambda_2(1+m,n)}{4n} + \frac{(n-1)(n-2)}{2^2 n(2n-3)(2n-5)}\lambda_3(1+m,n) \\ &\quad + \frac{(n-1)(n-3)}{2^4 n(2n-5)(2n-7)}\lambda_4(1+m,n) \\ &\quad + \sum_{k=5}^{m+1} \frac{(n-1)(n-2)\cdots(n+2-k)(n+1-k)}{2^{k-1}n(2n-k)(2n-5)\cdots(2n+1-2k)}\lambda_k(1+m,n).\end{aligned}$$

From Lemma 1, we conclude that there exists a polynomial Q_{2m+2} of integer coefficients and degree at most $2m+2$ such that

$$\Omega_{2m+1}(n) = Q_{2m+2}(n)(n+1)C_nC_{n-2}, \quad n \in \mathbb{N},$$

for $m \leqslant 3$. For $m \geqslant 4$, there exist polynomials $\tilde{Q}_{2m+2}$ and $(R_{k,m})_{5 \leqslant k \leqslant m+1}$ of integer coefficients and degree at most $2m+2$ and $2m+3$ respectively, such that

$$\frac{1}{(n+1)C_nC_{n-2}}\Omega_{2m+1}(n) = \tilde{Q}_{2m+2}(n) + \sum_{k=5}^{m+1} \frac{R_{k,m}(n)}{2n-k}.$$

Thus, there exists a polynomial R_{3m-1} of integer coefficients and degree at most $3m-1$ such that

$$\Omega_{2m+1}(n) = \frac{R_{3m-1}(n)}{\prod_{l=1}^{m-3}(2n-(l+4))}(n+1)C_nC_{n-2}, \quad n \in \mathbb{N},$$

for $m \geqslant 4$. □

Remark 3. The following values of Ω_{2m},

$$\Omega_0(n) = C_{2n-1},$$
$$\Omega_2(n) = \frac{(3n-2)n}{4n-3}C_{2n-1},$$
$$\Omega_4(n) = \frac{(15n^3-30n^2+16n-2)n}{(4n-3)(4n-5)}C_{2n-1},$$
$$\Omega_6(n) = \frac{(105n^5-420n^4+588n^3-356n^2+96n-10)n}{(4n-3)(4n-5)(4n-7)}C_{2n-1},$$

are obtained in [6, Theorem 2.1] using the WZ-theory. Note that the degree of polynomial P_{3m+1} in all these cases is $2m$. It is a open problem to show that in general P_{3m+1} has degree $2m$.

In a similar way, the first values of Ω_{2m+1} are given in [6, Theorem 2.2] and [1, Section 5],

$$\Omega_1(n) = (2n-3)(n+1)C_nC_{n-2},$$
$$\Omega_3(n) = n(2n-3)(n+1)C_nC_{n-2},$$
$$\Omega_5(n) = n\big(3n^2-5n+1\big)(n+1)C_nC_{n-2},$$
$$\Omega_7(n) = n\big(6n(n-1)^2-1\big)(n+1)C_nC_{n-2},$$
$$\Omega_9(n) = \frac{n(30n^5-150n^4+252n^3-185n^2+65n-9)}{2n-5}(n+1)C_nC_{n-2}.$$

Observe that, in these cases the degrees of polynomials Q_{2m+2} and R_{3m-1} are $m+1$ and $2m-2$, respectively.

2. Even moments of squares of combinatorial numbers

In this section we calculate the even moments $\{\Phi_{2m}\}_{m\geqslant 0}$ and $\{\Psi_{2m}\}_{m\geqslant 0}$ defined in the Introduction. In Theorem 6 we express $\{\Psi_{2m}\}_{m\geqslant 0}$ in terms of the Catalan numbers. The following result is inspired in [1, Theorem 1].

Theorem 4. *The following equalities hold*

(i) $$\Phi_{2m}(n) = 2^{2m}\sum_{k=0}^{m}\frac{\lambda_k(m, n+\frac{1}{2})}{2n+1-2k}\binom{4n+2-2k}{2n+1-k},$$

(ii) $$\Psi_{2m}(n) = \frac{2^{2m+2}}{(2n+1)^2}\sum_{k=0}^{m+1}\frac{\lambda_k(m+1, n+\frac{1}{2})}{2n+1-2k}\binom{4n+2-2k}{2n+1-k},$$

for $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

Proof. From formula (6), we have

$$\left(p-\frac{1}{2}\right)^{2m}=\sum_{k=0}^{m}(-1)^k\langle n+p\rangle_k\langle n-p+1\rangle_k\sigma_{k,m-k}\left(n+\frac{1}{2}\right), \tag{9}$$

for $p\in\mathbb{N}$. Hence, it follows

$$\begin{aligned}
\Phi_{2m}(n)&=\frac{1}{2}\sum_{p=-n}^{n+1}(2p-1)^{2m}\binom{2n+1}{n+1-p}^2\\
&=2^{2m-1}\sum_{k=0}^{m}(-1)^k\sigma_{k,m-k}\left(n+\frac{1}{2}\right)\sum_{p=-n}^{n+1}\langle n+p\rangle_k\langle n-p+1\rangle_k\binom{2n+1}{n+1-p}^2\\
&=2^{2m-1}\sum_{k=0}^{m}(-1)^k\sigma_{k,m-k}\left(n+\frac{1}{2}\right)\langle 2n+1\rangle_k^2\sum_{p=-n}^{n+1}\binom{2n+1-k}{n+p-k}\binom{2n+1-k}{n-p-k-1}\\
&=2^{2m-1}\sum_{k=0}^{m}(-1)^k\sigma_{k,m-k}\left(n+\frac{1}{2}\right)\langle 2n+1\rangle_k^2\binom{4n+2-2k}{2n+1-2k}
\end{aligned}$$

where we have applied the Chu–Vandermonde convolution formula (see for example [11]) in the latter equality. From formulae (7) and (8), we have

$$\sigma_{k,m-k}(y)=\frac{2(-1)^k}{\langle 2y\rangle_{2k+1}}\lambda_k(m,y),\quad y\in\mathbb{R}.$$

Therefore

$$\Phi_{2m}(n)=2^{2m}\sum_{k=0}^{m}\frac{\lambda_k(m,n+\frac{1}{2})}{2n+1-2k}\binom{4n+2-2k}{2n+1-k},$$

and the proof of item (i) is concluded. Now the proof of item (ii) follows from formula (5) and the above. □

Remark 5. In fact, we may write Ψ_{2m} in terms of the Catalan numbers in this way,

$$\Psi_{2m}(n)=\frac{2^{2m+2}}{(2n+1)^2}\sum_{k=0}^{m+1}\lambda_k\left(m+1,n+\frac{1}{2}\right)\frac{2n+2-k}{2n+1-2k}C_{2n+1-k}, \tag{10}$$

for $m\in\mathbb{N}_0$ and $n\in\mathbb{N}$.

Theorem 6. *Given $m\in\mathbb{N}_0$, there exists a polynomial of integer coefficients and degree at most $3m+1$, R_{3m+1}, such that*

$$\Psi_{2m}(n)=\frac{R_{3m+1}(n)}{\prod_{l=0}^{m-1}(4n-(2l+1))}C_{2n},\quad n\in\mathbb{N}.$$

Proof. It is direct to check that $C_{2n+1} = \frac{4n+1}{n+1} C_{2n}$ and

$$C_{2n+1-k} = \frac{(2n+1)(2n)\cdots(2n-k+3)}{2^{k-1}(4n-1)(4n-3)\cdots(4n+3-2k)} C_{2n}, \quad k \geqslant 2.$$

We use formula (10) to obtain

$$\begin{aligned}\Psi_{2m}(n) &= (2n+1)^{2m}(4n+1)C_{2n} - \frac{1}{2}\big((2n+1)^{2m+2} - (2n-1)^{2m+2}\big)C_{2n} \\ &\quad + C_{2n} 2^m \sum_{k=2}^{m+1} 2^{m+3-k} \frac{\lambda_k(m+1, n+\frac{1}{2})}{(2n+1)(2n+1-2k)} \frac{\prod_{j=0}^{k-2}(2n-j)}{\prod_{j=0}^{k-2}(4n-(2j+1))}.\end{aligned}$$

From Lemma 1, $(2n+1)(2n+1-2k)$ divides $\lambda_k(m+1, n+\frac{1}{2})$. Thus, there exists a polynomial of integer coefficients and degree at most $3m+1$, R_{3m+1}, such that

$$\Psi_{2m}(n) = \frac{R_{3m+1}(n)}{\prod_{l=0}^{m-1}(4n-(2l+1))} C_{2n}, \quad n \in \mathbb{N},$$

and we conclude the proof. $\square$

Remark 7. Particular values of Ψ_{2m} are obtained using item (ii) of Theorem 4,

$$\begin{aligned}\Psi_0(n) &= C_{2n}, \\ \Psi_2(n) &= \frac{-1+4n+12n^2}{4n-1} C_{2n}, \\ \Psi_4(n) &= \frac{3-16n-104n^2+240n^4}{(4n-1)(4n-3)} C_{2n}, \\ \Psi_6(n) &= \frac{-15+92n+1116n^2+2080n^3-4368n^4-6720n^5+6720n^6}{(4n-1)(4n-3)(4n-5)} C_{2n}.\end{aligned}$$

It is natural to conjecture this equality

$$\Psi_{2m}(n) = \frac{P_{2m}(n)}{\prod_{l=0}^{m-1}(4n-(2l+1))} C_{2n}, \quad m \in \mathbb{N}_0,\ n \in \mathbb{N},$$

where P_{2m} is a polynomial of integer coefficients and degree $2m$.

3. Odd moments of squares of combinatorial numbers

In a similar way to the previous section, we now calculate the odd moments $\{\Phi_{2m+1}\}_{m\geqslant 0}$ and $\{\Psi_{2m+1}\}_{m\geqslant 0}$. We express the value of $\{\Psi_{2m+1}\}_{m\geqslant 0}$ in terms of the Catalan numbers in Theorem 10.

Theorem 8. *The following equalities hold*

(i) $$\Phi_{2m+1}(n) = 2^{2m+1}(n+1)\binom{2n+1}{n} \sum_{k=0}^{m} \frac{\lambda_k(m, n+\frac{1}{2})}{2n+1-k} \binom{2n-2k}{n-k},$$

(ii) $$\Psi_{2m+1}(n)=\frac{2^{2m+2}(n+1)}{(2n+1)^2}\binom{2n+1}{n}\sum_{k=0}^{m+1}\frac{\lambda_k(m+1,n+\frac{1}{2})}{2n+1-2k}\binom{2n-2k}{n-k},$$

for $m\in\mathbb{N}_0$ and $n\in\mathbb{N}$.

Proof. We apply formula (9), and we have

$$\begin{aligned}\Phi_{2m+1}(n)&=2^{2m+1}\sum_{p=1}^{n+1}\left(p-\frac{1}{2}\right)\left(p-\frac{1}{2}\right)^{2m}\binom{2n+1}{n+1-p}^2\\&=2^{2m+1}\sum_{p=1}^{n+1}\left(p-\frac{1}{2}\right)\left(\sum_{k=0}^{m}(-1)^k\langle n+p\rangle_k\langle n-p+1\rangle_k\sigma_{k,m-k}\left(n+\frac{1}{2}\right)\right)\binom{2n+1}{n+1-p}^2\\&=2^{2m+1}\sum_{k=0}^{m}(-1)^k\sigma_{k,m-k}\left(n+\frac{1}{2}\right)\sum_{p=1}^{n+1}\left(p-\frac{1}{2}\right)\langle n+p\rangle_k\langle n-p+1\rangle_k\binom{2n+1}{n+1-p}^2\\&=2^{2m+1}\sum_{k=0}^{m}(-1)^k\sigma_{k,m-k}\left(n+\frac{1}{2}\right)\langle 2n+1\rangle_k^2\sum_{p=1}^{n+1}\left(p-\frac{1}{2}\right)\binom{2n+1-k}{n+1-p}\binom{2n+1-k}{n+p},\end{aligned}$$

where we have used the following equality,

$$\langle n+p\rangle_k\langle n-p+1\rangle_k\binom{2n+1}{n+1-p}^2=\langle 2n+1\rangle_k^2\binom{2n+1-k}{n+1-p}\binom{2n+1-k}{n+p}.$$

Now, we write $p-\frac{1}{2}=\frac{1}{2}((n+p)-(n-p+1))$ to conclude that

$$\sum_{p=1}^{n+1}\left(p-\frac{1}{2}\right)\binom{2n+1-k}{n+1-p}\binom{2n+1-k}{n+p}=\frac{2n+1-k}{2}\binom{2n-k}{n}^2.$$

Thus,

$$\begin{aligned}\Phi_{2m+1}(n)&=2^{2m+1}\sum_{k=0}^{m}(-1)^k\sigma_{k,m-k}\left(n+\frac{1}{2}\right)\langle 2n+1\rangle_k^2\frac{2n+1-k}{2}\binom{2n-k}{n}^2\\&=2^{2m+1}\sum_{k=0}^{m}\frac{\langle 2n+1\rangle_k\langle 2n+1\rangle_{k+1}}{\langle 2n+1\rangle_{2k+1}}\lambda_k\left(m,n+\frac{1}{2}\right)\binom{2n-k}{n}^2.\end{aligned}$$

From the equality

$$\frac{\langle 2n+1\rangle_k\langle 2n+1\rangle_{k+1}}{\langle 2n+1\rangle_{2k+1}}\binom{2n-k}{n}^2=(n+1)\binom{2n+1}{n}\frac{1}{2n-k+1}\binom{2n-2k}{n-k},$$

we conclude that

$$\Phi_{2m+1}(n)=2^{2m+1}(n+1)\binom{2n+1}{n}\sum_{k=0}^{m}\frac{\lambda_k(m,n+\frac{1}{2})}{2n-k+1}\binom{2n-2k}{n-k}.$$

We use formula (5) to prove item (ii) from (i). □

Remark 9. From item (i) of Theorem 8, we write for $m \in \mathbb{N}_0$,

$$\Psi_{2m+1}(n) = 2^{2m+2}(n+1)C_n \sum_{k=0}^{m+1} \frac{\lambda_k(m+1, n+\frac{1}{2})(n-k+1)}{(2n+1)(2n+1-2k)} C_{n-k}, \quad n \in \mathbb{N}.$$

Theorem 10. *There exists a polynomial of integer coefficients and degree at most $3m+2$, P_{3m+2}, such that*

$$\Psi_{2m+1}(n) = (n+1)C_n C_{n-1} \frac{P_{3m+2}(n)}{\prod_{j=2}^{m+1}(2n-(2j-1))}, \quad n \in \mathbb{N},$$

for $m \geqslant 1$.

Proof. From $C_n = \frac{2(2n-1)}{n+1} C_{n-1}$, and

$$C_{n-k} = \frac{n(n-1)\cdots(n+2-k)}{2^{k-1}(2n-3)(2n-5)\cdots(2n+1-2k)} C_{n-1}, \quad k \geqslant 2,$$

we have

$$\begin{aligned}\Psi_{2m+1}(n) &= (n+1)C_n C_{n-1}(2n+1)^{2m+1}(2n-1) \\ &\quad - \frac{(n+1)C_n C_{n-1}}{2}\big((2n+1)^{2m+2} - (2n-1)^{2m+2}\big)n \\ &\quad + \frac{(n+1)C_n C_{n-1}}{2n+1} \sum_{k=2}^{m+1} \frac{2^{2m-k+3}\lambda_k(m+1, n+\frac{1}{2})n(n-1)\cdots(n-k+1)}{(2n-3)(2n-5)\cdots(2n+1-2k)(2n+1-2k)}.\end{aligned}$$

Now, from Lemma 1, there exists a polynomial of integer coefficients $Q_{k,2m+3}$ and degree at most $2m+3$ such that

$$\frac{2^{2m-k+3}\lambda_k(m+1, n+\frac{1}{2})n(n-1)\cdots(n-k+1)}{(2n+1)(2n-3)(2n-5)\cdots(2n+1-2k)(2n+1-2k)} = \frac{2^{2m-2k+3}Q_{k,2m+3}(n)}{2n+1-2k}$$

for $2 \leqslant k \leqslant m+1$. Then, we conclude that

$$\Psi_{2m+1}(n) = (n+1)C_n C_{n-1} \frac{P_{3m+2}(n)}{(2n-3)\cdots(2n-2m-1)}, \quad n \in \mathbb{N},$$

where P_{3m+2} is a polynomial of degree at most $3m+2$. $\square$

Remark 11. Particular values of Ψ_{2m+1} are obtained,

$$\begin{aligned}\Psi_1(n) &= (n+1)C_n C_{n-1}(4n-2), \\ \Psi_3(n) &= (n+1)C_n C_{n-1}\big(16n^2-2\big), \\ \Psi_5(n) &= (n+1)C_n C_{n-1}\big(96n^3+32n^2-4n-2\big), \\ \Psi_7(n) &= (n+1)C_n C_{n-1}\frac{1536n^5-1536n^4-960n^3-160n^2+20n+6}{2n-3}.\end{aligned}$$

Therefore, we conjecture that

$$\Psi_{2m+1}(n) = (n+1)C_n C_{n-1} \frac{R_{2m-1}(n)}{\prod_{l=3}^{m}(2n-(2l-3))}, \quad n \in \mathbb{N},\ m \geqslant 3,$$

where R_{2m-1} is a polynomial of integer coefficients and degree $2m-1$.

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