ELSEVIER

Journal of Computational and Applied Mathematics 96 (1998) 1–12

JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

Avoiding the computation of the second Fréchet-derivative in the convex acceleration of Newton's method

J.A. Ezquerro *, M.A. Hernández

Department of Mathematics and Computation, University of La Rioja, C/Luis de Ulloa s/n. 26004 Logroño, Spain

Received 10 April 1997; received in revised form 6 April 1998

Abstract

We introduce a new two-step method to approximate a solution of a nonlinear operator equation in a Banach space. An existence-uniqueness theorem and error estimates are provided for this iteration using Newton–Kantorovich-type assumptions and a technique based on a new system of recurrence relations. For a special choice of the parameter involved we use, our method is of fourth order. © 1998 Elsevier Science B.V. All rights reserved.

AMS classification: 47H17; 65J15

Keywords: Nonlinear equations in Banach spaces; Two-point iteration; Third-order process; Convergence theorem; Recurrence relations; A priori error bounds

1. Introduction

Many scientific and engineering problems can be brought in the form of a nonlinear equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on an open convex subset Ω of a Banach space X with values in another Banach space Y. In the last years, several papers dealing with one-point iterations of order three have appeared [2, 3, 5, 7, 14]. The study of those methods are based on the well-known Newton–Kantorovich-type assumptions [9].

On the other hand, multipoint methods are defined as iterations which use new information at a number of points. In [13] it is imposed the restriction on one-point iteration of order N is that they must depend explicitely on the first $N-1$ derivatives of F. This implies that their informational

* Corresponding author. E-mail: jezquer@siur.unirioja.es.

0377-0427/98/$19.00
PII S0377-0427(98)00083-1

efficiency is less than or equal to unity. Those restrictions are relieved in only small measure by turning to one-point iterations with memory.

Neither of these restrictions need hold for multipoint methods, that is, for iterations which sample F and its derivatives at a number of values of the independent variable. We shall show that there exists a two-point method of R-order three which necessitates no evaluations of the second derivative. Moreover, for a special choice of the parameter involved we use, our method is of order four.

Traub [13] has shown how to construct useful multipoint iterations which are very efficient when equation (1) is such that the derivative F' can be rapidly evaluated compared with F itself. An example of this occurs when F is defined by an integral.

Next, we derive a new family of two-step methods from one of the most famous one-point iteration of order three called the Convex Acceleration of Newton's method or super-Halley method [3, 6]:

$$G(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x),$$

$$x_{n+1} = x_n - [I + \tfrac{1}{2}G(x_n)(I - G(x_n))^{-1}]F'(x_n)^{-1}F(x_n), \quad n \geqslant 0.$$

From Taylor's formula, we have

$$F'(z_n) = F'(x_n) + F''(x_n)(z_n - x_n) + \int_{x_n}^{z_n} F'''(x)(z_n - x)\,\mathrm{d}x$$

where $z_n = x_n + p(y_n - x_n)$ and $p \in (0, 1]$. We can now approximate

$$F''(x_n)(y_n - x_n) \approx \frac{1}{p}[F'(z_n) - F'(x_n)]$$

and derive the following two-point iteration function of R-order three:

$$y_n = x_n - F'(x_n)^{-1}F(x_n),$$

$$H(x_n, y_n) = \frac{1}{p}F'(x_n)^{-1}\left[F'(x_n + p(y_n - x_n)) - F'(x_n)\right], \tag{2}$$

$$x_{n+1} = y_n - \tfrac{1}{2}H(x_n, y_n)[I + H(x_n, y_n)]^{-1}(y_n - x_n), \quad n \geqslant 0,$$

where $p \in (0, 1]$, to approximate a zero x^* of (1).

For the special choice of $p = \frac{2}{3}$, we obtain the known Jarratt method whose order of convergence is four (see [1]).

We analyse, under certain assumptions of the pair (F, x_0), the convergence of (2) to a unique zero x^* of (1), by using a technique consisting of a new system of real sequences which simplifies those given by other authors [2, 8, 12]. We also provide some error estimates on the distances $\|x^* - x_n\|$ for all $n \geqslant 0$. From this analysis it follows a semilocal convergence result for the Jarratt method under mild differentiability conditions.

Denote $\overline{B(x,r)} = \{y \in X; \|y - x\| \leqslant r\}$ and $B(x,r) = \{y \in X; \|y - x\| < r\}$.

2. Preliminaries

Let F be a nonlinear twice Fréchet-differentiable operator defined on some open convex subset Ω of a Banach space X with values in another Banach space Y. Let $x_0 \in \Omega$ and suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathscr{L}(Y,X)$ exists, where $\mathscr{L}(Y,X)$ is the set of bounded linear operators from Y into X.

Let us assume that

(c_1) $\|\Gamma_0\| \leqslant \beta$,
(c_2) $\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta$,
(c_3) $\|F''(x)\| \leqslant M, \quad x \in \Omega$,
(c_4) $\|F''(x) - F''(y)\| \leqslant K\|x-y\|, \quad x,y \in \Omega, \quad K \geqslant 0$.

Denote $a_0 = M\beta\eta$ and $b_0 = K\beta\eta^2$ and define the sequences

$$a_{n+1} = a_n f(a_n)^2 g_p(a_n, b_n) \tag{3}$$

and

$$b_{n+1} = b_n f(a_n)^3 g_p(a_n, b_n)^2, \tag{4}$$

where

$$f(x) = \frac{2(1-x)}{x^2 - 4x + 2} \tag{5}$$

and

$$g_p(x,y) = \frac{3x^3 + 2y(1-x)[(1-6p)x + (2+3p)]}{24(1-x)^2}. \tag{6}$$

Firstly, it is provided a technical lemma whose proof is trivial.

Lemma 2.1. *Let f and g_p the two real functions given in (5) and (6), respectively. Then*
(i) *f is increasing and $f(x) > 1$ in $(0, \frac{1}{2})$,*
(ii) *g_p is increasing in its first and second arguments for $x \in (0, \frac{1}{2})$ and $y > 0$,*
(iii) *$f(\gamma x) < f(x)$ and $g_p(\gamma x, \gamma^2 y) < \gamma^2 g_p(x,y)$ for $\gamma \in (0,1)$ and $x \in (0,1)$.*

Some properties for the sequences $\{a_n\}$ and $\{b_n\}$ given, respectively, by (3) and (4) are now provided.

Lemma 2.2. *Let f and g_p the two real functions given by (5) and (6), respectively. Let*

$$h_p(x) = \frac{3(2x-1)(x-2)(x-3+\sqrt{5})(x-3-\sqrt{5})}{2(1-x)((1-6p)x + 2 + 3p)}. \tag{7}$$

If $a_0 \in (0, 1/2)$ and $b_0 < h_p(a_0)$, then
(i) *$f(a_0)^2 g_p(a_0, b_0) < 1$,*
(ii) *the sequences $\{a_n\}$ and $\{b_n\}$ are decreasing,*

(iii) $a_n\left(1+\dfrac{a_n}{2(1-a_n)}\right)<1$ *for all* $n\geqslant 0$.

Proof. From the hypotheses, (i) follows immediately. We show (ii) by mathematical induction on n. The facts that $0<a_1<a_0$ and $0<b_1<b_0$ follow by previous (i) and Lemma 2.1(i). Next, it is supposed that $0<a_j<a_{j-1}$ and $0<b_j<b_{j-1}$ for $j=1,2,\ldots,n$. Then

$$a_{n+1}=a_nf(a_n)^2g_p(a_n,b_n)<a_nf(a_0)^2g_p(a_0,b_0)<a_n,$$

since f is increasing and g_p is also increasing in its first and second arguments. We have

$$b_{n+1}=b_nf(a_n)^3g_p(a_n,b_n)^2<b_nf(a_n)^4g_p(a_n,b_n)^2<b_n$$

by the same reasoning as before and the fact that $f(x)>1$ in $(0,\frac{1}{2})$.

Finally, for all $n\geqslant 0$, we have

$$a_n\left(1+\frac{a_n}{2(1-a_n)}\right)<a_0\left(1+\frac{a_0}{2(1-a_0)}\right)<1,$$

since $\{a_n\}$ is a decreasing sequence and $a_0\in(0,\frac{1}{2})$. □

Lemma 2.3. *Let us suppose the hypotheses of Lemma* 2.2 *and define* $\gamma=a_1/a_0$. *Then*
(i_n) $a_n<\gamma^{3^{n-1}}a_{n-1}<\gamma^{(3^n-1)/2}a_0$ *and* $b_n<(\gamma^{3^{n-1}})^2b_{n-1}<\gamma^{3^n-1}b_0$, *for all* $n\geqslant 2$,
(ii_n) $f(a_n)g_p(a_n,b_n)<\gamma^{3^n-1}f(a_0)g_p(a_0,b_0)=\gamma^{3^n}/f(a_0)$, *for all* $n\geqslant 1$.

Proof. We prove (i_n) following an inductive procedure. So, $a_1=\gamma a_0$ and $b_1=b_0f(a_0)^3g_p(a_0,b_0)^2<\gamma^2b_0$ if and only if $f(a_0)>1$, and by Lemma 2.1 the result holds. If we suppose that (i_n) is true, then

$$\begin{aligned}a_{n+1}&=a_nf(a_n)^2g_p(a_n,b_n)<\gamma^{3^{n-1}}a_{n-1}f(\gamma^{3^{n-1}}a_{n-1})^2g_p(\gamma^{3^{n-1}}a_{n-1},(\gamma^{3^{n-1}})^2b_{n-1})\\&<\gamma^{3^{n-1}}a_{n-1}f(a_{n-1})^2(\gamma^{3^{n-1}})^2g_p(a_{n-1},b_{n-1})=\gamma^{3^n}a_n.\end{aligned}$$

On the other hand, we have

$$b_{n+1}=b_nf(a_n)^3g_p(a_n,b_n)^2<\left(\frac{a_{n+1}}{a_n}\right)^2b_n$$

if and only if

$$a_n^2f(a_n)^3g_p(a_n,b_n)^2<a_{n+1}^2=a_n^2f(a_n)^4g_p(a_n,b_n)^2,$$

and it is true since $f(a_n)>1$. Now, $b_{n+1}<(\gamma^{3^n})^2b_n$ since $a_{n+1}/a_n<\gamma^{3^n}$. Moreover,

$$a_{n+1}<\gamma^{3^n}a_n<\gamma^{3^n}\gamma^{3^{n-1}}a_{n-1}<\cdots<\gamma^{(3^{n+1}-1)/2}a_0$$

and

$$b_{n+1}<(\gamma^{3^n})^2b_n<(\gamma^{3^n})^2(\gamma^{3^{n-1}})^2b_{n-1}<\cdots<\gamma^{3^{n+1}-1}b_0.$$

Then we observe that

$$f(a_n)g_p(a_n,b_n) < f(\gamma^{(3^n-1)/2}a_0)g_p(\gamma^{(3^n-1)/2}a_0,\gamma^{3^n-1}b_0) < \gamma^{3^n-1}f(a_0)g_p(a_0,b_0) = \gamma^{3^n}/f(a_0).$$

The proof is complete. □

After that, taking into account initial hypotheses (c_1)–(c_4) and assuming that $y_0 \in \Omega$, we have

$$\|H(x_0,y_0)\| \leqslant M\|\Gamma_0\|\|y_0-x_0\| \leqslant a_0 \quad \text{and} \quad K\|\Gamma_0\|\|y_0-x_0\|^2 \leqslant b_0.$$

Hence x_1 is well defined and

$$\|x_1-x_0\| \leq \|I-\tfrac{1}{2}H(x_0,y_0)[I+H(x_0,y_0)]^{-1}\|\,\|y_0-x_0\| \leq \left(1+\frac{a_0}{2(1-a_0)}\right)\|y_0-x_0\|.$$

Next, we prove the following items are true for all $n \geqslant 1$:

[I_n] $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \leqslant f(a_{n-1})\|\Gamma_{n-1}\|$,

[II_n] $\|y_n-x_n\| = \|\Gamma_nF(x_n)\| \leqslant f(a_{n-1})g_p(a_{n-1},b_{n-1})\|y_{n-1}-x_{n-1}\|$,

[III_n] $\|H(x_n,y_n)\| \leqslant M\|\Gamma_n\|\|y_n-x_n\| \leqslant a_n$,

[IV_n] $K\|\Gamma_n\|\|y_n-x_n\|^2 \leqslant b_n$,

[V_n] $\|x_{n+1}-x_n\| \leq \left(1+\dfrac{a_n}{2(1-a_n)}\right)\|y_n-x_n\|$.

We use mathematical induction on n.

[I_1]: Observe that if $x_1 \in \Omega$,

$$\|I-\Gamma_0F'(x_1)\| \leqslant \|\Gamma_0\|\|F'(x_0)-F'(x_1)\| \leqslant M\|\Gamma_0\|\|x_1-x_0\| \leqslant a_0\left(1+\frac{a_0}{2(1-a_0)}\right) < 1$$

and, by the Banach lemma, Γ_1 exists and

$$\|\Gamma_1\| \leq \frac{\|\Gamma_0\|}{1-\|I-\Gamma_0F'(x_1)\|} \leqslant f(a_0)\|\Gamma_0\|.$$

[II_1]: Using Taylor's formula and (2), we have if $y_0 \in \Omega$

$$F(x_1) = F(y_0)+F'(y_0)(x_1-y_0)+\int_{y_0}^{x_1}F''(x)(x_1-x)\,\mathrm{d}x.$$

As

$$\begin{aligned}F(y_0)+F'(y_0)(x_1-y_0) &= F(y_0)-\tfrac{1}{2}F'(y_0)H(x_0,y_0)[I+H(x_0,y_0)]^{-1}(y_0-x_0)\\ &\quad \pm\tfrac{1}{2}F'(x_0)H(x_0,y_0)[I+H(x_0,y_0)]^{-1}(y_0-x_0)\\ &= F(y_0)-\tfrac{1}{2}[F'(y_0)-F'(x_0)]H(x_0,y_0)[I+H(x_0,y_0)]^{-1}(y_0-x_0)\\ &\quad -\tfrac{1}{2}F'(x_0)H(x_0,y_0)[I+H(x_0,y_0)]^{-1}(y_0-x_0)\\ &= \int_0^1 F''(x_0+t(y_0-x_0))(1-t)\,\mathrm{d}t(y_0-x_0)^2\\ &\quad -\tfrac{1}{2}\int_0^1 F''(x_0+t(y_0-x_0))\,\mathrm{d}t(y_0-x_0)H(x_0,y_0)[I+H(x_0,y_0)]^{-1}(y_0-x_0)\end{aligned}$$

$$-\frac{1}{2}\left(\int_0^1 F''(x_0+pt(y_0-x_0))\,\mathrm{d}t(y_0-x_0)^2\right.$$
$$\left.-\int_0^1 F''(x_0+pt(y_0-x_0))\,\mathrm{d}t(y_0-x_0)H(x_0,y_0)[I+H(x_0,y_0)]^{-1}(y_0-x_0)\right),$$

it follows that

$$\begin{aligned}F(x_1)=&\int_0^1 [F''(x_0+t(y_0-x_0))(1-t)-\tfrac{1}{2}F''(x_0+pt(y_0-x_0))]\mathrm{d}t(y_0-x_0)^2\\&\pm\tfrac{1}{2}F''(x_0)(y_0-x_0)^2+\frac{1}{2}\int_0^1[F''(x_0+pt(y_0-x_0))\\&-F''(x_0+t(y_0-x_0))]\,\mathrm{d}t(y_0-x_0)H(x_0,y_0)[I+H(x_0,y_0)]^{-1}(y_0-x_0)\\&+\int_0^1 F''(y_0+t(x_1-y_0))(1-t)\,\mathrm{d}t(x_1-y_0)^2\\=&\int_0^1[F''(x_0+t(y_0-x_0))-F''(x_0)](1-t)\,\mathrm{d}t(y_0-x_0)^2\\&+\frac{1}{2}\int_0^1[F''(x_0)-F''(x_0+pt(y_0-x_0))]\,\mathrm{d}t(y_0-x_0)^2\\&+\frac{1}{2}\int_0^1[F''(x_0+pt(y_0-x_0))-F''(x_0+t(y_0-x_0))]\,\mathrm{d}t(y_0-x_0)H(x_0,y_0)\\&\times[I+H(x_0,y_0)]^{-1}(y_0-x_0)\\&+\int_0^1 F''(y_0+t(x_1-y_0))(1-t)\,\mathrm{d}t(x_1-y_0)^2.\end{aligned}$$

Taking norms we infer that

$$\|F(x_1)\|\leqslant\left[\frac{2+3p}{12}K\eta^2+\frac{1-p}{4}K\eta^2\frac{a_0}{1-a_0}+\frac{M\eta}{8}\left(\frac{a_0}{1-a_0}\right)^2\right]\|y_0-x_0\|$$

So

$$\begin{aligned}\|y_1-x_1\|&=\|\Gamma_1F(x_1)\|\leqslant\|\Gamma_1\|\|F(x_1)\|\leqslant f(a_0)\|\Gamma_0\|\|F(x_1)\|\\&\leqslant f(a_0)g_p(a_0,b_0)\|y_0-x_0\|.\end{aligned}$$

[III$_1$]: If $y_1\in\Omega$,

$$\begin{aligned}\|H(x_1,y_1)\|&\leqslant\frac{1}{p}\|\Gamma_1\|\|F'(x_1+p(y_1-x_1))-F(x_1)\|\\&\leqslant M\|\Gamma_1\|\|y_1-x_1\|\leqslant M\|\Gamma_0\|\|y_0-x_0\|f(a_0)^2g_p(a_0,b_0)\leqslant a_1.\end{aligned}$$

[IV$_1$]:

$$K\|\Gamma_1\|\|y_1-x_1\|^2\leqslant K\|\Gamma_0\|\|y_0-x_0\|^2f(a_0)^3g_p(a_0,b_0)^2\leqslant b_1.$$

[V_1]:

$$\begin{aligned}\|x_2 - x_1\| &\leqslant \|I - \tfrac{1}{2}H(x_1,y_1)[I + H(x_1,y_1)]^{-1}\| \, \|y_1 - x_1\| \\ &\leqslant \left(1 + \frac{a_1}{2(1-a_1)}\right)\|y_1 - x_1\|.\end{aligned}$$

Assuming now [I_n]–[V_n] are true for a fixed $n \geqslant 1$, we can prove [I_{n+1}]–[V_{n+1}]. Then the induction is complete.

3. Convergence theorem

We can already show the next convergence theorem.

Theorem 3.1. *Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain Ω. Let us assume that $\Gamma_0 = F'(x_0)^{-1} \in \mathscr{L}(Y,X)$ exists at some $x_0 \in \Omega$ and (c_1)–(c_4) are satisfied. Let us denote $a_0 = M\beta\eta$ and $b_0 = K\beta\eta^2$. Suppose that $a_0 \in (0, \frac{1}{2})$ and $b_0 < h_p(a_0)$ where h_p is defined in* (7)*. Then, if $\overline{B(x_0, \eta/a_0)} \subseteq \Omega$, the sequence $\{x_n\}$ defined in* (2) *and starting at x_0 converges R-cubically at least to a solution x^* of* (1)*. In that case, the solution x^* and the iterates x_n and y_n belong to $\overline{B(x_0, \eta/a_0)}$. Moreover, the solution x^* is unique in $B(x_0, \eta/a_0)$.*

Furthermore, we can give the following error estimates:

$$\|x^* - x_n\| \leqslant \left(1 + \frac{a_0\,\gamma^{(3^n-1)/2}}{2(1-a_0)}\right)\gamma^{\frac{3^n-1}{2}}\,\frac{\Delta^n}{1-\Delta}\eta, \quad n \geqslant 0. \tag{8}$$

Proof. Firstly, we prove that $\{x_n\}$ is a Cauchy sequence. Observe that for $i \geqslant 0$:

$$\begin{aligned}&\left(1 + \frac{a_{n+i}}{2(1-a_{n+i})}\right)\|y_{n+i} - x_{n+i}\| \\ &\quad\leqslant \left(1 + \frac{a_{n+i}}{2(1-a_{n+i})}\right) f(a_{n+i-1})g_p(a_{n+i-1}, b_{n+i-1})\|y_{n+i-1} - x_{n+i-1}\| \\ &\quad\leqslant \cdots \leqslant \left(1 + \frac{a_{n+i}}{2(1-a_{n+i})}\right)\|y_0 - x_0\| \prod_{j=0}^{n+i-1} f(a_j)g_p(a_j,b_j)\end{aligned}$$

as a consequence of estimate [II_n]. We now have, from Lemma 2.3,

$$\prod_{j=0}^{n+i-1} f(a_j)g_p(a_j,b_j) \leq \prod_{j=0}^{n+i-1} (\gamma^{3^j}\Delta) = \gamma^{(3^{n+i}-1)/2}\Delta^{n+i},$$

where $\gamma = a_1/a_0 < 1$ and $\Delta = 1/f(a_0) < 1$. So

$$\begin{aligned}\|x_{n+m} - x_n\| &\leqslant \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leqslant \left(1 + \frac{a_{n+m-1}}{2(1-a_{n+m-1})}\right)\|y_0 - x_0\| \prod_{j=0}^{n+m-2} f(a_j)g_p(a_j,b_j)\end{aligned}$$

$$+\cdots+\left(1+\frac{a_n}{2(1-a_n)}\right)\|y_0-x_0\|\prod_{j=0}^{n-1}f(a_j)g_p(a_j,b_j)$$
$$\leqslant\left(1+\frac{a_n}{2(1-a_n)}\right)\left(\gamma^{(3^{n+m-1}-1)/2}\Delta^{n+m-1}+\cdots+\gamma^{\frac{3^n-1}{2}}\Delta^n\right)\|y_0-x_0\|$$
$$<\left(1+\frac{a_0\gamma^{\frac{3^n-1}{2}}}{2(1-a_0)}\right)\frac{1-\Delta^m}{1-\Delta}\gamma^{\frac{3^n-1}{2}}\Delta^n\eta, \tag{9}$$

since $a_n<a_0\gamma^{(3^n-1)/2}\leqslant a_0$ and

$$\frac{a_n}{2(1-a_n)}<\frac{a_0\gamma^{\frac{3^n-1}{2}}}{2(1-a_0)}.$$

For $n=0$, we obtain

$$\|x_m-x_0\|<\left(1+\frac{a_0}{2(1-a_0)}\right)\frac{1-\Delta^m}{1-\Delta}\eta<\left(1+\frac{a_0}{2(1-a_0)}\right)\frac{\eta}{1-\Delta}=\eta/a_0.$$

By letting $m\to\infty$ in (9), we get (8). Similarly, we have $y_n\in B(x_0,\eta/a_0)$ for all $n\geqslant 0$.

To see that x^* is a solution of (1), we have $\|\Gamma_nF(x_n)\|\to 0$ as $n\to\infty$. Taking into account that $\|F(x_n)\|\leqslant\|F'(x_n)\|\|\Gamma_nF(x_n)\|$ and the sequence $\{\|F'(x_n)\|\}$ is bounded, we infer that $\|F(x_n)\|\to 0$ as $n\to\infty$. Consequently, we obtain $F(x^*)=0$ by the continuity of F.

To prove the uniqueness, assume some other solution y^* of (1) in $B(x_0,\eta/a_0)$. From the approximation

$$0=F(y^*)-F(x^*)=\int_0^1 F'(x^*+t(y^*-x^*))\,\mathrm{d}t\,(y^*-x^*),$$

we have to prove that the operator $\int_0^1 F'(x^*+t(y^*-x^*))\,\mathrm{d}t$ is invertible and then $y^*=x^*$. Indeed, from

$$\|\Gamma_0\|\int_0^1\|F'(x^*+t(y^*-x^*))-F'(x_0)\|\,\mathrm{d}t\leqslant M\beta\int_0^1\|x^*+t(y^*-x^*)-x_0\|\,\mathrm{d}t$$
$$\leqslant M\beta\int_0^1((1-t)\|x^*-x_0\|+t\|y^*-x_0\|)\,\mathrm{d}t<\frac{M\beta}{2}(\eta/a_0+\eta/a_0)=1,$$

it follows that $\left[\int_0^1 F'(x^*+t(y^*-x^*))\,\mathrm{d}t\right]^{-1}$ exists.

Finally, we deduce that the R-order of convergence [10] of sequence (2) is at least three. Indeed, from (8) it follows that

$$\|x^*-x_n\|\leqslant\left(1+\frac{a_0}{2(1-a_0)}\right)\frac{(\gamma^{1/2})^{3^n}}{\gamma^{1/2}(1-\Delta)}\eta.$$

The proof is complete. □

Remark 1. From the initial conditions on the pair (a_0,b_0), we have that if the point (a_0,b_0) lies in region I (see Fig. 1) for $\tilde{p}=1$, we can apply iteration (2) for all $p\in(0,1]$. Observe that if $p=0$,

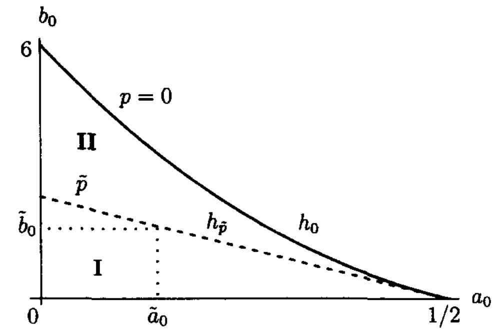

Fig. 1.

we obtain the Newton method whose order of convergence is two and, consequently, the speed of convergence is slower.

If the point (a_0, b_0) lies in region II, we locate the curve $h_{\tilde{p}}$ such that $h_{\tilde{p}}(a_0) = b_0$. In this case, we can apply iteration (2) for all $p \in (0, \tilde{p})$.

Note that the domain for initial conditions a_0 and b_0 is similar to the one obtained for the Newton method ($p = 0$), but for third-order iterations without the computation of the second Fréchet-derivative of F.

Observe that for the choice of $p = \frac{2}{3}$, the Jarratt method is obtained (see [1]) whose order of convergence is four. In addition, the iteration considered for solving (1) is one with $\tilde{p} \geqslant \frac{2}{3}$ whenever it is possible.

4. Applications

We apply our new technique of convergence analysis to the following three examples. The two first appear also in [2, 4]. We compare some results with those obtained before.

Example 1. Firstly, we apply iteration (2) to the cubic function $F : [-4, 4] \to \mathbb{R}$ where $F(t) = t^3 - 10$ introduced by Döring [4]. The initial value $t_0 = 2.5$ is chosen. Then all the parameters appearing in Theorem 3.1 are easily found:

$$\beta = 0.053334, \quad \eta = 0.3, \quad M = 24 \quad \text{and} \quad K = 6.$$

In addition, $a_0 = M\beta\eta = 0.384 \in (0, \frac{1}{2})$ and $b_0 = K\beta\eta^2 = 0.0288$. Therefore,

$$0.0288 = b_0 < h_p(a_0) = \frac{1.68296}{2.384 + 0.696p}$$

and, consequently, we can apply (2), for all $p \in (0, 1]$, in order to approximate the solution $t^* = 2.154434690031884$ of $F(t) = 0$.

On the other hand, from the asymptotic error constant $C_p = 0.0359072|3p - 2|$, we observe that sequence (2) converges the fastest to x^* for $p = \frac{2}{3}$, since a fourth-order iterative method (Jarratt's method) is obtained. See Table 1.

Table 1
Error estimates when $p = \frac{2}{3}$ and $p = \frac{3}{4}$

n	$t^* - t_n\,(p = 2/3)$	$t^* - t_n\,(p = 3/4)$
0	0.345565309968116	0.345565309968116
1	0.000658865061671	0.000962300108230
2	$1.199040866595169 \times 10^{-14}$	$8.052225553001340 \times 10^{-12}$

In the next example, by this new technique, it is shown that we can improve the error bounds obtained by the classical one-point methods of order three.

Example 2. Consider the next integral equation also quoted in [4]:

$$F(x)(s) = x(s) - s + \frac{1}{2}\int_0^1 s\cos(x(t))\,\mathrm{d}t$$

in the space $X = C([0,1])$ of all continuous functions on the interval $[0,1]$ with the norm

$$\|x\| = \max_{s\in[0,1]} |x(s)|\,.$$

If we choose $x_0 = x_0(s) = s$, then all the parameters appearing in Theorem 3.1 are

$$\beta = 1.2705952, \quad \eta = 0.4953228 \quad \text{and} \quad M = 0.5 = K.$$

So, $a_0 = M\beta\eta = 0.3146773 \in (0, \frac{1}{2})$ and $b_0 = K\beta\eta^2 = 0.1558668$. We now obtain $p < 15.3598$ from the inequality $b_0 < h_p(a_0)$ where h_p is defined in (7). In consequence we can take any $p \in (0,1]$. For $p = \frac{2}{3}$, the value of the parameter p which provides the fastest iteration, we have that (2) converges to a unique solution x^* of $F(x) = 0$ in $B(x_0, 1.57407)$.

On the other hand, we get better error estimates than those obtained by other authors. For $10^{11}\|x^* - x_2\|$, where x_2 is the second iterate of (2), we have the upper bounds $L = 5825764$ when $p = \frac{2}{3}$ and $L = 9756584$ when $p = 1$. Instead of that, Candela and Marquina got $L = 14987029$ for the Halley method in [2].

Example 3. Finally, let us consider the system of equations $F(u,v) = 0$ where $F : [4,6]\times[5,7] \to \mathbb{R}^2$ and

$$F(u,v) = \left(u^2 - v - 19,\ v^3/6 - u^2 + v - 17\right).$$

Then we have

$$F'(u,v)^{-1} = \frac{1}{v^2}\begin{pmatrix} \dfrac{1+v^2/2}{u} & 1/u \\ 2 & 2 \end{pmatrix}$$

Table 2
Error estimates by Newton's method

n	$u^* - u_n$	$v^* - v_n$
0	0.500000000000000	0.500000000000000
1	0.026134122287968	0.037475345167652
2	0.000091036676663	0.000232132746151
3	$1.726783414298796 \times 10^{-9}$	$8.980472046340010 \times 10^{-9}$
4	$1.642326062324995 \times 10^{-18}$	$1.344147966902442 \times 10^{-17}$
5	$3.280946417712476 \times 10^{-36}$	$3.011222928213283 \times 10^{-35}$
6	$1.618890014525233 \times 10^{-71}$	$1.511243920566230 \times 10^{-70}$

if (u,v) does not belong to the lines $u = 0$ or $v = 0$. The second derivative is a bilinear operator on $\mathbb{R}^2$ given by

$$F''(u,v) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ \hline -2 & 0 \\ 0 & v \end{pmatrix}.$$

We take the max-norm in $\mathbb{R}^2$ and the norm $\|A\| = \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}$ for

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

As in [11] we define the norm of a bilinear operator B on $\mathbb{R}^2$ by

$$\|B\| = \sup_{\|u\|=1} \max_i \sum_{j=1}^{2} \left| \sum_{k=1}^{2} b_i^{jk} u_k \right|.$$

where $u = (u_1, u_2)$ and

$$B = \begin{pmatrix} b_1^{11} & b_1^{12} \\ b_1^{21} & b_1^{22} \\ \hline b_2^{11} & b_2^{12} \\ b_2^{21} & b_2^{22} \end{pmatrix}.$$

If we choose $\mathbf{x_0} = (u_0, v_0) = (5.5, 6.5)$, then

$$\beta = 0.0995159, \quad \eta = 0.473866, \quad M = 9 \quad \text{and} \quad K = 1.$$

Thus, $a_0 = 0.424415$, $b_0 = 0.0223462$ and $p < 94.713$. Therefore, any $p \in (0, 1]$ can be considered in (2) to approximate the solution $(u^*, v^*) = (5, 6)$ of $F(u,v) = 0$.

In Tables 2–4, we see that, under a similar operational cost to the one of Newton's method, the speed of convergence is quite increased by iteration (2), obtaining the fastest one for the Jarratt method.

Table 3
Error estimates by iteration (2) and $p = 3/4$

n	$u^* - u_n$	$v^* - v_n$
0	0.500000000000000	0.500000000000000
1	0.000096860900968	0.000290679411182
2	$2.845303226225787 \times 10^{-15}$	$2.844752710967945 \times 10^{-14}$
3	$2.664525110659125 \times 10^{-45}$	$2.664525110659105 \times 10^{-44}$

Table 4
Error estimates by the Jarratt method

n	$u^* - u_n$	$v^* - v_n$
0	0.500000000000000	0.500000000000000
1	0.000083774425044	0.000159130644820
2	$3.241970341093828 \times 10^{-19}$	$1.978989109938055 \times 10^{-18}$
3	$5.308320318383308 \times 10^{-75}$	$4.734003854818204 \times 10^{-74}$

Remark 4. As we can see in the three examples mentioned above, by a suitable choice of the starting point x_0 for iteration (2), we can usually consider (2) for any $p \in (0, 1]$. Therefore, we shall take $p = \frac{2}{3}$ as a consequence of the fact that (2) is of order four (see [1]).

References

[1] I.K. Argyros, D. Chen, Q. Qian, The Jarratt method in Banach space setting, J. Comput. Appl. Math. 51 (1994) 103–106.
[2] V. Candela, A. Marquina, Recurrence relations for rational cubic methods I: The Halley Method, Computing 44 (1990) 169–184.
[3] D. Chen, I.K. Argyros, Q. Qian, A local convergence theorem for the super-Halley method in a Banach space, Appl. Math. Lett. 7 (5) (1994) 49–52.
[4] B. Döring, Einige Sätze über das Verfahren der tangierenden Hyperbeln in Banach-Räumen, Aplikace Mat. 15 (1975) 418–464.
[5] J.M. Gutiérrez, M.A. Hernández, A Family of Chebyshev–Halley type Methods in Banach spaces, Bull. Austral. Math. Soc. 55 (1997) 113–130.
[6] M.A. Hernández, Newton–Raphson's method and convexity, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 22 (1) (1992) 159–166.
[7] M.A. Hernández, M.A. Salanova, A family of Chebyshev–Halley type methods, Internat. J. Comput. Math. 47 (1993) 59–63.
[8] L.V. Kantorovich, The Majorant Principle for Newton's method, Dokl. Akad. Nauk SSSR 76 (1951) 17–20.
[9] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.
[10] F.A. Potra, V. Pták, Nondiscrete Induction and Iterative Processes Pitman, New York, 1984.
[11] L.B. Rall, Quadratic equations in Banach spaces, Rend. Circ. Mat., Palermo 10 (2) (1961) 314–332.
[12] R.A. Safiev, On some iterative processes, Ž. Vyčcisl. Mat. Fiz. 4 (1964) 139–143 (Translated into English by L.B. Rall as MRC Technical Summary Report, No. 649) Univ. Wisconsin-Madison, 1966.
[13] J.F. Traub, Iterative Methods for the Solution of Equations, Prentice-Hall, Englewood Cliffs, NJ, 1964.
[14] T. Yamamoto, On the method of tangent hyperbolas in Banach spaces, J. Comput. Appl. Math. 21 (1988) 75–86.