A TRANSFORM INVOLVING CHEBYSHEV POLYNOMIALS AND ITS INVERSION FORMULA

ÓSCAR CIAURRI, LUIS M. NAVAS, AND JUAN L. VARONA

Abstract. We define a functional analytic transform involving the Chebyshev polynomials $T_n(x)$, with an inversion formula in which the Möbius function $\mu(n)$ appears. If $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, then given a bounded function from $[-1,1]$ into $\mathbb{C}$, or from $\mathbb{C}$ into itself, the following inversion formula holds:

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^s} f(T_n(x))$$

if and only if

$$f(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} g(T_n(x)).$$

Some other similar results are given.

1. Introduction and main results

If we have an arithmetical function $\alpha : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) and a function $f : (0,\infty) \to \mathbb{R}$ (or $\mathbb{C}$), we can define a new function $g = \alpha \circ f$ by taking

$$g(x) = (\alpha \circ f)(x) = \sum_{n=1}^{\infty} \alpha(n) f\left(\frac{x}{n}\right), \qquad x \in (0,\infty). \tag{1}$$

Moreover, let us suppose that α is invertible with respect to Dirichlet convolution (this happens if and only if $\alpha(1) \neq 0$). Then, it is well known that we have the inversion formula $f = \alpha^{-1} \circ g$. A typical case is when α is a completely multiplicative function; in this case, $\alpha^{-1}(n) = \mu(n)\alpha(n)$, where $\mu(n)$ is the Möbius function. Thus, we have

$$f(x) = \sum_{n=1}^{\infty} \mu(n)\alpha(n) g\left(\frac{x}{n}\right), \qquad x \in (0,\infty)$$

(see, for instance, [1]). A common example of a completely multiplicative function is $\alpha(n) = n^{-s}$, $s \in \mathbb{C}$, which gives rise to Dirichlet series.

In this paper we present a new transform/inverse pair in which both the Chebyshev polynomials $\{T_n(x)\}_{n=1}^{\infty}$ and the Möbius function $\mu(n)$ appear. The Chebyshev polynomials satisfy many identities and orthogonal conditions, but for our purposes only the property

$$T_m(T_n(x)) = T_{mn}(x) \tag{2}$$

is essential. For $x \in [-1,1]$, this formula is clear from $T_k(x) = \cos(k \arccos x)$ and, for $x \in \mathbb{C}$, it follows by analytic continuation. It is interesting to note that, up to a linear change of variable, $\{x^n\}$ and the Chebyshev polynomials are the unique families of polynomials that satisfy an identity similar to (2) (see [2, Chapter 4, Th. 4.4]

THIS PAPER HAS BEEN PUBLISHED IN: *J. Math. Anal. Appl.* **323** (2006), 57–62.

2000 *Mathematics Subject Classification.* Primary 11A25; Secondary 33C45.

Key words and phrases. Chebyshev polynomials, Möbius function, Möbius transform, Dirichlet convolution, inversion formula.

Research of the first and third authors supported by grant BFM2003-06335-C03-03 of the DGI.

for details); in particular, similar inversion formulas can be given for expansions in $\{x^n\}$ on $[0,1]$.

Prior to continuing, let us establish our notation. For $x \in [-1,1]$, we have $T_n(x) = \cos(n \arccos x)$ so $T_n : [-1,1] \to [-1,1]$. But we can also consider T_n both as $T_n : \mathbb{R} \to \mathbb{R}$ or $T_n : \mathbb{C} \to \mathbb{C}$. Moreover, $T_n(x) \in \mathbb{Z}[x]$, so we also have $T_n : \mathbb{Z} \to \mathbb{Z}$ and $T_n : \mathbb{Q} \to \mathbb{Q}$. Let us then use Δ to denote $[-1,1]$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, or $\mathbb{Q}$, accordingly. Thus, for functions f of type $f : \Delta \to \mathbb{R}$ (or $\mathbb{C}$), the composition $f(T_n(x))$ is well defined for every n. (Some perhaps more "esoteric" choices can be taken into account for Δ, such us $[1,\infty)$, $\mathbb{N}$, or the algebraic numbers.)

The main result of this paper is the following:

Theorem 1. *Let $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ and $\Delta = [-1,1]$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, or $\mathbb{Q}$. If f is a bounded function defined on Δ, then the series*

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^s} f(T_n(x)), \qquad x \in \Delta, \tag{3}$$

is absolutely convergent, the function g is bounded, and we can recover f as

$$f(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} g(T_n(x)), \qquad x \in \Delta. \tag{4}$$

Conversely, if we have a bounded function g on Δ, the function f defined as in (4) *is bounded and fulfills* (3)*.*

We also study some other conditions that yield similar results. In particular, in Section 3 we give a more general approach to our inversion formula.

2. The transform and the inversion formula: proof of the main theorem

Let us begin by defining an operation $\odot$ similar to the $\circ$ in (1), but properly adapted to our circumstances. Given a function f on Δ and an arithmetical function $\alpha : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$), we define the transform

$$g(x) = (\alpha \odot f)(x) = \sum_{n=1}^{\infty} \alpha(n) f(T_n(x)), \tag{5}$$

provided that the series converges.

Let us suppose that we have another arithmetical function $\beta : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) that is inverse to α with respect to Dirichlet convolution, i.e., $\alpha * \beta = \delta$ with $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$. Let us calculate $(\beta \odot g)(x)$, at least formally, from (5). If the formal manipulations that follow are analytically justified, we can reorder series, group the terms such than $nm = k$, use (2), $\alpha * \beta = \delta$, and $T_1(x) = x$, so

$$\begin{aligned}
(\beta \odot g)(x) &= \sum_{n \in \mathbb{N}} \beta(n) (\alpha \odot f)(T_n(x)) \\
&= \sum_{n \in \mathbb{N}} \beta(n) \sum_{m \in \mathbb{N}} \alpha(m) f(T_m(T_n(x))) \\
&= \sum_{n,m \in \mathbb{N}} \beta(n)\alpha(m) f(T_{mn}(x)) \\
&= \sum_{k \in \mathbb{N}} \left(\sum_{nm=k} \beta(n)\alpha(m) \right) f(T_k(x)) \\
&= \sum_{k \in \mathbb{N}} (\alpha * \beta)(k) f(T_k(x)) \\
&= f(x).
\end{aligned} \tag{6}$$

Thus, we have found the inversion formula. It remains to determine conditions under which the series that define $(\alpha \odot f)(x)$ and $(\beta \odot g)(x)$ converge and the manipulations in (6) can be justified.

Some simple assumptions guaranteeing this are the following:

Proposition 1. *Let α and β be two arithmetical functions related by $\alpha * \beta = \delta$, and such that $\sum_{n=1}^{\infty} |\alpha(n)| < \infty$ and $\sum_{n=1}^{\infty} |\beta(n)| < \infty$; let Δ be $[-1,1]$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, or $\mathbb{Q}$. If f is a bounded function defined on Δ, then the series*

$$g(x) = \sum_{n=1}^{\infty} \alpha(n) f(T_n(x)), \qquad x \in \Delta, \tag{7}$$

is absolutely convergent, the function g is bounded by

$$\sup_{x\in\Delta} |g(x)| \le \Big(\sum_{n=1}^{\infty} |\alpha(n)| \Big) \sup_{x\in\Delta} |f(x)|, \tag{8}$$

and we can recover f as

$$f(x) = \sum_{n=1}^{\infty} \beta(n) g(T_n(x)), \qquad x \in \Delta. \tag{9}$$

Conversely, if we have a bounded function g on Δ, the function f defined as in (9) *is bounded in a similar way and fulfills* (7).

With this, we have,

Proof of Theorem 1. In the proposition, take $\alpha(n) = \alpha_s(n) = n^{-s}$, which is a completely multiplicative function whose inverse is $\alpha^{-1}(n) = \mu(n)n^{-s}$. As $\mathrm{Re}(s) > 1$, it follows that $\sum_{n=1}^{\infty} |a(n)| = \sum_{n=1}^{\infty} |n^{-s}| = \zeta(\mathrm{Re}(s))$, where $\zeta(s)$ denote the Riemann's zeta function. The inversion part is similar. □

Another example. Let us consider the Liouville function $\lambda(n)$, defined by

$$\lambda(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^{a_1+\cdots+a_k}, & \text{if } n = p_1^{a_1} \cdots p_k^{a_k} \end{cases}$$

(where $p_1^{a_1} \cdots p_k^{a_k}$ denotes the decomposition of n into prime factors). $\lambda(n)$ is a completely multiplicative whose inverse function is $\lambda^{-1}(n) = \mu(n)\lambda(n) = |\mu(n)|$. Then, in a similar way to Theorem 1, for $\mathrm{Re}(s) > 1$ and bounded functions, we have

$$g(x) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} f(T_n(x)), \qquad x \in \Delta,$$

if and only if

$$f(x) = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} g(T_n(x)), \qquad x \in \Delta.$$

3. A more general approach

The assumptions in Proposition 1 are very demanding. Here we study other general conditions under which the transformation formula holds.

For an arithmetical function ρ, we say that $f \in L(\Delta, \rho)$ if

$$\sum_{n=1}^{\infty} |\rho(n) f(T_n(x))| < \infty \quad \forall x \in \Delta$$

(recall that we are using Δ to denote $[-1,1]$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ or $\mathbb{Q}$). In particular, $f \in L(\Delta, \alpha)$ means that (5) converges absolutely for every $x \in \Delta$.

Once again we use the arithmetical function δ defined by $\delta(1) = 1$ and $\delta(n) = 0$ for all $n > 1$. The relation $\delta \odot f = f$ follows easily from $T_1(x) = x$.

Analogously to the mixed associative property between $\circ$ and Dirichlet convolution $*$, we have the following version between $\odot$ and $*$. The proof is straightforward, because the absolute convergence allows the rearrangement of the sums.

Proposition 2. *Let α, β be two arithmetical functions, $f : \Delta \to \mathbb{R}$ (or $\mathbb{C}$), and suppose at a given $x \in \Delta$*

$$\sum_{n,m\in\mathbb{N}} |\alpha(n)\beta(m)f(T_{nm}(x))| = \sum_{k\in\mathbb{N}} (|\alpha| * |\beta|)(k)|f(T_k(x))| < \infty. \tag{10}$$

Then, all the series involved in the definitions of $(\alpha\odot(\beta\odot f))(x)$ and $((\alpha*\beta)\odot f)(x)$ are absolutely convergent, and

$$(\alpha \odot (\beta \odot f))(x) = ((\alpha * \beta) \odot f)(x).$$

In particular, if $f \in L(\Delta, |\alpha| * |\beta|)$, then $\alpha \odot (\beta \odot f) = (\alpha * \beta) \odot f$.

In this general context, the inversion formula becomes

Proposition 3. *Let α be an arithmetical funcion with Dirichlet convolution inverse α^{-1}. Given a function $f : \Delta \to \mathbb{R}$ (or $\mathbb{C}$), with $f \in L(\Delta, |\alpha|*|\alpha^{-1}|)$, the transform $g(x) = (\alpha \odot f)(x)$ is defined for all $x \in \Delta$. Moreover, if $g \in L(\Delta, |\alpha| * |\alpha^{-1}|)$, then $f(x) = (\alpha^{-1} \odot g)(x)$ for all $x \in \Delta$.*

Proof. By Proposition 2,

$$\alpha^{-1} \odot g = \alpha^{-1} \odot (\alpha \odot f) = (\alpha^{-1} * \alpha) \odot f = \delta \odot f = f.$$

For the second part, recall that $|\alpha| * |\alpha^{-1}| = |\alpha^{-1}| * |\alpha|$. □

In general, it does not seem easy to check that the condition $f \in L(\Delta, |\alpha|*|\alpha^{-1}|)$ implies $g \in L(\Delta, |\alpha| * |\alpha^{-1}|)$; this—if true—would mean that the inversion formula $\alpha^{-1} \odot g$ is defined without this extra hypothesis.

The following special case of Proposition 3 has special interest:

Proposition 4. *Let α be a completely multiplicative arithmetical function, $f : \Delta \to \mathbb{R}$ (or $\mathbb{C}$), and suppose that $f \in L(\Delta, \alpha d)$ (where $d(n)$ is the number of divisors of n). Then*

$$g(x) = \sum_{n\in\mathbb{N}} \alpha(n)f(T_n(x))$$

is defined for all $x \in \Delta$. Moreover, if $g \in L(\Delta, \alpha d)$, then

$$f(x) = \sum_{n\in\mathbb{N}} \mu(n)\alpha(n)g(T_n(x))$$

for all $x \in \Delta$.

Proof. If α is completely multiplicative, then $\alpha^{-1}(n) = \mu(n)\alpha(n)$. Moreover,

$$\begin{aligned}(|\alpha| * |\alpha^{-1}|)(k) &= \sum_{nm=k} |\alpha(n)\mu(m)\alpha(m)| \\ &\le \sum_{nm=k} |\alpha(nm)| = d(k)|\alpha|(k) = |d(k)\alpha(k)|,\end{aligned}$$

so the hypothesis $f \in L(\Delta, \alpha d)$ allows us to apply Proposition 3. The same holds with respect to $g \in L(\Delta, \alpha d)$. □

Remark. As commented previously, it does not seem easy to check if $g \in L(\Delta, \alpha d)$ given that $f \in L(\Delta, \alpha d)$. However, we claim that something weaker is true:

$$f \in L(\Delta, \alpha d^2) \;\Rightarrow\; g \in L(\Delta, \alpha d).$$

To prove this, take into account that $d(n) \le d(k)$ when $n \mid k$, and notice also that $|\alpha|$ is completely multiplicative. Thus

$$\begin{aligned}\sum_{n\in\mathbb{N}} d(n)|\alpha(n) g(T_n(x))| &= \sum_{n\in\mathbb{N}} d(n)|\alpha(n)| \left| \sum_{m\in\mathbb{N}} \alpha(m) f(T_m(T_n(x))) \right| \\ &\le \sum_{k\in\mathbb{N}} d(k) \left(\sum_{nm=k} |\alpha(n)\alpha(m)| \right) |f(T_k(x))| \\ &= \sum_{k\in\mathbb{N}} d(k)^2 |\alpha(k)| |f(T_k(x))| < \infty\end{aligned}$$

since $f \in L(\Delta, \alpha d^2)$, so the claim is proved. Actually, the extra factor $d(n)$ is not very troublesome, because $d(n) = o(n^r)$ for every $r > 0$ (see [1, Section 18.1, Theorem 15, p. 260]).

References

[1] G. M. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fifth ed., Oxford Univ. Press, 1979.

[2] T. J. Rivlin, *Chebyshev Polynomials: From Aproximation Theory to Algebra and Number Theory*, second ed., Wiley, 1990.

Departamento de Matemáticas y Computación, Universidad de La Rioja, Calle Luis de Ulloa s/n, 26004 Logroño, Spain
E-mail address: oscar.ciaurri@dmc.unirioja.es

Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain
E-mail address: navas@usal.es

Departamento de Matemáticas y Computación, Universidad de La Rioja, Calle Luis de Ulloa s/n, 26004 Logroño, Spain
E-mail address: jvarona@dmc.unirioja.es
URL: http://www.unirioja.es/dptos/dmc/jvarona/welcome.html