MEAN CESÀRO-TYPE SUMMABILITY OF FOURIER-NEUMANN SERIES

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Abstract. Let J_ν be the Bessel function of order ν. For $\alpha > -1$, the functions $x^{-\alpha-1}J_{\alpha+2n+1}(x)$, $n = 0, 1, 2 \dots$, form an orthogonal system in $L^2(x^{2\alpha+1}\,dx)$, but the span of such functions is not dense in this space. For a function f, let $S_k^\alpha f$ denote the kth partial sum of the Fourier-Neumann series of f. In this paper we provide the minimal conditions on a real γ and $1 < p < \infty$, for which the means $R_n^\alpha f = \frac{\lambda_0 S_0^\alpha f + \dots + \lambda_n S_n^\alpha f}{\lambda_0 + \dots + \lambda_n}$, $\lambda_k = 2(\alpha+2k+2)$, are uniformly bounded in the spaces $L^p(x^{2(\alpha+\gamma)+1}\,dx)$. Clearly, the convergence $R_n^\alpha f \to f$ holds only for functions from the closure of the linear span of the orthogonal system in these spaces. As a byproduct of the main result, we obtain a characterization of the closure of the span in terms of functions whose modified Hankel transforms of order α are supported on the interval $[0, 1]$.

1. Introduction and statement of results

Let J_ν stand for the Bessel function of the first kind of order ν. For $\alpha > -1$, the formula

$$\int_0^\infty J_{\alpha+2n+1}(x)J_{\alpha+2m+1}(x)\frac{dx}{x} = \frac{\delta_{nm}}{2(2n+\alpha+1)}, \qquad n, m = 0, 1, 2, \dots,$$

provides an orthonormal system $\{j_n^\alpha\}_{n=0}^\infty$ in $L^2((0,\infty), x^{2\alpha+1}\,dx)$ ($L^2(d\mu_\alpha)$ and, more generally, $L^p(d\mu_\alpha)$ from now on) given by

$$j_n^\alpha(x) = \sqrt{2(2n+\alpha+1)}\,J_{\alpha+2n+1}(x)x^{-\alpha-1}, \qquad n = 0, 1, 2, \dots.$$

For a function f, provided that the coefficients

$$c_k^\alpha(f) = \int_0^\infty f(y)j_k^\alpha(y)y^{2\alpha+1}\,dy, \qquad k = 0, 1, 2, \dots,$$

exist, consider its formal expansion $f \sim \sum_{k=0}^\infty c_k^\alpha(f)j_k^\alpha(x)$ and the partial sum operators

$$S_n^\alpha(f, x) = \sum_{k=0}^n c_k^\alpha(f)j_k^\alpha(x), \qquad n = 0, 1, 2, \dots.$$

Series of the form $\sum_{n\ge 0} a_n J_{\alpha+n}$ are usually called the Neumann series, hence we refer to $\sum_{k=0}^\infty c_k^\alpha(f)j_k^\alpha(x)$ as to the Fourier-Neumann series.

For α and γ, $\alpha > -1$, $\gamma > -1-\alpha$, let

$$p_0(\alpha,\gamma) = \max\left\{1, \frac{4(\alpha+\gamma+1)}{2\alpha+3}\right\}, \qquad p_1(\alpha,\gamma) = \begin{cases} \frac{4(\alpha+\gamma+1)}{2\alpha+1}, & \alpha > -1/2, \\ \infty, & -1 < \alpha \le -1/2. \end{cases}$$

In the case $\gamma = 0$ we simply write $p_i(\alpha)$ in place of $p_i(\alpha, 0)$, $i = 1, 2$.

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The essential aim of this paper is to study the convergence $R_n^\alpha f \to f$ in the $L^p(d\mu_{\alpha+\gamma})$ spaces (note that it is equivalent to the study of the convergence in the weighted $L^p(x^{2\gamma} d\mu_\alpha)$ spaces), $1 < p < \infty$, where R_n^α are means of the partial sum operators $\{S_k^\alpha\}_{k=0}^n$, cf. (6). To pose the problem correctly, we must impose some natural conditions on the parameters involved. First, we have the requirement $j_n^\alpha \in L^p(d\mu_{\alpha+\gamma})$. Second, to ensure the existence of the coefficients $c_n^\alpha(f)$ for every $f \in L^p(d\mu_{\alpha+\gamma})$, we must require $j_n^\alpha \in L^{p'}(x^{-2\gamma p'}\, d\mu_{\alpha+\gamma})$. However, assuming $1 < p < \infty$, we have $j_n^\alpha \in L^p(d\mu_{\alpha+\gamma}) \cap L^{p'}(x^{-2\gamma p'}\, d\mu_{\alpha+\gamma})$ for every $n = 0, 1, 2, \dots$ if and only if $0 < \alpha + \gamma + 1 < (\alpha + 1)p$ and $p_0(\alpha, \gamma) < p < p_1(\alpha, \gamma)$, and these are precisely the minimal assumptions we impose. The last equivalence easily follows by using the following well-known asymptotics for the Bessel functions:

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} + O(x^{\nu+2}), \qquad x \to 0^+; \tag{1}$$

$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left[\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1})\right], \qquad x \to \infty. \tag{2}$$

Most orthogonal systems considered in the literature, for instance, the trigonometric, Jacobi, Laguerre and Hermite systems as well as the ones related to Freud weights, Bessel and Dini, are complete. Moreover, the span of each of these systems is dense in the corresponding L^p spaces, $1 \le p < \infty$. Thus, the L^p-convergence of the associated Fourier series is equivalent to the uniform boundedness of the partial sum operators. However, this is not the present case since the span of $\{j_n^\alpha\}$ is not dense in $L^p(d\mu_\alpha)$. Therefore, it is important to identify the closure in $L^p(d\mu_\alpha)$ of the subspace spanned by j_n^α, $n = 0, 1, 2, \dots$, or, more generally, the space

$$B_{p,\alpha,\gamma} = \overline{\operatorname{span}\{j_n^\alpha : n = 0, 1, 2, \dots\}} \qquad (\text{closure in } L^p(d\mu_{\alpha+\gamma})),$$

because the uniform boundedness of the partial sum operators S_n^α, or the Cesàro-type means R_n^α, would imply the L^p-convergence only for functions in $B_{p,\alpha,\gamma}$. Note that the spaces $B_{p,\alpha,\gamma}$ are well defined if $j_n^\alpha \in L^p(d\mu_{\alpha+\gamma})$ and this holds if and only if $\alpha + \gamma + 1 > 0$ and $p_0(\alpha, \gamma) < p$ (again by using (1) and (2), or see [2]).

Identifying the spaces $B_{p,\alpha,\gamma}$ with those defined in (5), we will use the modified Hankel transform H_α given by

$$H_\alpha(f, x) = \int_0^\infty \frac{J_\alpha(xy)}{(xy)^\alpha} f(y) y^{2\alpha+1}\, dy, \qquad x > 0. \tag{3}$$

For $\alpha \ge -1/2$, (1), (2) and Hölder's inequality show that (3) is well defined for every $f \in L^p(d\mu_\alpha)$ with $1 \le p < p_0(\alpha)$. Furthermore, it is easy to see that H_α is a bounded operator from $L^1(d\mu_\alpha)$ into $L^\infty(d\mu_\alpha)$. It is also well known that H_α extends to an isometric isomorphism from $L^2(d\mu_\alpha)$ onto itself and $H_\alpha \circ H_\alpha = \mathrm{Id}$ (for $-1 < \alpha < -1/2$, the isometric extension of H_α to $L^2(d\mu_\alpha)$ can also be done; see [7, 1]). As a consequence, by interpolation, for $\alpha \ge -1/2$ we consider H_α to be a bounded operator from $L^p(d\mu_\alpha)$ into $L^{p'}(d\mu_\alpha)$, $1 \le p \le 2$.

The Hankel transform of the function j_n^α is

$$H_\alpha(j_n^\alpha, x) = \sqrt{2(\alpha + 2n + 1)}\, P_n^{(\alpha,0)}(1 - 2x^2) \chi_{[0,1]}(x),$$

where $P_n^{(\alpha,\beta)}(x)$ is the n-th Jacobi polynomial of order (α, β), cf. [3], a remark on p. 127. The important fact here is that $H_\alpha(j_n^\alpha)$ is supported on $[0, 1]$; then, only functions f having this property could be approximated by the sequence $S_n^\alpha f$. We describe this property using the language of the modified Hankel transform.

Thus, define the operator M_α, associated with H_α, as the multiplier operator

$$M_\alpha f = H_\alpha(\chi_{[0,1]} H_\alpha f).$$

Applying Fubini's theorem, we have $M_\alpha(f,x) = \int_0^\infty M_\alpha(x,y) f(y)\, d\mu_\alpha(y)$, where

$$M_\alpha(x,y) = (xy)^{-\alpha} \int_0^1 J_\alpha(xs) J_\alpha(ys) s\, ds. \tag{4}$$

It is known that M_α is a bounded operator on $L^p(d\mu_\alpha)$ if and only if $p_0(\alpha) < p < p_1(\alpha)$, cf. [9, 14, 4, 5, 1].

For $\alpha \geq -1/2$ and $p_0(\alpha) < p < p_1(\alpha)$ define the following closed subspace of $L^p(d\mu_\alpha)$:

$$E_{p,\alpha} = \{f \in L^p(d\mu_\alpha) : M_\alpha f = f\} = M_\alpha(L^p(d\mu_\alpha)). \tag{5}$$

The spaces $E_{p,\alpha}$ enjoy some nice properties: for $1 < p < q < \infty$, $E_{p,\alpha} \subset E_{q,\alpha}$ and the inclusion is continuous and dense. Besides, the dual space of $E_{p,\alpha}$ is naturally identified with $E_{p',\alpha}$, cf. [14] for details.

Note that we assume $\alpha > -1$ in the definition of $B_{p,\alpha,\gamma}$. We require, however, $\alpha \geq -1/2$ for $E_{p,\alpha}$. Actually, the definition of $E_{p,\alpha}$ can be extended to the whole range of $\alpha > -1$. But in the case $\alpha < -1/2$, H_α is not a bounded operator from L^1 into L^∞ since $J_\alpha(t)t^{-\alpha}$ is not a bounded function. As a consequence, the spaces $E_{p,\alpha}$ do not behave for $\alpha < -1/2$ like those for $\alpha \geq -1/2$. Thus, some of the results in this paper will be established for $\alpha > -1$, but we will require $\alpha \geq -1/2$ when $E_{p,\alpha}$ appears.

It is clear from the very definition that for $f \in E_{p,\alpha} \cap L^2(d\mu_\alpha)$ the Hankel transform of f is supported on $[0,1]$. When $p = 2$, using the fact that H_α is an isomorphism and the completeness of the Jacobi system, we easily get that $\{j_n^\alpha\}$ is, indeed, a basis for $E_{2,\alpha}$. But this cannot be proved in such a direct way for other values of p.

In [14], one of the authors (actually, in that paper there is a change of notation that is, basically, a change of variable $x \mapsto x^2$; see [5, Remark 1] for appropriate comments on the change of notation) proved the following: by using the basic case $p = 2$, and the properties of $E_{p,\alpha}$, it follows that $B_{p,\alpha,0} = E_{p,\alpha}$ for $2 \leq p < p_1(\alpha)$. On the other hand, it was also shown that, for $\alpha \geq -1/2$,

$$\|S_n^\alpha f\|_{L^p(d\mu_\alpha)} \leq C \|f\|_{L^p(d\mu_\alpha)}, \qquad n = 0, 1, 2, \dots,$$

if and only if $\max\{p_0(\alpha), 4/3\} < p < \min\{p_1(\alpha), 4\}$ (for a more general result in a weighted setting, see [3, Theorem 1]). Then, for this range of p's, $\|S_n^\alpha f - f\|_{L^p(d\mu_\alpha)} \to 0$ for every f from the space $B_{p,\alpha,0}$, already identified as $E_{p,\alpha}$ for $2 \leq p < p_1(\alpha)$. By using duality and the uniform boundedness of the partial sum operators S_n^α, it also follows that $B_{p,\alpha,0} = E_{p,\alpha}$ when $\max\{p_0(\alpha), 4/3\} < p < 2$.

If $-1/2 \leq \alpha < 0$, we have $p_0(\alpha) < 4/3$. Thus, there is a gap $p_0(\alpha) < p < 4/3$, for which the previous argument does not show if $B_{p,\alpha,0}$ and $E_{p,\alpha}$ are equal. Filling this gap is another aim of this paper (see Theorem 2 with $\gamma = 0$).

To overcome the difficulty, we will use means equivalent to Cesàro $(C,1)$ means instead of partial sums. This will allow us to eliminate the condition $4/3 < p < 4$ and, again by using a duality argument, will enable us to identify $B_{p,\alpha,0}$ with $E_{p,\alpha}$ in the whole range of $p_0(\alpha) < p < p_1(\alpha)$. Actually, instead of using Cesàro means, we will use the means

$$R_n^\alpha = \frac{\lambda_0 S_0^\alpha + \cdots + \lambda_n S_n^\alpha}{\lambda_0 + \cdots + \lambda_n}, \qquad \lambda_k = 2(\alpha + 2k + 2). \tag{6}$$

A reason for this is that for such means it is possible to write an exact expression for their kernels. This summation method is sometimes known as the Riesz method (R, λ_k); see, for instance, [10, § 59, p. 470]. The convergence in this method is equivalent to the convergence of Cesàro $(C,1)$ means (see, for instance, [8, § 3.8, Th. 14]). Thus, we refer to this method as to a *Cesàro-type* summation method.

Investigating the uniform boundedness and convergence of R_n^α reveals that the kernels of these operators remind those for the Hankel transform partial sum operators, cf. [5]; boundedness properties of the corresponding operators were studied there by using a convolution structure for the Hankel transform. Unfortunately, in the case of kernels considered here, an extra factor of size xy/n^2 appears, and the method used in [5] didn't turn out to be useful. On the other hand, however, the kernels have a similar behaviour as the ones that appear in [6]; boundedness properties of the corresponding operators were established without using properties of the aforementioned convolution. It occurs that the proofs may be adapted to the present case and the extra factor can be properly annihilated. This is particularly well visible in the proof of Proposition 4.

Finally, some comments seem to be relevant. Although, in many aspects, the situation of the present setting is similar to the one of the Laguerre expansions, for instance, the condition $4/3 < p < 4$ disappears when we pass from the partial sum operators to the $(C,1)$ means, cf. [11] (see [13] for a motivation of such a comparison), there are important differences. First, the $(C,1)$ (and also (C,δ)) summability of the Laguerre series can be deduced from an associated convolution structure, see [12]. Second, we always restrict the p's interval to $1 < p < \infty$, not including $p = 1$ or $p = \infty$. This is explained by the fact that already in the unweigted case, $\gamma = 0$, the system j_n^α, $n = 0, 1, 2, \dots$, is not contained in $L^1(d\mu_\alpha)$.

The main result of the paper states the uniform boundedness of R_n^α in $L^p(d\mu_{\alpha+\gamma})$.

Theorem 1. *Let $\alpha > -1$ and $\gamma \in \mathbb{R}$. Assume that*

$$0 < \alpha + \gamma + 1 < (\alpha + 1)p \tag{7}$$

and

$$p_0(\alpha, \gamma) < p < p_1(\alpha, \gamma). \tag{8}$$

Then

$$\|R_n^\alpha f\|_{L^p(d\mu_{\alpha+\gamma})} \le C \|f\|_{L^p(d\mu_{\alpha+\gamma})},$$

with C independent of $n = 0, 1, 2, \dots$ and f.

Note, that the assumptions that appear in Theorem 1 are, according to what was said earlier, "natural" (minimal).

The second main result of the paper concerns the spaces $B_{p,\alpha,\gamma}$.

Theorem 2. *Let $\alpha > -1$, $\alpha + \gamma \ge -1/2$, $p_0(\alpha + \gamma) < p < p_1(\alpha + \gamma)$ and $\gamma < 1/2$. If $p < 2$, assume further that $p_0(\alpha, \gamma) < p$ and $\alpha + \gamma + 1 < \frac{p}{4}(2\alpha + 4\gamma + 3)$. Then $B_{p,\alpha,\gamma} = E_{p,\alpha+\gamma}$.*

The proofs of Theorems 1 and 2 are postponed to Sections 3 and 4. Theorem 2 imediately shows that $B_{p,\alpha,0} = E_{p,\alpha}$ in the whole range of p's in which both spaces are well defined.

By using Theorems 1 and 2, the proof of the following result is standard.

Corollary 3. *Let $\alpha > -1$, $\alpha + \gamma \ge -1/2$, $p_0(\alpha + \gamma) < p < p_1(\alpha + \gamma)$, $\gamma < 1/2$, and (7) and (8) are satisfied. If $p < 2$, assume further $\alpha + \gamma + 1 < \frac{p}{4}(2\alpha + 4\gamma + 3)$. Then, $R_n^\alpha f \to f$ in $L^p(d\mu_{\alpha+\gamma})$ for every $f \in E_{p,\alpha+\gamma}$.*

Apart of the expansions based on the system $\{j_n^\alpha\}_{n=0}^\infty$, there is another setting in which the Fourier-Neumann series may be investigated. The system of functions

$$\varphi_n^\alpha(x) = \sqrt{2(2n + \alpha + 1)}\, J_{\alpha+2n+1}(x) x^{-1/2}, \qquad n = 0, 1, 2, \dots$$

is orthonormal (but not complete) in $L^2((0,\infty), dx)$; in what follows we simply write $L^2(dx)$ or, more generally, $L^p(x^\gamma dx)$ for weighted L^p spaces. Hence, for any suitable function f on $(0,\infty)$ consider its expansion $f \sim \sum_{k=0}^{\infty} d_k^\alpha(f)\varphi_k^\alpha(x)$, $d_k^\alpha(f) = \int_0^\infty f(x)\varphi_k^\alpha(x)\,dx$, and the partial sum operators

$$\mathcal{S}_n^\alpha(f,x) = \sum_{k=0}^{n} d_k^\alpha(f)\varphi_k^\alpha(x), \qquad n = 0,1,2,\dots .$$

Analogously, define the means

$$\mathcal{R}_n^\alpha = \frac{\lambda_0\mathcal{S}_0^\alpha + \dots + \lambda_n\mathcal{S}_n^\alpha}{\lambda_0 + \dots + \lambda_n}, \qquad \lambda_k = 2(\alpha + 2k + 2).$$

Clearly enough, results concerning both types of expansions are linked together since $\mathcal{S}_n^\alpha(f,x) = x^{\alpha+1/2}S_n^\alpha((\cdot)^{-(\alpha+1/2)}f,x)$.

Therefore, Theorem 1 has its counterpart in the following.

Theorem 4. *Let $\alpha > -1$, $\gamma \in \mathbb{R}$ and $1 < p < \infty$. Assume that*

$$\max\{-1, -1 - p(\alpha + 1/2)\} < \gamma < \min\{-1 + p, -1 + p(\alpha + 3/2)\}. \tag{9}$$

Then

$$\|\mathcal{R}_n^\alpha f\|_{L^p(x^\gamma\,dx)} \le C\|f\|_{L^p(x^\gamma\,dx)},$$

with C independent of $n = 0,1,2,\dots$ and f.

As in the case of Theorem 1, it can be easily seen by a direct computation that the assumptions of Theorem 4 are minimal in the sense that for all $n = 0,1,\dots$, $\varphi_n^\alpha \in L^p(x^\gamma\,dx)$ and the coefficients $d_k^\alpha(f)$ exist for every $f \in L^p(x^\gamma\,dx)$ if and only if (9) is satisfied. Clearly, Theorem 4 and Theorem 1 are equivalent. Actually, the direct proof of Theorem 4 simplifies a little bit; for instance, the formula corresponding to that of Proposition 1 becomes

$$\mathcal{R}_n^\alpha(f,x) = \mathcal{M}_\alpha^0(f,x) - \frac{d_n}{2}\Big(x\mathcal{M}_{\alpha+1}^1(yf,x) - x\mathcal{M}_{\alpha+2n+3}^1(yf,x)\Big).$$

2. The kernel

In this section we find exact expressions for the kernel of the Cesàro-type means R_n^α. The partial sum operators S_n^α can be written as

$$S_n^\alpha(f,x) = \int_0^\infty S_n^\alpha(x,y)f(y)y^{2\alpha+1}\,dy \tag{10}$$

with the kernel

$$S_n^\alpha(x,y) = (xy)^{-(\alpha+1)}\sum_{k=0}^{n} 2(\alpha + 2k + 1)J_{\alpha+2k+1}(x)J_{\alpha+2k+1}(y).$$

Thus, by (6),

$$R_n^\alpha(f,x) = \int_0^\infty R_n^\alpha(x,y)f(y)y^{2\alpha+1}\,dy$$

with

$$R_n^\alpha(x,y) = d_n\left(\lambda_0 S_0^\alpha(x,y) + \dots + \lambda_n S_n^\alpha(x,y)\right), \qquad \lambda_k = 2(\alpha + 2k + 2),$$

where $d_n^{-1} = \sum_{k=0}^{n}\lambda_k = 4(n+1)(\alpha+n+2)$.

For $\nu > -1$, let

$$\mathcal{J}_\nu(x,y) = \int_0^1 J_\nu(xs)J_\nu(ys)s\,ds. \tag{11}$$

Then

$$\mathcal{J}_\nu(x,y) = \frac{1}{x^2 - y^2}\{xJ_{\nu+1}(x)J_\nu(y) - yJ_\nu(x)J_{\nu+1}(y)\} \tag{12}$$

and

$$\mathcal{J}_\nu(x,y) = \frac{1}{x^2-y^2}\{yJ'_\nu(y)J_\nu(x) - xJ'_\nu(x)J_\nu(y)\}. \tag{13}$$

We applied von Lommel's formula, cf. [3], and, for the last equality, the identity $zJ_{\nu+1}(z) = \nu J_\nu(z) - zJ'_\nu(z)$.

By using (12), it can be proved (see [3]) that

$$S_n^\alpha(x,y) = (xy)^{-\alpha}\{\mathcal{J}_\alpha(x,y) - \mathcal{J}_{\alpha+2n+2}(x,y)\}. \tag{14}$$

Then

$$\begin{aligned} R_n^\alpha(x,y) &= \frac{1}{2}d_n(xy)^{-\alpha}\sum_{k=0}^{n} 2(\alpha+2k+2)\{\mathcal{J}_\alpha(x,y) - \mathcal{J}_{\alpha+2k+2}(x,y)\} \\ &= (xy)^{-\alpha}\left[\mathcal{J}_\alpha(x,y) - d_n\int_0^1 s\left(\sum_{k=0}^{n} 2(\alpha+2k+2)J_{\alpha+2k+2}(xs)J_{\alpha+2k+2}(ys)\right)ds\right] \\ &= (xy)^{-\alpha}\left[\mathcal{J}_\alpha(x,y) - d_n\int_0^1 s(xsys)(\mathcal{J}_{\alpha+1}(xs,ys) - \mathcal{J}_{\alpha+2n+3}(xs,ys))\,ds\right]. \end{aligned}$$

Therefore, denoting

$$\xi_\nu(x,y) = 2\int_0^1 s^3\mathcal{J}_\nu(xs,ys)\,ds,$$

we finally obtain

$$R_n^\alpha(x,y) = (xy)^{-\alpha}\mathcal{J}_\alpha(x,y) - \frac{1}{2}d_n(xy)^{-\alpha+1}(\xi_{\alpha+1}(x,y) - \xi_{\alpha+2n+3}(x,y)). \tag{15}$$

In this decomposition, the first summand is just the integral kernel of the multiplier operator M_α, as one can observe by comparing (11) with (4). Its boundedness properties has already been studied; see for instance [3] (note also that this summand appears in the decomposition of the kernel $S_n^\alpha(x,y)$ used to study the uniform boundedness of S_n^α).

We will find another expression for $\xi_\nu(x,y)$. By using (11), the change of variable $t = w/s$ and Fubini's theorem easily produce

$$\xi_\nu(x,y) = \int_0^1 w(1-w^2)J_\nu(xw)J_\nu(yw)\,dw.$$

This expression for $\xi_\nu(x,y)$ will be useful for our purposes for the following reasons: let $\mathcal{H}_\nu$ denote the (non-modified) Hankel transform of order $\nu > -1$, i.e.,

$$\mathcal{H}_\nu(f,x) = \int_0^\infty (xy)^{1/2}J_\nu(xy)f(y)\,dy, \qquad x>0.$$

The Bochner-Riesz multiplier of order $\delta \ge 0$ for $\mathcal{H}_\nu$ is given by

$$\mathcal{M}_\nu^\delta(f,x) = \mathcal{H}_\nu(m^\delta\cdot\mathcal{H}_\nu f,x) = \int_0^\infty \mathcal{M}_\nu^\delta(x,y)f(y)\,dy, \tag{16}$$

where $m^\delta(y) = (1-y^2)_+^\delta$ and

$$\mathcal{M}_\nu^\delta(x,y) = \int_0^1 (1-w^2)^\delta(xw)^{1/2}J_\nu(xw)(yw)^{1/2}J_\nu(yw)\,dw. \tag{17}$$

Then, $\xi_\nu(x,y) = (xy)^{-1/2}\mathcal{M}_\nu^1(x,y)$. Pointwise estimates for $\mathcal{M}_\nu^1(x,y)$, as well as necessary and sufficient conditions for the uniform boundedness

$$\|\mathcal{M}_\nu^1(f,x)x^a\|_{L^p(dx)} \le C\|f(x)x^a\|_{L^p(dx)}, \qquad \nu\ge\alpha,$$

were found in [6]. We will partially use those results in Section 3 of this paper.

Actually, also the first summand in (15) can be described following the notation of (16): the multiplier M_α (for the modified Hankel transform H_α) is related to the multiplier $\mathcal{M}_\alpha = \mathcal{M}_\alpha^0$ (for the Hankel transform $\mathcal{H}_\alpha$) by mean of $\mathcal{M}_\alpha(f,x) = x^{\alpha+1/2}M_\alpha((\cdot)^{-(\alpha+1/2)}f,x)$, with the analogous relation for the kernels $\mathcal{M}_\alpha(x,y)$ and $M_\alpha(x,y)$.

Summarizing, we found the following decomposition.

Proposition 1. *Let $\alpha > -1$ and $n = 0,1,2\dots$. Then*

$$R_n^\alpha(f,x) = x^{-\alpha-1/2}\mathcal{M}_\alpha^0(y^{\alpha+1/2}f,x) - \frac{d_n}{2}\Big(x^{-\alpha+1/2}\mathcal{M}_{\alpha+1}^1(y^{\alpha+3/2}f,x) - x^{-\alpha+1/2}\mathcal{M}_{\alpha+2n+3}^1(y^{\alpha+3/2}f,x)\Big)$$

with $d_n^{-1} = \sum_{k=0}^n \lambda_k = 4(n+1)(\alpha+n+2)$.

3. Proof of Theorem 1

The main idea of the proof is the following. The first summand in the decomposition of $R_n^\alpha(f,x)$, $x^{-\alpha-1/2}\mathcal{M}_\alpha^0(y^{\alpha+1/2}f,x)$, may be easily treated since we know appropriate weighted bounds for the operator $\mathcal{M}_\alpha^0$. Thus we are immediately left with the operators

$$Q_n^\alpha(f,x) = R_n^\alpha(f,x) - x^{-\alpha-1/2}\mathcal{M}_\alpha^0(y^{\alpha+1/2}f,x).$$

The relevant uniform inequalities for them,

$$\|Q_n^\alpha f\|_{L^p(d\mu_{\alpha+\gamma})} \le C\|f\|_{L^p(d\mu_{\alpha+\gamma})}, \qquad n = 1,2,\dots,$$

will be obtained by a proper splitting of $(0,\infty)\times(0,\infty)$ and then by treating each resulting region separately. It is important to stress here the fact that in the region that corresponds to large x and y we will go back to the original definition of R_n^α as a Cesàro type mean of $\{S_k^\alpha\}$. In consequence, the results of Section 2 will allow us to write $Q_n^\alpha f$ as a mean of $x^{-\alpha-1/2}\mathcal{M}_{\alpha+2k+2}^0(y^{\alpha+1/2}f,x)$, $k = 0,1,\dots,n$, and then to treat each resulting term in the aforementioned region separately.

The weighted bound for $\mathcal{M}_\alpha^0$ is contained in the following (see, with different degree of generality, [9], [4], [13] and, in particular, [3, § 6.4]).

Proposition 2. *Let $\alpha > -1$, $\gamma \in \mathbb{R}$ and $1 < p < \infty$. Then*

$$\|x^{-\alpha-1/2}\mathcal{M}_\alpha^0(y^{\alpha+1/2}f,x)\|_{L^p(d\mu_{\alpha+\gamma})} \le C\|f(x)\|_{L^p(d\mu_{\alpha+\gamma})}$$

if and only if (7) *and* (8) *are satisfied.*

Let $\nu = \nu(n) = \alpha+2n+3$, $n = 0,1,\dots$, and $A = (0,4\nu)$, $B = (2\nu,\infty)$. Since

$$\chi_{(0,\infty)\times(0,\infty)} = \chi_{A\times A} + \chi_{A^c\times B^c} + \chi_{B^c\times A^c} + \chi_{B\times B} - \chi_{(A\cap B)\times(A\cap B)},$$

by using the notation $Q_{n,T,S}^\alpha f = \chi_T Q_n^\alpha(\chi_S f)$ we have

$$\|Q_n^\alpha f\|_{L^p(d\mu_{\alpha+\gamma})} \le \|Q_{n,A,A}^\alpha f\|_{L^p(d\mu_{\alpha+\gamma})} + \|Q_{n,A^c,B^c}^\alpha f\|_{L^p(d\mu_{\alpha+\gamma})} + \|Q_{n,B^c,A^c}^\alpha f\|_{L^p(d\mu_{\alpha+\gamma})} + \|Q_{n,B,B}^\alpha f\|_{L^p(d\mu_{\alpha+\gamma})} + \|Q_{n,A\cap B,A\cap B}^\alpha f\|_{L^p(d\mu_{\alpha+\gamma})}.$$

We will bound each of the five quantities above by $C\|f\|_{L^p(d\mu_{\alpha+\gamma})}$ with C independent of n and f. In the rest of this section we write $\|\cdot\|_p$ for the unweighted L^p-norm on $(0,\infty)$.

3.1. Estimating $\|Q_{n,A,A}^\alpha f\|_{L^p(d\mu_{\alpha+\gamma})}$.

Noting that $d_n \sim \nu(n)^{-2}$, it is enough to show the uniform bounds:

$$\nu^{-2}\|x^{-\alpha+1/2}\chi_A(x)\mathcal{M}_\nu^1(y^{\alpha+3/2}\chi_A f,x)\|_{L^p(d\mu_{\alpha+\gamma})} \le C\|\chi_A f\|_{L^p(d\mu_{\alpha+\gamma})}, \quad \nu \ge \alpha+3,$$

and the analogous bound with $\mathcal{M}_\nu^1$ frozen to $\mathcal{M}_{\alpha+1}^1$ (but with A associated to ν, not to $\alpha+1$!). In both cases, taking $g(y) = y^{\alpha+3/2}f(y)$ and using the notation

$$a = -\alpha - 1/2 + (2(\alpha+\gamma)+1)/p, \tag{18}$$

proving the former bounds reduces to showing that

$$\nu^{-2}\|x\chi_A(x)\mathcal{M}_\nu^1(\chi_A g, x)x^a\|_p \le C\|y^{-1}\chi_A g y^a\|_p, \quad \nu \ge \alpha+3,$$

and the analogous bound with $\mathcal{M}_\nu^1$ replaced by $\mathcal{M}_{\alpha+1}^1$. These bounds, due to $x \le 4\nu$ and $(4\nu)^{-1} \le y^{-1}$, are further reduced to

$$\|\chi_A(x)\mathcal{M}_\nu^1(\chi_A g, x)x^a\|_p \le C\|\chi_A g y^a\|_p, \quad \nu \ge \alpha+3, \tag{19}$$

and the one with $\mathcal{M}_\nu^1$ replaced by $\mathcal{M}_{\alpha+1}^1$.

In [6], the following result on the uniform boundedness of the Bochner-Riesz operators for the Hankel transform was found (actually, in a more general setting).

Proposition 3 ([6, Theorem 1])**.** *Let $\beta > -1$ and $1 < p < \infty$. The Bochner-Riesz multipliers of order one for $\mathcal{H}_\nu$ given by* (16)*, satisfy the uniform estimate*

$$\|\mathcal{M}_\nu^1(g,x)x^a\|_p \le C\|g(x)x^a\|_p, \qquad \nu \ge \beta,$$

if and only if $-1/p-(\beta+1/2) < a < 1-1/p+(\beta+1/2)$ and $-1/p-1 < a < 2-1/p$.

Thus, taking $\beta = \alpha+1$ gives (19) and its companion once we assume that

$$\max\left\{1, \frac{4(\alpha+\gamma+1)}{2\alpha+5}\right\} < p < \begin{cases} \frac{4(\alpha+\gamma+1)}{2\alpha-1}, & \text{if } \alpha > 1/2, \\ \infty, & \text{if } -1 < \alpha \le 1/2, \end{cases}$$

and

$$0 < \alpha+\gamma+1 < (\alpha+3/2)p$$

are satisfied. Finally, note that the two conditions are implied by (7) and (8).

3.2. **Estimating $\|Q^\alpha_{n,B^c,A^c} f\|_{L^p(d\mu_{\alpha+\gamma})}$ and $\|Q^\alpha_{n,A^c,B^c} f\|_{L^p(d\mu_{\alpha+\gamma})}$.** Applying an argument from the previous subsection and taking $g(y) = y^{\alpha+1/2}f(y)$ it is enough to prove the uniform inequalities

$$\nu^{-2}\|x\chi_{B^c}(x)\mathcal{M}_\nu^1(y\chi_{A^c} g, x)x^a\|_p \le C\|g(y)y^a\|_p, \quad \nu \ge \alpha+3, \tag{20}$$

and

$$\nu^{-2}\|x\chi_{A^c}(x)\mathcal{M}_\nu^1(y\chi_{B^c} g, x)x^a\|_p \le C\|g(y)y^a\|_p, \quad \nu \ge \alpha+3, \tag{21}$$

with a given by (18), and its counterparts with $\mathcal{M}_\nu^1$ replaced by $\mathcal{M}_{\alpha+1}^1$. This will be done in Propositions 4 and 5 below by applying uniform pointwise estimates of Bessel functions which were already used in [6]: with a constant C, independent of ν and x,

$$|x^{1/2}J_\nu(x)| \le C\Phi_\nu(x) \tag{22}$$

where, for $\nu \ge 1$,

$$\Phi_\nu(x) = \begin{cases} \frac{x^{\nu+1/2}}{\Gamma(\nu+1)2^\nu}, & 0 < x < \nu/2, \\ \nu^{1/4}(\nu^{1/3}+|x-\nu|)^{-1/4}, & \nu/2 < x < 2\nu, \\ 1, & 2\nu < x < \infty, \end{cases}$$

and, for $-1 < \nu < 1$,

$$\Phi_\nu(x) = \begin{cases} \frac{x^{\nu+1/2}}{\Gamma(\nu+1)}, & 0 < x < 1/2, \\ 1, & 1/2 < x < \infty. \end{cases}$$

Moreover, there exists a constant C independent of $\nu > -1$, such that, for $(x,y) \in B^c \times A^c$ or $(x,y) \in A^c \times B^c$, we have

$$|\mathcal{M}_\nu^1(x,y)| \le C\frac{\Phi_\nu(x)\Phi_\nu(y)}{|x-y|^2}; \tag{23}$$

see [6, Lemma 4] and a slight modification of [6, Lemma 2].

Proposition 4. *Let $\alpha > -1$, $\alpha + \gamma > -1$, $1 < p < \infty$, and a be given by* (18). *If $p < p_1(\alpha, \gamma)$, then the inequality* (20) *and its aforementioned counterpart are satisfied.*

Proof. We consider only the case of (20). The proof of its mutant, with $\mathcal{M}^1_{\alpha+1}$ in place of $\mathcal{M}^1_\nu$, requires minor modifications only. For $(x, y) \in B^c \times A^c$, $x \le 2\nu$. Thus

$$\left(\int_0^{2\nu} \left|\frac{x}{\nu^2}\mathcal{M}^1_\nu(yg\chi_{A^c}, x)x^a\right|^p dx\right)^{1/p} \le \frac{C}{\nu}\left(\int_0^{2\nu} \left|\mathcal{M}^1_\nu(yg\chi_{A^c}, x)x^a\right|^p dx\right)^{1/p}.$$

Moreover, in that region, $y - x \ge y/2$, so $1/|x-y|^2 \le 4/y^2$. Then, by using (23) and Hölder's inequality,

$$\begin{aligned}\left|\chi_{B^c}(x)\mathcal{M}^1_\nu(yg\chi_{A^c}, x)x^a\right| &\le C\chi_{B^c}(x)x^a\Phi_\nu(x)\left(\int_0^\infty \chi_{A^c}(y)y^{-1}|g(y)|\Phi_\nu(y)\,dy\right)\\ &\le C\chi_{B^c}(x)x^a\Phi_\nu(x)\,\|g(y)y^a\|_p\,\|\chi_{A^c}(y)y^{-a-1}\Phi_\nu(y)\|_{p'}\end{aligned}$$

and thus

$$\begin{aligned}&\nu^{-2}\|\chi_{B^c}(x)\mathcal{M}^1_\nu(yg\chi_{A^c}, x)x^a\|_p\\ &\qquad \le C\nu^{-1}\,\|g(y)y^a\|_p\,\|\chi_{A^c}(y)y^{-a-1}\Phi_\nu(y)\|_{p'}\,\|\chi_{B^c}(x)x^a\Phi_\nu(x)\|_p.\end{aligned}$$

Therefore, to establish the proposition we only need to check that the product of the last two terms is bounded by $C\nu$. This easily follows, by a direct calculation, once we have $(-a-1)p' < -1$ (i.e., $p < p_1(\alpha, \gamma)$), from the definition of Φ_ν. □

Proposition 5. *Let $\alpha > -1$, $\alpha + \gamma > -1$, $1 < p < \infty$ and a be given by* (18). *If $p_0(\alpha, \gamma) < p$, then the inequality* (21) *and its aforementioned counterpart are satisfied.*

Proof. The proof is very similar to that of the previous proposition. Moreover, a duality argument can also be used, because the operators from Propositions 4 and 5 are adjoint in the following sense:

$$\langle x\chi_{B^c}\mathcal{M}^1_\nu(y^{-a}yg_1\chi_{A^c}, x)x^a, h_1\rangle = \langle y\chi_{A^c}\mathcal{M}^1_\nu(x^a xh_1\chi_{B^c}, y)y^{-a}, g_1\rangle.$$

Thus, the hypothesis $(-a-1)p' < -1$ from the proof of Proposition 4 becomes $(a-1)p < -1$, i.e., $p_0(\alpha, \gamma) < p$. □

3.3. **Estimating $\|Q^\alpha_{n,B,B}f\|_{L^p(d\mu_{\alpha+\gamma})}$ and $\|Q^\alpha_{n,A\cap B,A\cap B}f\|_{L^p(d\mu_{\alpha+\gamma})}$.** Start with the case of $B \times B$. Here we do not use the decomposition of $Q^\alpha_n(f, x)$ that follows from Proposition 1. Instead, we apply earlier results of Section 2 that lead to writing $Q^\alpha_n(f, x)$ as the mean

$$Q^\alpha_n(f, x) = -\frac{d_n}{2}\sum_{k=0}^n \lambda_k x^{-\alpha-1/2}\mathcal{M}^0_{\alpha+2k+2}(y^{\alpha+1/2}f, x).$$

The required bound

$$\|Q^\alpha_{n,B,B}f\|_{L^p(d\mu_{\alpha+\gamma})} \le C\|f\|_{L^p(d\mu_{\alpha+\gamma})}$$

then follows from the following.

Proposition 6. *Let $\alpha > -1$, $\alpha + \gamma > -1$, $1 < p < \infty$ and, for simplicity, write $\kappa = \alpha + 2k + 2$. If the assumption* (8) *is satisfied then*

$$\|x^{-\alpha-1/2}\chi_B(x)\mathcal{M}^0_\kappa(y^{\alpha+1/2}\chi_B f, x)\|_{L^p(d\mu_{\alpha+\gamma})} \le C\|f(x)\|_{L^p(d\mu_{\alpha+\gamma})}, \quad 0 \le k \le n,$$

with a constant C independent of n.

Proof. Denoting by $H(g,x)$ the Hilbert transform of g,

$$H(g,x) = \text{P.V.} \int_0^\infty \frac{g(y)}{x-y}\,dy,$$

and performing the change of variables $x \mapsto x^{1/2}$ and $y \mapsto y^{1/2}$ in (13), it is then enough to prove

$$\int_0^\infty \left|\chi_B(\sqrt{x})x^{-\alpha/2}J_\kappa(\sqrt{x})H(\chi_B(\sqrt{y})f(\sqrt{y})y^{(1+\alpha)/2}J'_\kappa(\sqrt{y}),x)\right|^p x^{\alpha+\gamma}\,dx \le C\int_0^\infty \left|f(\sqrt{x})\right|^p x^{\alpha+\gamma}\,dx \tag{24}$$

and the analogous inequality with $J_\kappa(\sqrt{x})$ replaced by $J'_\kappa(\sqrt{x})$ and $J'_\kappa(\sqrt{y})$ replaced by $J_\kappa(\sqrt{y})$. We treat (24) only, the proof of its companion is completely analogous. By (22) (and similar bounds for J'_ν, see [6]), it is clear that $|J_\kappa(x)| \le Cx^{-1/2}$ and $|J'_\kappa(x)| \le Cx^{-1/2}$ for $x \ge 2\nu$ with a constant C independent of x, κ and ν ($\nu \ge \alpha+2 > 1$).

We will use the fact that the Hilbert transform is a bounded operator on weighted L^p spaces with weights from A_p. It is clear that $x^{\alpha+\gamma\pm p/4-\alpha p/2} \in A_p$ if and only if $-1 < \alpha+\gamma \pm p/4 - \alpha p/2 < p-1$. Here, the relevant inequalities are $-1 < \alpha+\gamma-p/4-\alpha p/2$ and $\alpha+\gamma+p/4-\alpha p/2 < p-1$, which are equivalent to (8) when considered together. Take

$$g(y) = \chi_B(\sqrt{y})f(\sqrt{y})y^{(1+\alpha)/2}J'_\kappa(\sqrt{y}).$$

To prove (24) we apply successively $|J_\kappa(\sqrt{x})| \le Cx^{-1/4}$, $x^{\alpha+\gamma-p/4-\alpha p/2} \in A_p$, and $|J'_\kappa(\sqrt{x})| \le Cx^{-1/4}$, obtaining

$$\begin{aligned}\int_0^\infty \left|\chi_B(\sqrt{x})x^{-\alpha/2}J_\kappa(\sqrt{x})H(g,x)\right|^p x^{\alpha+\gamma}\,dx &\le C\int_0^\infty |H(g,x)|^p x^{\alpha+\gamma-p/4-\alpha p/2}\,dx\\ &\le C\int_0^\infty |g(x)|^p x^{\alpha+\gamma-p/4-\alpha p/2}\,dx\\ &\le C\int_0^\infty \left|f(\sqrt{x})\right|^p x^{\alpha+\gamma}\,dx.\end{aligned}$$

□

The bound for $\|Q^\alpha_{n,A\cap B,A\cap B}f\|_{L^p(d\mu_{\alpha+\gamma})}$ follows from the uniform estimate

$$\|x^{-\alpha-1/2}\chi_{A\cap B}(x)\mathcal{M}^0_\kappa(y^{\alpha+1/2}\chi_{A\cap B}f,x)\|_{L^p(d\mu_{\alpha+\gamma})} \le C\|f(x)\|_{L^p(d\mu_{\alpha+\gamma})},$$

$0 \le k \le n$, with C independent of n. This can be obtained by applying Proposition 6 to the function $f(y)\chi_{A\cap B}(y)$ and taking into account that

$$|x^{-\alpha-1/2}\chi_{A\cap B}(x)\mathcal{M}^0_\kappa(y^{\alpha+1/2}\chi_{A\cap B}f,x)| \le |x^{-\alpha-1/2}\chi_B(x)\mathcal{M}^0_\kappa(y^{\alpha+1/2}\chi_{A\cap B}f,x)|.$$

The proof of Theorem 1 is completed.

4. Proof of Theorem 2

We follow closely the proof of Theorem 4 in [3], where we refer the reader for details. Clearly, the assumptions $p_0(\alpha+\gamma) < p < p_1(\alpha+\gamma)$ and $\alpha+\gamma \ge -1/2$ are needed to have the space $E_{p,\alpha+\gamma}$ well defined and assure good properties of it, see a comment in Section 1. Moreover, in our argument the case $p=2$ is crucial since it is required when considering other cases, $p<2$ and $p>2$. Therefore, in the range of p's, $p_0(\alpha,\gamma) < p$, for which the spaces $B_{p,\alpha,\gamma}$ are well defined, $p=2$ must be included, which forces us to impose the condition $\gamma < 1/2$. Clearly, with the last assumption the spaces $B_{p,\alpha,\gamma}$ are well defined for all $p \ge 2$. Thus, apart of $\alpha > -1$, in what follows we assume the conditions $\alpha+\gamma \geq -1/2$, $p_0(\alpha+\gamma) < p < p_1(\alpha+\gamma)$, and $\gamma < 1/2$ to hold.

The formula (which holds for $\alpha > -1$, $\gamma < 1$ and $\alpha+\gamma > -1$, see [3, Lemma 3])

$$H_{\alpha+\gamma}(j_n^\alpha, x) = 2^{1/2+\gamma} \frac{\sqrt{\alpha+2n+1}\,\Gamma(n+1)}{\Gamma(n+1-\gamma)}(1-x^2)^{-\gamma} P_n^{(\alpha+\gamma,-\gamma)}(1-2x^2)\chi_{[0,1]}(x).$$

shows the inclusion $B_{p,\alpha,\gamma} \subseteq E_{p,\alpha+\gamma}$. It remains to check that in fact the inclusion becomes the identity (with the additional assumption $\alpha+\gamma+1 < \frac{p}{4}(2\alpha+4\gamma+3)$ if $p < 2$).

Case $p = 2$. The above formula, the completeness of the Jacobi system and the fact that $H_{\alpha+\gamma}$ is an automorphism of $L^2(d\mu_{\alpha+\gamma})$ prove the claim, i.e. $B_{2,\alpha,\gamma} = E_{2,\alpha+\gamma}$.

Case $p > 2$. Here the crucial fact is the result of Case $p = 2$ and continuity and density of the inclusion $E_{2,\alpha+\gamma} \subset E_{p,\alpha+\gamma}$.

Case $p < 2$. The required equality follows by showing that the only functional $T \in (E_{p,\alpha+\gamma})'$ such that $T(f) = 0$ for all $f \in B_{p,\alpha,\gamma}$ is $T = 0$. Since dual of $E_{p,\alpha+\gamma}$ is naturally identified with $E_{p',\alpha+\gamma}$, there exists $g \in E_{p',\alpha+\gamma}$, such that $T(f) = \int_0^\infty g(x)f(x)\,d\mu_{\alpha+\gamma}(x)$, for every $f \in E_{p,\alpha+\beta}$. With the additional assumption $\alpha+\gamma+1 < \frac{p}{4}(2\alpha+4\gamma+3)$, we have $j_n^{\alpha+\gamma} \in B_{p,\alpha,\gamma}$, see [3, Lemma 2]. Therefore, in particular, $c_n^{\alpha+\gamma}(g) = \int_0^\infty g(x) j_n^{\alpha+\gamma}(x)\,d\mu_{\alpha+\gamma}(x) = 0$, for $n = 0, 1, 2, \dots$. Now, from the preceding case, we have $B_{p',\alpha+\gamma,0} = E_{p',\alpha+\gamma}$, so that $g \in B_{p',\alpha+\gamma,0}$. This fact, taking into account that with the hypothesis $p_0(\alpha,\gamma) < p < 2$ we can apply Theorem 1, together with $c_n^{\alpha+\gamma}(g) = 0$ gives $0 = \lim_{n\to\infty} R_n^{\alpha+\gamma} g = g$ in $L^{p'}(d\mu_{\alpha+\gamma})$.

The proof of Theorem 2 is completed.

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