WEAK BEHAVIOUR OF FOURIER-NEUMANN SERIES

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Abstract. Let J_μ denote the Bessel function of order μ. The functions $x^{-\alpha-1}J_{\alpha+2n+1}(x)$, $n = 0, 1, 2, \dots$, form an orthogonal system in the space $L^2((0,\infty), x^{2\alpha+1}dx)$ when $\alpha > -1$. In this paper we prove that the Fourier series associated to this system is of restricted weak type for the endpoints of the interval of mean convergence, while it is not of weak type if $\alpha \geq 0$.

1. Introduction and results

Given a positive measure σ on some space and an orthonormal system $\{\varphi_n\}_{n\geq 0}$ in $L^2(\sigma)$, the Fourier series associated to $\{\varphi_n\}_{n\geq 0}$ is the sequence of operators S_n defined by

$$S_n f = \sum_{k=0}^{n} c_k(f)\varphi_k, \qquad f \in L^2(\sigma),$$

where $c_k(f) = \int f\varphi_k\, d\sigma$. The elementary property that $\|S_n f - f\|_{L^2(\sigma)} \longrightarrow 0$ for every $f \in \overline{\operatorname{span}\{\varphi_n\}_{n\geq 0}}$ raises the same question with the $L^2(\sigma)$ norm replaced by the $L^p(\sigma)$ norm, $1 \leq p \leq \infty$. By the Banach-Steinhaus theorem, this is equivalent to the uniform boundedness $\|S_n f\|_{L^p(\sigma)} \leq C\|f\|_{L^p(\sigma)}$, $f \in L^p(\sigma)$, $n \geq 0$.

Needless to say, the most important case is the trigonometric system on the unit circle $\mathbb{T}$, for which the boundedness holds if $1 < p < \infty$ [19]. For $p = \infty$ the answer is definitely negative, while for $p = 1$ the boundedness fails but there is a weak substitute in terms of the Lorentz space $L^{1,\infty}(\mathbb{T}, d\theta)$:

$$\|S_n f\|_{L^{1,\infty}(\mathbb{T},d\theta)} \leq C\|f\|_{L^1(\mathbb{T},d\theta)}, \qquad f \in L^1(\mathbb{T}, d\theta),\ n \geq 0.$$

Here,

$$\|f\|_{L^{p,\infty}(\sigma)} = \sup_{y>0} y\lambda(y)^{1/p} = \|t^{1/p} f^*(t)\|_{L^\infty(\mathbb{R}^+,dt)}, \qquad 1 \leq p < \infty$$

$$\|f\|_{L^{p,r}(\sigma)} = \left(\frac{r}{p}\int_0^\infty [t^{1/p} f^*(t)]^r \frac{dt}{t}\right)^{1/r} \qquad 1 \leq p < \infty,\ 1 \leq r < \infty,$$

where λ is the distribution function and f^* the nonincreasing rearrangement of f. There is a Hölder's inequality $\|f\|_{p_1,p_2} \leq C\|f\|_{q_1,q_2}\|f\|_{r_1,r_2}$, $1/p_i = 1/q_i + 1/r_i$. Also, $\|f\|_{p,\infty} \leq C\|f\|_{p,p} = C\|f\|_p \leq C_1\|f\|_{p,1}$. The reader is referred to [12] or [21] for further details on $L^{p,r}$ spaces.

After the trigonometric system, the convergence of Fourier series has been studied for a number of orthonormal systems, including Jacobi polynomials [17, 18, 14,

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4, 10], Hermite and Laguerre polynomials [15, 16], generalized Jacobi polynomials [1] and Bessel functions on $(0,1)$ [9].

In [23, 11] the authors characterized the L^p convergence for the Fourier-Neumann series, i.e., the Fourier expansion associated to the functions

$$\mathbf{j}_n^\alpha(x) = \sqrt{4n+2\alpha+2}\; J_{\alpha+2n+1}(x)x^{-\alpha-1}, \qquad n = 0, 1, \dots$$

which are orthonormal on $L^2((0,\infty), x^{2\alpha+1}\,dx)$ [$L^2(x^{2\alpha+1})$, from now on], see [24, § 13.41 (7), p. 404] and [24, § 13.42 (1), p. 405]. Here, $\alpha > -1$ and J_ν is the Bessel function of order ν.

For each suitable function f, let $S_n f$ be the n-th partial sum of its Fourier series with respect to the system $\{\mathbf{j}_n^\alpha\}_{n=0}^\infty$, i.e.,

$$S_n(f,x) = \int_0^\infty f(t)K_n(x,t)t^{2\alpha+1}\,dt, \qquad K_n(x,t) = \sum_{k=0}^n \mathbf{j}_k^\alpha(x)\mathbf{j}_k^\alpha(t).$$

In this paper, we study the weak and restricted weak behaviour of these series, i.e., the uniform boundedness

$$\|S_n f\|_{L^{p,\infty}(x^{2\alpha+1})} \le C\|f\|_{L^p(x^{2\alpha+1})}, \qquad f \in L^p(x^{2\alpha+1}),\ n \ge 0$$

or

$$\|S_n f\|_{L^{p,\infty}(x^{2\alpha+1})} \le C\|f\|_{L^{p,1}(x^{2\alpha+1})}, \qquad f \in L^{p,1}(x^{2\alpha+1}),\ n \ge 0.$$

Let us focus on the weak boundedness. The a priori assumption that $\mathbf{j}_n^\alpha \in L^q(x^{2\alpha+1})$ ($n = 0, 1, \dots,\ 1/p + 1/q = 1$) should be made so as to guarantee the existence of the Fourier coefficients for any $f \in L^p(x^{2\alpha+1})$. Also, we must assume that $\mathbf{j}_n^\alpha \in L^{p,\infty}(x^{2\alpha+1})$ if we want $S_n f$ to be in $L^{p,\infty}(x^{2\alpha+1})$. By Lemmas 1 and 2 below, these assumptions hold if and only if $p_1 \le p < p_2$, where $p_1 = 4(\alpha+1)/(2\alpha+3)$, $p_2 = 4(\alpha+1)/(2\alpha+1)$ if $\alpha \ge 0$, and $p_1 = 4/3$, $p_2 = 4$ if $-1 < \alpha < 0$. For the restricted weak boundedness, the same arguments lead to the a priori assumptions that $p_1 \le p \le p_2$. Since the L^p-L^p boundedness holds if and only if $p_1 < p < p_2$, and it implies the L^p-$L^{p,\infty}$ and $L^{p,1}$-$L^{p,\infty}$ boundedness, the real interest is in both endpoints. We obtain a negative answer for the weak type in the case $\alpha \ge 0$, and a positive answer for the restricted weak type.

Theorem. *Let $\alpha \ge 0$, $p_1 = 4(\alpha+1)/(2\alpha+3)$, $p_2 = 4(\alpha+1)/(2\alpha+1)$. Then the partial sum operators S_n, $n = 0, 1, \dots,$ are not uniformly bounded as operators from $L^{p_i}(x^{2\alpha+1})$ into $L^{p_i,\infty}(x^{2\alpha+1})$ but are uniformly bounded as operators from $L^{p_i,1}(x^{2\alpha+1})$ into $L^{p_i,\infty}(x^{2\alpha+1})$, $i = 1, 2$. In the case $-1 < \alpha < 0$ the second statement holds with $p_1 = 4/3$ and $p_2 = 4$.*

There is a close connection between S_n and the Hankel transform $\mathcal{H}_\alpha$ given by

$$\mathcal{H}_\alpha f(x) = \int_0^\infty \frac{J_\alpha(xy)}{(xy)^\alpha} f(y)y^{2\alpha+1}\,dy. \tag{1}$$

It turns out from [24, § 5.1 (8), p. 134] and [7, proof of Lemma 4] that $S_n f = \mathcal{H}_\alpha(\chi_{[0,1]}\mathcal{H}_\alpha f) - \mathcal{H}_{\alpha,n}(\chi_{[0,1]}\mathcal{H}_{\alpha,n} f) \equiv M_\alpha f - M_{\alpha,n} f$, where $\mathcal{H}_{\alpha,n}$ is given by (1) with $J_{\alpha+2n+2}$ in place of J_α. Now, Kenig and Thomas [13] proved that the multiplier M_α is not bounded from $L^p(x^{2\alpha+1})$ into $L^{p,\infty}(x^{2\alpha+1})$, $p = 4(\alpha+1)/(2\alpha+3)$. Chanillo [5] proved that M_α is bounded from $L^{p,1}(x^{2\alpha+1})$ into $L^{p,\infty}(x^{2\alpha+1})$, $p = 4(\alpha+1)/(2\alpha+3)$. Some related uniform estimates were obtained by Carbery, Romera and Soria [20, 3] in the context of the disc multiplier.

Throughout this paper, unless otherwise stated, we use C, C_1 to denote positive constants independent of n (and all other variables), which can assume different values in different occurrences. As usual, we write $f = O(g)$ in a given domain if $|f| \le Cg$. Finally, the standard notation $a_+ = \max\{a, 0\}$ will be used.

2. Auxiliary results

Some appropriate estimates for Bessel functions will be needed. For instance,

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} + O(x^{\nu+2}), \quad x \to 0+, \tag{2}$$

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1})\right], \qquad x \to \infty, \tag{3}$$

where the O terms depend on ν (see [24, § 3.1 (8), p. 40] and [24, § 7.21 (1), p. 199]). Some bounds for J_ν and J'_ν with constants independent of ν are also available. If $\nu > 0$, $0 < x \le \nu/2$, and $a \ge -\nu$ then there exists some constant C_a depending only on a, such that

$$|J_\nu(x)| x^a \le C_a \nu^{a-1/2} \left(\frac{e}{4}\right)^\nu \tag{4}$$

(see [24, § 3.31, p. 49]). The formula $2J'_\nu = J_{\nu-1} - J_{\nu+1}$ proves the same bound for $J'_\nu(x)$, as well as the analogs to (2) and (3):

$$J'_\nu(x) = \frac{x^{\nu-1}}{2^\nu \Gamma(\nu)} + O(x^{\nu+1}), \quad x \to 0+,$$

$$J'_\nu(x) = \sqrt{\frac{2}{\pi x}} \left[-\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1})\right], \qquad x \to \infty.$$

It is easy to deduce from (4) and bounds done by Barceló and Córdoba (see [2, p. 661], [8, p. 24]) that

$$|J_\nu(x)| \le C x^{-1/4} \left(|x-\nu| + \nu^{1/3}\right)^{-1/4}, \qquad x \in (0,\infty), \quad \nu > 1 \tag{5}$$

$$|J'_\nu(x)| \le C x^{-3/4} \left(|x-\nu| + \nu^{1/3}\right)^{1/4}, \qquad x \in (0,\infty), \quad \nu > 1 \tag{6}$$

with some constant C independent on ν. As a consequence, the following estimate for the norm of $x^b J_\nu(x)$ and $x^b J'_\nu(x)$ in $L^p(x^{2\alpha+1})$ and $L^{p,\infty}(x^{2\alpha+1})$ can be given.

Lemma 1. *Let $1 \le p < \infty$, $\alpha > -1$, $b \in \mathbb{R}$ and $\nu > 1$. Let $\lambda(4,\nu) = (\log\nu)^{1/4}$, $\lambda(p,\nu) = 1$ if $p \ne 4$. Then*

(a) *$x^b J_\nu(x) \in L^p(x^{2\alpha+1})$ if and only if $p(b+\nu) + 2\alpha + 2 > 0$ and $p(b - \frac{1}{2}) + 2\alpha + 2 < 0$. In that case, $\|x^b J_\nu(x)\|_{L^p(x^{2\alpha+1})} \le C\lambda(p,\nu)\nu^{2\frac{\alpha+1}{p} + b - \frac{1}{2} + \frac{2}{3}(\frac{1}{4} - \frac{1}{p})_+}$.*

(b) *$x^b J_\nu(x) \in L^{p,\infty}(x^{2\alpha+1})$ if and only if $p(b+\nu) + 2\alpha + 2 \ge 0$ and $p(b - \frac{1}{2}) + 2\alpha + 2 \le 0$. In that case, $\|x^b J_\nu(x)\|_{L^{p,\infty}(x^{2\alpha+1})} \le C\nu^{2\frac{\alpha+1}{p} + b - \frac{1}{2} + \frac{2}{3}(\frac{1}{4} - \frac{1}{p})_+}$.*

(c) *$x^b J'_\nu(x) \in L^p(x^{2\alpha+1})$ if and only if $p(b+\nu-1) + 2\alpha + 2 > 0$ and $p(b - \frac{1}{2}) + 2\alpha + 2 < 0$. In that case, $\|x^b J'_\nu(x)\|_{L^p(x^{2\alpha+1})} \le C\nu^{2\frac{\alpha+1}{p} + b - \frac{1}{2}}$.*

(d) *$x^b J'_\nu(x) \in L^{p,\infty}(x^{2\alpha+1})$ if and only if $p(b+\nu-1) + 2\alpha + 2 \ge 0$ and $p(b - \frac{1}{2}) + 2\alpha + 2 \le 0$. In that case, $\|x^b J'_\nu(x)\|_{L^{p,\infty}(x^{2\alpha+1})} \le C\nu^{2\frac{\alpha+1}{p} + b - \frac{1}{2}}$.*

Similar results can be found in [2, 22], so we will omit the proof (details are given in [6, Chapter 2]). Let us just mention that the L^p and $L^{p,\infty}$ conditions follow easily from (2), (3), and the analogs for $J'_\nu(x)$, while the norm estimates are a consequence of (4), (5), and the analogs for $J'_\nu(x)$.

Next lemma is the main step in the proof of the uniform restricted weak type:

Lemma 2. *Let $\nu > 1$, $1 < p < \infty$, and $L_\nu(f,x) = J_\nu(x^{1/2})H(t^{1/2}J'_\nu(t^{1/2})f(t),x)$, where H denotes the Hilbert transform. Then, there exists a constant C, independent of ν, such that*

(a) $\|L_\nu f\|_{L^p(dx)} \le C\|f\|_{L^p(dx)}$, *$f \in L^p(dx)$, if $p < 4$,*
(b) $\|L_\nu f\|_{L^{4,\infty}(dx)} \le C\|f\|_{L^{4,1}(dx)}$, *$f \in L^{4,1}(dx)$.*

Proof. (a) It follows from (5) that

$$\|L_\nu f\|_{L^p(dx)} \le C\|H(t^{1/2}J'_\nu(t^{1/2})f(t),x)\|_{L^p(x^{-p/8}(|x^{1/2}-\nu|+\nu^{1/3})^{-p/4})}.$$

Now, $x^{-p/8}(|x^{1/2}-\nu|+\nu^{1/3})^{-p/4} \in A_p$ uniformly in ν if $p<4$ (see [7], [11] or [23]). Thus, H is a bounded operator on $L^p(x^{-p/8}(|x^{1/2}-\nu|+\nu^{1/3})^{-p/4})$ and this, together with (6), proves (a).

(b) Let us write $L_\nu(f,x) = L_{\nu,1}(f,x) + L_{\nu,2}(f,x)$, where

$$L_{\nu,1}(f,x) = J_\nu(x^{1/2})(|x^{1/2}-\nu|+\nu^{1/3})^{1/4} H\left(\frac{t^{1/2}J'_\nu(t^{1/2})f(t)}{(|t^{1/2}-\nu|+\nu^{1/3})^{1/4}}, x\right),$$

$$L_{\nu,2}(f,x) = J_\nu(x^{\frac12}) H\left(\frac{t^{\frac12}J'_\nu(t^{\frac12})f(t)[(|t^{\frac12}-\nu|+\nu^{\frac13})^{\frac14} - (|x^{\frac12}-\nu|+\nu^{\frac13})^{\frac14}]}{(|t^{\frac12}-\nu|+\nu^{\frac13})^{\frac14}}, x\right).$$

The term $L_{\nu,1}(f,x)$ is easy to handle: we have

$$|J_\nu(x^{1/2})|(|x^{1/2}-\nu|+\nu^{1/3})^{1/4} \le Cx^{-1/8}$$

and $x^{-1/2} \in A_4(0,\infty)$, so that

$$\begin{aligned}\|L_{\nu,1}f\|_{L^{4,\infty}(dx)} &\le C\left\|H\left(\frac{t^{1/2}J'_\nu(t^{1/2})f(t)}{(|t^{1/2}-\nu|+\nu^{1/3})^{1/4}}, x\right)\right\|_{L^4(x^{-1/2})} \\ &\le C\left\|\frac{x^{1/2}J'_\nu(x^{1/2})f(x)}{(|x^{1/2}-\nu|+\nu^{1/3})^{1/4}}\right\|_{L^4(x^{-1/2})} \le C\|f\|_{L^{4,1}(dx)}.\end{aligned}$$

Let us consider now $L_{\nu,2}f$. The elementary inequality $|a^{1/4}-b^{1/4}| \le a^{-3/4}|a-b|$, which holds for every $a,b>0$, gives

$$\left|(|t^{\frac12}-\nu|+\nu^{\frac13})^{\frac14} - (|x^{\frac12}-\nu|+\nu^{\frac13})^{\frac14}\right| \le t^{-1/2}(|t^{\frac12}-\nu|+\nu^{\frac13})^{-\frac34}|t-x|.$$

Thus,

$$\begin{aligned}|L_{\nu,2}(f,x)| &\le |J_\nu(x^{1/2})| \int_0^\infty |J'_\nu(t^{1/2})|(|t^{\frac12}-\nu|+\nu^{\frac13})^{-1}|f(t)|\,dt \\ &\le |J_\nu(x^{1/2})|\,\|J'_\nu(t^{1/2})(|t^{\frac12}-\nu|+\nu^{\frac13})^{-1}\|_{L^{4/3,\infty}(dt)}\|f\|_{L^{4,1}(dt)}.\end{aligned}$$

Finally, Lemma 1(b) and a small change in Lemma 1(d) give $\|J_\nu(x^{\frac12})\|_{L^{4,\infty}(dx)} \le C$ and $\|J'_\nu(x^{\frac12})(|x^{\frac12}-\nu|+\nu^{\frac13})^{-1}\|_{L^{4/3,\infty}(dx)} \le C$. □

3. Weak boundedness

Using that, for $\alpha > -1$,

$$\sum_{k=0}^{n} 2(\alpha+2k+1)J_{\alpha+2k+1}(x)J_{\alpha+2k+1}(t) = \frac{xt}{x^2-t^2}\,[xJ_{\alpha+1}(x)J_\alpha(t)-tJ_\alpha(x)J_{\alpha+1}(t) + xJ'_{\alpha+2n+2}(x)J_{\alpha+2n+2}(t) - tJ_{\alpha+2n+2}(x)J'_{\alpha+2n+2}(t)]$$

(see [11, 23]), we have $S_n f = W_1 f - W_2 f + W_{3,n} f - W_{4,n} f$, with

$$W_1(f,x) = \tfrac{1}{2}x^{-\alpha+1}J_{\alpha+1}(x)\,H\big(t^{\alpha/2}J_\alpha(t^{1/2})f(t^{1/2}),x^2\big),$$
$$W_2(f,x) = \tfrac{1}{2}x^{-\alpha}J_{\alpha}(x)\,H\big(t^{\alpha/2+1/2}J_{\alpha+1}(t^{1/2})f(t^{1/2}),x^2\big),$$
$$W_{3,n}(f,x) = \tfrac{1}{2}x^{-\alpha+1}J'_{\nu}(x)\,H\big(t^{\alpha/2}J_\nu(t^{1/2})f(t^{1/2}),x^2\big),$$
$$W_{4,n}(f,x) = \tfrac{1}{2}x^{-\alpha}J_{\nu}(x)\,H\big(t^{\alpha/2+1/2}J'_{\nu}(t^{1/2})f(t^{1/2}),x^2\big),$$

and $\nu = \alpha+2n+2$. Here, H denotes the Hilbert transform on $(0,\infty)$. The following was proved in [11, Theorem 1]:

$$\|W_1 f\|_{L^p(x^{2\alpha+1})} \le C\|f\|_{L^p(x^{2\alpha+1})}, \quad \frac{2\alpha-1}{4(\alpha+1)} < \frac{1}{p} < \frac{2\alpha+3}{4(\alpha+1)}; \tag{7}$$

$$\|W_2 f\|_{L^p(x^{2\alpha+1})} \le C\|f\|_{L^p(x^{2\alpha+1})}, \quad \frac{2\alpha+1}{4(\alpha+1)} < \frac{1}{p} < \frac{2\alpha+5}{4(\alpha+1)}; \tag{8}$$

$$\|W_{3,n} f\|_{L^p(x^{2\alpha+1})} \le C\|f\|_{L^p(x^{2\alpha+1})}, \quad \frac{2\alpha-\frac{1}{2}}{4(\alpha+1)} < \frac{1}{p} < \min\left\{\frac{2\alpha+3}{4(\alpha+1)},\frac{3}{4}\right\}; \tag{9}$$

$$\|W_{4,n} f\|_{L^p(x^{2\alpha+1})} \le C\|f\|_{L^p(x^{2\alpha+1})}, \quad \max\left\{\frac{2\alpha+1}{4(\alpha+1)},\frac{1}{4}\right\} < \frac{1}{p} < \frac{2\alpha+\frac{9}{2}}{4(\alpha+1)}. \tag{10}$$

Now, let $\alpha \ge 0$. As mentioned in the introduction, the S_n are not bounded operators from $L^p(x^{2\alpha+1})$ into $L^{p,\infty}(x^{2\alpha+1})$ if $p = 4(\alpha+1)/(2\alpha+1)$, so we only need to prove here that the uniform weak boundedness fails for $p = 4(\alpha+1)/(2\alpha+3)$.

It follows from (8) and (10) that W_2 and $W_{4,n}$ are uniformly bounded from $L^p(x^{2\alpha+1})$ into itself, $p = 4(\alpha+1)/(2\alpha+3)$. Thus, it will be enough to find a sequence of functions $\{f_n\}$ such that the inequality

$$\|W_1 f_n + W_{3,n} f_n\|_{L^{p,\infty}(x^{2\alpha+1})} \le C\|f_n\|_{L^p(x^{2\alpha+1})} \tag{11}$$

fails for every constant C. Let $f_n(t) = \operatorname{sgn}(J_\alpha(t))t^{-\frac{2\alpha+3}{2}}\chi_{[1,n]}(t)$. Then,

$$\|f_n\|_{L^p(x^{2\alpha+1})} = C(\log n)^{\frac{1}{p}}, \qquad p = 4(\alpha+1)/(2\alpha+3).$$

Now, for $\nu = \alpha+2n+2$ and $x > 2\nu$ we have

$$|W_{3,n}(f_n,x)| \le Cx^{-\alpha-1}|J'_\nu(x)| \int_1^{n^2} t^{-\frac{3}{4}}|J_\nu(t^{\frac{1}{2}})|\,dt \le Cx^{-\alpha-\frac{3}{2}}\left(\frac{e}{4}\right)^{2n},$$

where the last step follows from (6) and (4). Thus,

$$\|\chi_{(2\nu,\infty)}W_{3,n}f_n\|_{L^{p,\infty}(x^{2\alpha+1})} \le C\left(\frac{e}{4}\right)^{2n}, \qquad p = 4(\alpha+1)/(2\alpha+3). \tag{12}$$

On the other hand, for $x > 2\nu$ we have

$$|W_1(f_n,x)| \ge Cx^{-\alpha-1}|J_{\alpha+1}(x)| \int_1^{n^2} t^{-\frac{3}{4}}|J_\alpha(t^{\frac{1}{2}})|\,dt \ge C(\log n)x^{-\alpha-1}|J_{\alpha+1}(x)|,$$

the last step following from (3). Therefore,

$$\|\chi_{(2\nu,\infty)}(x)W_1(f_n,x)\|_{L^{p,\infty}(x^{2\alpha+1})} \geq C\log n. \tag{13}$$

Putting (12) and (13) together, we get

$$\|W_1 f_n + W_{3,n} f_n\|_{L^{p,\infty}(x^{2\alpha+1})} \geq C\log n, \qquad p = 4(\alpha+1)/(2\alpha+3)$$

and (11) indeed fails. □

4. Restricted weak boundedness

By duality, we only need to prove that the self adjoint operators S_n are uniformly of restricted weak type in two cases: $\alpha \geq 0$, $p = \frac{4(\alpha+1)}{2\alpha+1}$, and $-1 < \alpha < 0$, $p = 4$.

Case $\alpha \geq 0$ and $p = \frac{4(\alpha+1)}{2\alpha+1}$. From (7) and (9), we conclude that W_1 and $W_{3,n}$ are uniformly bounded from $L^p(x^{2\alpha+1})$ into itself. Therefore, it is enough to prove that W_2 and $W_{4,n}$ are uniformly bounded from $L^{p,1}(x^{2\alpha+1})$ into $L^{p,\infty}(x^{2\alpha+1})$, i.e.,

$$\|W_{4,n} f\|_{L^{p,\infty}(x^{2\alpha+1})} \leq C\|f\|_{L^{p,1}(x^{2\alpha+1})}, \ f \in L^{p,1}(x^{2\alpha+1})$$

and the same inequality for W_2. We will consider only $W_{4,n}$, since the boundedness of W_2 is completely analogous. Let $f \in L^{p,1}(x^{2\alpha+1})$ and, for each $k \in \mathbb{Z}$,

$$I_k = [2^k, 2^{k+1}), \qquad f_1^k = f\chi_{(0,2^{k-1})\cup[2^{k+2},\infty)}, \qquad f_2^k = f\chi_{[2^{k-1},2^{k+2})}.$$

Thus, $f = f_1^k + f_2^k$ for each $k \in \mathbb{Z}$ and

$$|W_{4,n}(f,x)| \leq \sum_{k\in\mathbb{Z}} |W_{4,n}(f_1^k,x)|\chi_{I_k}(x) + \sum_{k\in\mathbb{Z}} |W_{4,n}(f_2^k,x)|\chi_{I_k}(x). \tag{14}$$

Let $x \in I_k$. Then, it is easy to check that $|x^2-y^2| \geq \frac{3}{4}y^2$ if $y \in (0,2^{k-1})\cup[2^{k+2},\infty)$. Hence, after a change of variable we get

$$|W_{4,n}(f_1^k,x)| \leq Cx^{-\alpha}|J_\nu(x)| \int_0^\infty y^\alpha |J'_\nu(y)||f(y)|\,dy$$

and

$$\sum_{k\in\mathbb{Z}} |W_{4,n}(f_1^k,x)|\chi_{I_k}(x) \leq Cx^{-\alpha}|J_\nu(x)|\, \|x^{-\alpha-1}J'_\nu(x)\|_{L^{q,\infty}(x^{2\alpha+1})}\|f\|_{L^{p,1}(x^{2\alpha+1})},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, the first term in (14) is bounded:

$$\|\sum_{k\in\mathbb{Z}} |W_{4,n}(f_1^k,x)|\chi_{I_k}(x)\|_{L^{p,\infty}(x^{2\alpha+1})} \leq C\|f\|_{L^{p,1}(x^{2\alpha+1})},$$

by Lemma 1(b) and (d). Let us consider now the second term. If $x \in I_k$,

$$|W_{4,n}(f_2^k,x)| \leq C2^{-\alpha k}\left|L_\nu(t^{\frac{\alpha}{2}} f_k^2(t^{1/2}), x^2)\right|.$$

From Lemma 2 it follows that

$$\begin{aligned}\|W_{4,n}(f_2^k,x)\chi_{I_k}(x)\|_{L^{p,\infty}(x^{2\alpha+1})} &\leq C2^{-\alpha k+\frac{2\alpha k}{p}}\|\chi_{I_k}(x)L_\nu(t^{\frac{\alpha}{2}} f_k^2(t^{1/2}),x^2)\|_{L^{p,\infty}(x\,dx)}\\ &\leq C2^{-\alpha k+\frac{2\alpha k}{p}}\|x^{\frac{\alpha}{2}} f_k^2(x^{1/2})\|_{L^{p,1}(dx)} \leq C\|f\chi_{[2^{k-2},2^{k+2})}\|_{L^{p,1}(x^{2\alpha+1})}.\end{aligned}$$

Then,

$$\|\sum_{k\in\mathbb{Z}} |W_{4,n}(f_2^k,x)|\chi_{I_k}(x)\|_{L^{p,\infty}(x^{2\alpha+1})} \leq C\|f\|_{L^{p,1}(x^{2\alpha+1})}. \tag{15}$$

Case $-1 < \alpha < 0$ *and* $p = 4$. Now, W_1, W_2, and $W_{3,n}$ are uniformly bounded from $L^4(x^{2\alpha+1})$ into itself (see (7), (8), (9)), so we only need to prove that

$$\|W_{4,n}f\|_{L^{4,\infty}(x^{2\alpha+1})} \le C\|f\|_{L^{4,1}(x^{2\alpha+1})}, \ f \in L^{4,1}(x^{2\alpha+1}).$$

The above proof of (15) remains valid, while only minor changes are necessary for the first term in (14): it is not difficult to check that

$$\frac{y^{2+\frac{\alpha}{2}}x^{-\frac{\alpha}{2}}}{|x^2-y^2|} \le \frac{4}{3}$$

if $x \in I_k$ and $y \in (0, 2^{k-1}) \cup [2^{k+2}, \infty)$. Then, it follows that

$$|W_{4,n}(f_1^k, x)| \le Cx^{-\frac{\alpha}{2}}|J_\nu(x)| \int_0^\infty y^{\frac{\alpha}{2}}|J_\nu'(y)|\,|f(y)|\,dy. \quad \square$$

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