

# AN UNIFORM BOUNDEDNESS FOR BOCHNER-RIESZ OPERATORS RELATED TO THE HANKEL TRANSFORM

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ABSTRACT. Let  $\mathcal{H}_\alpha$  be the modified Hankel transform

$$\mathcal{H}_\alpha(f, x) = \int_0^\infty \frac{J_\alpha(xt)}{(xt)^\alpha} f(t)t^{2\alpha+1} dt,$$

defined for suitable functions and extended to some  $L^p((0, \infty), x^{2\alpha+1})$  spaces. Given  $\delta > 0$ , let  $M_\alpha^\delta$  be the Bochner-Riesz operator for the Hankel transform. Also, we take the following generalization

$$\mathcal{H}_\alpha^k(f, x) = \int_0^\infty \frac{J_{\alpha+k}(xt)}{(xt)^\alpha} f(t)t^{2\alpha+1} dt, \quad k = 0, 1, 2, \dots$$

for the Hankel transform, and define  $M_{\alpha,k}^\delta$  as

$$M_{\alpha,k}^\delta f = \mathcal{H}_\alpha^k \left( (1-x^2)_+^\delta \mathcal{H}_\alpha f \right), \quad k = 0, 1, 2, \dots$$

(thus, in particular,  $M_\alpha^\delta = M_{\alpha,0}^\delta$ ). In the paper, we study the uniform boundedness of  $\{M_{\alpha,k}^\delta\}_{k \in \mathbb{N}}$  in  $L^p((0, \infty), x^{2\alpha+1})$  spaces when  $\alpha \geq 0$ . We found that, for  $\delta > (2\alpha+1)/2$  (the critical index), the uniform boundedness of  $\{M_{\alpha,k}^\delta\}_{k=0}^\infty$  is satisfied for every  $p$  in the range  $1 \leq p \leq \infty$ . And, for  $0 < \delta \leq (2\alpha+1)/2$ , the uniform boundedness happens if and only if

$$\frac{4(\alpha+1)}{2\alpha+3+2\delta} < p < \frac{4(\alpha+1)}{2\alpha+1-2\delta}.$$

In the paper, the case  $\delta = 0$  (the corresponding generalization of the  $\chi_{[0,1]}$ -multiplier for the Hankel transform) is previously analyzed; here, for  $\alpha > -1$ . For this value of  $\delta$ , the uniform boundedness of  $\{M_{\alpha,k}^0\}_{k=0}^\infty$  is related to the convergence of Fourier-Neumann series.

## 1. INTRODUCTION

Let  $\alpha \geq -1/2$ . For a function  $f(t)$  on the interval  $(0, \infty)$ , the so-called modified Hankel transform  $\mathcal{H}_\alpha(f, x)$ ,  $x > 0$ , of order  $\alpha$  is given by

$$(1) \quad \mathcal{H}_\alpha(f, x) = \int_0^\infty \frac{J_\alpha(xt)}{(xt)^\alpha} f(t)t^{2\alpha+1} dt,$$

where  $J_\alpha(x)$  is the Bessel function of the first kind of order  $\alpha$ . Well-known bounds for Bessel function and Hölder's inequality show that (1) is well defined for every  $f \in L^p((0, \infty), x^{2\alpha+1} dx)$  ( $L^p(x^{2\alpha+1})$ , from now on) with  $1 \leq p < \frac{4(\alpha+1)}{(2\alpha+3)}$ .

Furthermore, it is easy to see that  $\mathcal{H}_\alpha$  is a bounded operator from  $L^1(x^{2\alpha+1})$  into  $L^\infty(x^{2\alpha+1})$ . Also, as usual, the expression (1) is extended by continuity to different

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$L^p(x^{2\alpha+1})$  spaces. For instance, it is well known that  $\mathcal{H}_\alpha$  is an isomorphism from  $L^2(x^{2\alpha+1})$  into itself and  $\mathcal{H}_\alpha \circ \mathcal{H}_\alpha = \text{Id}$  (for  $\alpha > -1$ , the extension of  $\mathcal{H}_\alpha$  to  $L^2(x^{2\alpha+1})$  can also be done; see [7, 8, 23, 2]). As a consequence, of course, we can get the corresponding interpolation result that we do not detail here.

Associated with  $\mathcal{H}_\alpha$ , we can define the ball multiplier  $M_\alpha$  as the operator that verifies the relation  $\mathcal{H}_\alpha(M_\alpha f) = \chi_{[0,1]} \mathcal{H}_\alpha f$ ; or, in other words,

$$M_\alpha f = \mathcal{H}_\alpha(\chi_{[0,1]} \mathcal{H}_\alpha f).$$

This operator is bounded from  $L^p(x^{2\alpha+1})$  into itself if and only if  $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$ . The first proof of this fact can be found in [11]. For this and related properties on Hankel multipliers see also [14, 9, 23, 20, 10, 4, 5].

Now, let us take the following generalization for the Hankel transform. We consider

$$(2) \quad \mathcal{H}_\alpha^k(f, x) = \int_0^\infty \frac{J_{\alpha+k}(xt)}{(xt)^\alpha} f(t) t^{2\alpha+1} dt, \quad k = 0, 1, 2, \dots$$

In this way, the operator  $M_\alpha$  can be also generalized by taking

$$(3) \quad M_{\alpha,k} f = \mathcal{H}_\alpha^k(\chi_{[0,1]} \mathcal{H}_\alpha^k f), \quad k = 0, 1, 2, \dots$$

The study of the uniform boundedness of these operators in  $L^p(x^{2\alpha+1})$  spaces is very useful. As we can see in [23, 10, 4], to prove the mean convergence of Fourier-Neumann series is reduced to prove the uniform boundedness of the operators  $\{M_{\alpha,k} f\}_{k \in \mathbb{N}}$ . There, it is proved that the uniform boundedness in  $L^p(x^{2\alpha+1})$  is equivalent to  $\max\{\frac{4}{3}, \frac{4(\alpha+1)}{2\alpha+3}\} < p < \min\{4, \frac{4(\alpha+1)}{2\alpha+1}\}$ . Actually, in this article, we will explicitly show the uniform boundedness of  $\{M_{\alpha,k} f\}_{k \in \mathbb{N}}$  only because it plays an important role in the proof of some other results of the paper. Moreover, the boundedness for  $M_{\alpha,k}$  is shown not only for  $\alpha \geq -1/2$ , but also for  $\alpha > -1$ .

Another multiplier for the Hankel transform is the Bochner-Riesz multiplier; of course, it is similar to the well known Bochner-Riesz multiplier for the Fourier transform (see, for instance, [25, 16] or [18, Ch. IX, § 2.2]). Taking  $\delta > 0$ , the operator  $M_\alpha^\delta$  is the one that makes  $\mathcal{H}_\alpha(M_\alpha^\delta f) = (1-x^2)_+^\delta \mathcal{H}_\alpha f$ , being  $(1-x^2)_+ = \max\{0, 1-x^2\}$  (of course,  $M_\alpha^\delta$  for  $\delta = 0$  would be  $M_\alpha$ ). Also, it can be described as

$$(4) \quad M_\alpha^\delta f = \mathcal{H}_\alpha((1-x^2)_+^\delta \mathcal{H}_\alpha f).$$

Again, Bochner-Riesz multiplier can be generalized by using (2). Thus, we take

$$(5) \quad M_{\alpha,k}^\delta f = \mathcal{H}_\alpha^k((1-x^2)_+^\delta \mathcal{H}_\alpha^k f), \quad k = 0, 1, 2, \dots$$

Similarly to the ball multiplier generalization, we can wonder if these operators are uniformly bounded in  $L^p(x^{2\alpha+1})$ . This paper is devoted to the study of this fact. We will only deal with the case  $\alpha \geq 0$ .

We found that there exists an index  $\delta_0 = (2\alpha + 1)/2$  such that, for  $\delta > \delta_0$ , the uniform boundedness is true for every  $p$  in the range  $1 \leq p \leq \infty$ ; however, for  $0 < \delta \leq \delta_0$ , the uniform boundedness only happens in a finite range of  $p$ . Moreover, as we will see, the proofs for both facts are different. This value  $\delta_0$  is called the critical index. The existence of a critical index is also a common fact in the study of the boundedness of the Bochner-Riesz operator for the Fourier transform (see, for instance, [19])

The paper is organized as follows. In Section 2, we give the main results of the paper. First, the uniform boundedness for  $\{M_{\alpha,k}\}_{k \in \mathbb{N}}$  (Theorem 1); then, the uniform boundedness for  $\{M_{\alpha,k}^\delta\}_{k \in \mathbb{N}}$ , both for  $\delta > \delta_0$  (Theorem 2) and for  $0 < \delta \leq \delta_0$  (Theorem 3). In Section 3, we give the proof of Theorem 1, which is reduced to known results that already appear in papers related with the uniform boundedness of Fourier-Neumann series. In Section 4 we give the proof of Theorem 2. To obtain it, we will use some results about the translation for the Hankel transform and the corresponding convolution product. Finally, in Section 5, we give the proof of Theorem 3. In this case, we will apply an interpolation result for families of analytic operators from Stein [17], and the uniform boundedness of  $\{M_{\alpha,k}\}_{k \in \mathbb{N}}$  that appears in Theorem 1.

Throughout this paper, we will use  $C$  (or  $C'$ ) to denote a positive constant independent of  $f$  and  $k$  (and of any other variable, if it is the case), which can assume different values in different occurrences. When  $C$  has a subindex, it depends only on the parameters that appear in the subindex.

Some of the operators defined in this paper have an integral expression only for suitable functions, and then they are extended by density and continuity. Usually, a class  $S^+$  of smooth functions in  $(0, \infty)$ , related with Schwartz class  $S$ , is used ( $S^+$  are the even functions of  $S$  restricted to  $(0, \infty)$ ); see, for instance, [8, 6] for details (and [9, 20] for related density results). Moreover, with these functions, Fubini's Theorem can be applied when necessary. This is a standard technique; we will implicitly use this kind of arguments sometimes without notice it again.

*Remark 1.* Note that, although we have referred to some results that appeared in [23, 10, 4, 5], the Hankel transform used in these papers (as well as by other authors such as [7, 8]) was somewhat different. In these papers, instead of (1), it was used

$$(6) \quad \mathcal{H}_\alpha(f, x) = \frac{x^{-\alpha/2}}{2} \int_0^\infty J_\alpha(\sqrt{xt}) g(t) t^\alpha dt,$$

defined in  $L^p(x^\alpha)$ . And the ball multiplier  $M_\alpha$  (and  $M_{\alpha,k}$ ) was also defined for this Hankel transform. It is clear that, by the changes of variables  $t \mapsto t^2$  and  $x \mapsto x^2$ , the Hankel transform (6) in  $L^p(x^\alpha)$  becomes (1) in  $L^p(x^{2\alpha+1})$ . Of course, the range of  $p$  for which there exists boundedness is preserved.

Nowadays, it seems that the notation in (1) is more used (see, for instance [13, 14, 9, 20, 15, 6, 1, 2], although it is sometimes called Fourier-Bessel transform), so we have adopted it in this paper.

When studying the boundedness of  $M_\alpha$  and  $M_{\alpha,k}$ , the notation in (6) was more handy than (1): the operators that appear are, directly, Hilbert transforms with weights (with (1), in the denominator, an  $x^2 - t^2$  arises instead of a  $x - t$ , and this requires extra work). However, in this paper, is more suitable to use (1) and its corresponding  $M_\alpha^\delta$  and  $M_{\alpha,k}^\delta$ .

## 2. MAIN RESULTS

Let us begin to show an expression for the  $\chi_{[0,1]}$ -multiplier generalization  $M_{\alpha,k} f$  described in (3); or, in other words,  $M_{\alpha,k}^\delta f$  for  $\delta = 0$ . We will study  $\{M_{\alpha,k}\}_{k \in \mathbb{N}}$  not only for  $\alpha \geq -1/2$ , but also for  $\alpha > -1$ . By using (3) and Fubini's Theorem

we obtain

$$(7) \quad M_{\alpha,k}(f, x) = x^{-\alpha} \int_0^\infty \left( \int_0^1 J_{\alpha+k}(ts) J_{\alpha+k}(xs) s ds \right) t^{\alpha+1} f(t) dt.$$

The result that shows the uniform boundedness of this family of operators is the following:

**Theorem 1.** *Let  $\alpha > -1$ ,  $1 < p < \infty$ , and the family of operators  $\{M_{\alpha,k}\}_{k \in \mathbb{N}}$  defined as in (3). Then,*

$$\|M_{\alpha,k} f\|_{L^p(x^{2\alpha+1})} \leq C \|f\|_{L^p(x^{2\alpha+1})}, \quad k \in \mathbb{N},$$

if and only if

$$\begin{cases} 4/3 < p < 4, & \text{if } -1 < \alpha < 0, \\ \frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}, & \text{if } 0 \leq \alpha. \end{cases}$$

Now, let us consider the generalized Bochner-Riesz multiplier  $M_{\alpha,k}^\delta$  described in (5). Let us take  $\delta > 0$ . An expression that allows us to show  $M_{\alpha,k}^\delta$  as an operator with a kernel  $K_{\alpha+k}^\delta$  is the following:

$$M_{\alpha,k}^\delta(f, x) = x^{-\alpha} \int_0^\infty f(t) K_{\alpha+k}^\delta(x, t) t^{\alpha+1} dt$$

with

$$(8) \quad K_{\alpha+k}^\delta(x, t) = \int_0^1 s(1-s^2)^\delta J_{\alpha+k}(xs) J_{\alpha+k}(ts) ds.$$

Again, it can be easily deduced by applying Fubini's Theorem to the definition for  $M_{\alpha,k}^\delta$  given in (5).

The main results in the paper are the ones that show uniform boundedness for  $\{M_{\alpha,k}^\delta\}_{k \in \mathbb{N}}$ . When  $\delta$  exceeds the critical index, we have

**Theorem 2.** *Let  $\alpha \geq 0$ ,  $\delta > \delta_0 = (2\alpha + 1)/2$  and  $1 \leq p \leq \infty$ , and the family of operators  $\{M_{\alpha,k}^\delta\}_{k \in \mathbb{N}}$  defined as in (5). Then,*

$$\|M_{\alpha,k}^\delta f\|_{L^p(x^{2\alpha+1})} \leq C \|f\|_{L^p(x^{2\alpha+1})}, \quad k \in \mathbb{N}.$$

And, for  $\delta$  below the critical index, we have

**Theorem 3.** *Let  $\alpha \geq 0$ ,  $0 < \delta \leq \delta_0 = (2\alpha + 1)/2$  and  $1 \leq p \leq \infty$ , and the family of operators  $\{M_{\alpha,k}^\delta\}_{k \in \mathbb{N}}$  defined as in (5). Then,*

$$\|M_{\alpha,k}^\delta f\|_{L^p(x^{2\alpha+1})} \leq C \|f\|_{L^p(x^{2\alpha+1})}, \quad k \in \mathbb{N},$$

if and only if

$$(9) \quad \frac{4(\alpha+1)}{2\alpha+3+2\delta} < p < \frac{4(\alpha+1)}{2\alpha+1-2\delta}.$$

3. PROOF OF THEOREM 1 (I.E., UNIFORM BOUNDEDNESS FOR  $\delta = 0$ )

First, we are going to establish some new expressions for (7) that will be more useful to study the uniform boundedness of the operators.

We will use von Lommel's formula

$$\begin{aligned} \int_0^1 J_\nu(ts)J_\nu(xs)s ds &= \frac{1}{x^2-t^2} (J_\nu(x)tJ'_\nu(t) - J_\nu(t)xJ'_\nu(x)) \\ &= \frac{1}{x^2-t^2} (xJ_\nu(t)J_{\nu+1}(x) - tJ_{\nu+1}(t)J_\nu(x)) \end{aligned}$$

(for the last equality, use  $zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z)$ ). By applying it to (7) we get

$$\begin{aligned} (10) \quad M_{\alpha,k}(f,x) &= x^{-\alpha+1}J_{\alpha+k+1}(x) \int_0^\infty \frac{t^{-\alpha}J_{\alpha+k}(t)}{x^2-t^2} f(t)t^{2\alpha+1} dt \\ &\quad - x^{-\alpha}J_{\alpha+k}(x) \int_0^\infty \frac{t^{-\alpha+1}J_{\alpha+k+1}(t)}{x^2-t^2} f(t)t^{2\alpha+1} dt \\ &= W_{1,k}(f,x) - W_{2,k}(f,x) \end{aligned}$$

or, also,

$$\begin{aligned} (11) \quad M_{\alpha,k}(f,x) &= x^{-\alpha}J_{\alpha+k}(x) \int_0^\infty \frac{t^{-\alpha+1}J'_{\alpha+k}(t)}{x^2-t^2} f(t)t^{2\alpha+1} dt \\ &\quad - x^{-\alpha+1}J'_{\alpha+k}(x) \int_0^\infty \frac{t^{-\alpha}J_{\alpha+k}(t)}{x^2-t^2} f(t)t^{2\alpha+1} dt \\ &= \widetilde{W}_{1,k}(f,x) - \widetilde{W}_{2,k}(f,x). \end{aligned}$$

Now, we have all the necessary for

*Proof of Theorem 1.* These are the operators that appear in the decomposition of the partial sums of Fourier-Neumann series to prove their uniform boundedness in  $L^p(x^{2\alpha+1})$  spaces, such as it is studied in [23, 10, 4]. Actually, with this notation, the partial sums are  $S_n f = W_{1,0}f - W_{2,0}f + \widetilde{W}_{2,2n+2}f - \widetilde{W}_{1,2n+2}f$  (take into account that the notation in these papers is a little different and, also, that a change of variable for  $x$  and  $t$  is being used, as described in Remark 1).

In this way, a sketch of the proof for the uniform boundedness of  $M_{\alpha,k}$  is as follows:

For  $k = 0$ , the decomposition (10) is used. Then, well-known bounds for  $|J_\alpha|$  and  $|J_{\alpha+1}|$  are applied, and so the proof of the boundedness of  $W_{2,0}$  and  $W_{1,0}$  is reduced to the boundedness of the Hilbert transform with weights. We get that this boundedness is true for  $1 < p < \infty$  when  $-1 < \alpha < -1/2$ ; and for  $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$  when  $\alpha \geq -1/2$ . (The case  $\alpha < -1/2$  is studied only in [10].)

For  $k > 0$ , the decomposition (11) is used. Now, suitable bounds for  $|J_{\alpha+k}|$  and  $|J'_{\alpha+k}|$  are applied. Then,  $\widetilde{W}_{1,k}$  and  $\widetilde{W}_{2,k}$  are bounded by Hilbert transforms with weights (that depend of  $k$ ). Finally, uniform  $A_p$  weights theory is used to find the uniform boundedness of these Hilbert transforms. Here, the condition  $4/3 < p < 4$  appears.  $\square$

*Remark 2.* As we have commented in the previous proof, the condition  $4/3 < p < 4$  does not arise if we only analyze the case  $k = 0$ . Then, for the boundedness of the

$\chi_{[0,1]}$ -multiplier  $M_\alpha$  in  $L^p(x^{2\alpha+1})$ ,  $1 < p < \infty$  and  $\alpha > -1$ , we have

$$\|M_\alpha f\|_{L^p(x^{2\alpha+1})} \leq C \|f\|_{L^p(x^{2\alpha+1})} \iff \begin{cases} 1 < p < \infty, & \text{if } -1 < \alpha < \frac{-1}{2}, \\ \frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}, & \text{if } \frac{-1}{2} \leq \alpha. \end{cases}$$

This result is already implicit in [10], but not explicitly stated. And, changing  $L^p(x^{2\alpha+1})$  by some other weighted  $L^p$  spaces, also [22] can be seen.

A completely different proof of this fact, based on multipliers for Fourier-Bessel expansions and transplantation theorems, can be found in [2].

#### 4. UNIFORM BOUNDEDNESS FOR $\delta$ OVER THE CRITICAL INDEX

First, let us describe the translation and convolution for the Hankel transform as well as some of its properties. A wider exposition of these results can be found in [13, 12, 3]. Mostly in this section, we can assume  $\alpha > -1/2$ , although the proof of Theorem 2 will require  $\alpha \geq 0$  (because (18) is not true for  $\alpha < 0$ ).

We consider the translation operator  $T^x$ , with  $x \geq 0$ , defined, for suitable functions  $h$ , by

$$T^x(h, t) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_0^\pi h(w) \sin^{2\alpha} \theta \, d\theta,$$

where  $w^2 = x^2 + t^2 - 2xt \cos \theta$ ,  $x, t \geq 0$ . Using this translation, the convolution operator can be described as

$$(12) \quad h * g(x) = \int_0^\infty g(t) T^x(h, t) t^{2\alpha+1} \, dt.$$

It is not difficult to check that, for  $\alpha > -1/2$ ,

$$(13) \quad \|h * g\|_{L^p(x^{2\alpha+1})} \leq \|h\|_{L^1(x^{2\alpha+1})} \|g\|_{L^p(x^{2\alpha+1})}$$

for  $1 \leq p \leq \infty$ . The convolution structure and its boundedness will play a significant role in the proof of Theorem 2.

Let us introduce some other notation. Let  $\{P_k^{(\alpha)}\}_{k=0}^\infty$  be the ultraspherical polynomials of order  $\alpha$  (also known as Gegenbauer polynomials). Here, and in the discussion that will follow, the usual modifications must be applied if  $\alpha = 0$  (i.e., the use of Chebyshev polynomials). A wide information on ultraspherical polynomials can be found in [21]. To simplify some expressions, we take

$$(14) \quad p_k^\alpha(x) = \frac{2^{\alpha-1} k! \Gamma(\alpha)}{\pi \Gamma(2\alpha + k)} P_k^{(\alpha)}(x).$$

For Bessel functions, we will denote

$$(15) \quad \mathcal{J}_\alpha^\delta(x) = 2^\delta \Gamma(\alpha + 1) \frac{J_{\alpha+\delta+1}(x)}{x^{\alpha+\delta+1}}.$$

Also, we will use the Sonine's formula (see [24, § 12.11, p. 373] or [19, Lemma 4.13])

$$(16) \quad J_{\nu+\mu+1}(z) = \frac{z^{\nu+1}}{2^\nu \Gamma(\nu + 1)} \int_0^1 J_\mu(zs) s^{\mu+1} (1 - s^2)^\nu \, ds,$$

which is valid for  $\operatorname{Re}(\mu) > -1$  and  $\operatorname{Re}(\nu) > -1$ .

Now, let us see a new expression for the Bochner-Riesz operator  $M_{\alpha,k}^\delta$  and its kernel  $K_{\alpha+k}^\delta$ . It is more useful for our purposes than the previous one we showed in (8).

**Lemma 1.** *Let  $\alpha > -1/2$  and  $\delta > 0$ . Then,*

$$M_{\alpha,k}^{\delta}(f, x) = x^{-\alpha} \int_0^{\infty} f(t) K_{\alpha+k}^{\delta}(x, t) t^{\alpha+1} dt$$

with

$$(17) \quad K_{\alpha+k}^{\delta}(x, t) = (xt)^{\alpha} \int_0^{\pi} \mathcal{J}_{\alpha}^{\delta}(w) p_k^{\alpha}(\cos \theta) \sin^{2\alpha} \theta d\theta,$$

being  $w^2 = x^2 + t^2 - 2xt \cos \theta$ .

*Proof.* We have already seen that  $M_{\alpha,k}^{\delta}(f, x)$  can be written with  $K_{\alpha+k}^{\delta}(x, t)$  as in (8). Now, let us transform this expression to become (17). For this, we will use

$$\int_0^{\pi} \frac{J_{\alpha}(w)}{w^{\alpha}} p_k^{\alpha}(\cos \theta) \sin^{2\alpha} \theta d\theta = \frac{J_{\alpha+k}(x)}{x^{\alpha}} \frac{J_{\alpha+k}(t)}{t^{\alpha}},$$

with  $p_k^{\alpha}$  as in (14); this formula, that is valid for  $\alpha > -1/2$ , can be found in [24, § 11.41, p. 367]. In this way,

$$\begin{aligned} K_{\alpha+k}^{\delta}(x, t) &= \int_0^1 s(1-s^2)^{\delta} J_{\alpha+k}(xs) J_{\alpha+k}(ts) ds \\ &= (xt)^{\alpha} \int_0^1 s^{2\alpha+1} (1-s^2)^{\delta} \left( \int_0^{\pi} \frac{J_{\alpha}(sw)}{(sw)^{\alpha}} p_k^{\alpha}(\cos \theta) \sin^{2\alpha} \theta d\theta \right) ds. \end{aligned}$$

Exchanging the order of integration and applying Sonine's formula (16), we can conclude

$$\begin{aligned} K_{\alpha+k}^{\delta}(x, t) &= (xt)^{\alpha} \int_0^{\pi} w^{-\alpha} p_k^{\alpha}(\cos \theta) \sin^{2\alpha} \theta \left( \int_0^1 s^{\alpha+1} (1-s^2)^{\delta} J_{\alpha}(sw) ds \right) d\theta \\ &= (xt)^{\alpha} \int_0^{\pi} \mathcal{J}_{\alpha}^{\delta}(w) p_k^{\alpha}(\cos \theta) \sin^{2\alpha} \theta d\theta. \end{aligned}$$

□

By using standard estimates for ultraspherical polynomials (see, for instance [21, Th. 7.33.1]) we have that, for  $\alpha \geq 0$ ,

$$(18) \quad |p_k^{\alpha}(x)| \leq 1, \quad x \in [-1, 1].$$

Also, the well-known estimates for Bessel functions (of order  $\nu > -1$ )

$$|J_{\nu}(t)| \leq C_{\nu} t^{\nu}, \quad t \in (0, 1), \quad |J_{\nu}(t)| \leq C_{\nu} t^{-1/2}, \quad t \in (1, \infty),$$

ensure that

$$(19) \quad \|\mathcal{J}_{\alpha}^{\delta}(x)\|_{L^1(x^{2\alpha+1})} = C_{\alpha,\delta} < \infty$$

for  $\delta > \delta_0 = (2\alpha + 1)/2$ .

Now, we have all that we will use for

*Proof of Theorem 2.* By Lemma 1, we can write

$$(20) \quad \|M_{\alpha,k}^{\delta} f\|_{L^p(x^{2\alpha+1})} = \left\| x^{-\alpha} \int_0^{\infty} f(t) K_{\alpha+k}^{\delta}(x, t) t^{\alpha+1} dt \right\|_{L^p(x^{2\alpha+1})}.$$

Moreover, by (17) and (18), we get

$$|K_{\alpha+k}^{\delta}(x, t)| \leq C (xt)^{\alpha} \int_0^{\pi} |\mathcal{J}_{\alpha}^{\delta}(w)| \sin^{2\alpha} \theta d\theta.$$

Then, by using this and (12), we have

$$\begin{aligned}
& \left| x^{-\alpha} \int_0^\infty f(t) K_{\alpha+k}^\delta(x, t) t^{\alpha+1} dt \right| \\
& \leq C \int_0^\infty |f(t)| \left( \int_0^\pi |\mathcal{J}_\alpha^\delta(w)| \sin^{2\alpha} \theta d\theta \right) t^{2\alpha+1} dt \\
& = C' \int_0^\infty |f(t)| T^x(|\mathcal{J}_\alpha^\delta|, t) t^{2\alpha+1} dt \\
& \leq C' (|\mathcal{J}_\alpha^\delta| * |f|)(x).
\end{aligned}$$

Finally, by applying (20), (13) and (19), we conclude

$$\begin{aligned}
\|M_{\alpha,k}^\delta f\|_{L^p(x^{2\alpha+1})} & \leq C \| |\mathcal{J}_\alpha^\delta| * |f| \|_{L^p(x^{2\alpha+1})} \\
& \leq C \|\mathcal{J}_\alpha^\delta\|_{L^1(x^{2\alpha+1})} \|f\|_{L^p(x^{2\alpha+1})} \\
& \leq C' \|f\|_{L^p(x^{2\alpha+1})}.
\end{aligned}$$

□

## 5. UNIFORM BOUNDEDNESS FOR $\delta$ BELOW THE CRITICAL INDEX

The proof of Theorem 3 needs some prior machinery. In particular, we will use Stein's theorem on interpolation of analytic families of operators; it can be seen in [17]. Here, we will show this result adapted to our spaces  $L^p(x^{2\alpha+1})$ .

First, let us consider the notion of analytic family of operators. A family of operators  $\{T_z\}$  depending on a complex parameter  $z$  that runs in  $0 \leq \operatorname{Re}(z) \leq 1$  is called analytic if:

- (a) For each  $z$ ,  $T_z$  is a linear transformation of simple functions on  $(0, \infty)$  into measurable functions on  $(0, \infty)$ .
- (b) If  $\phi$  and  $\psi$  are simple functions on  $(0, \infty)$ , then

$$\Phi(z) = \int_0^\infty T_z(\psi, x) \phi(x) dx$$

is analytic in  $0 < \operatorname{Re}(z) < 1$  and continuous in  $0 \leq \operatorname{Re}(z) \leq 1$ .

We say that an analytic family  $\{T_z\}$  is of admissible growth if  $\Phi(z)$  is of admissible growth; that is, if

$$\sup_{|y| \leq s} \sup_{0 \leq x \leq 1} \log |\Phi(x + iy)| \leq Ae^{as},$$

where  $a < \pi$  and  $A$  is a constant. Both  $A$  and  $a$  may depend on the functions  $\phi$  and  $\psi$ .

The interpolation result is therefore:

**Theorem 4.** *Let  $\{T_z\}$  be an analytic family of linear operators of admissible growth, defined in the strip  $0 \leq \operatorname{Re}(z) \leq 1$ . Suppose that  $1 \leq p_0, p_1, q_0, q_1 \leq \infty$  and*

$$\frac{1}{p} = \frac{1-r}{p_0} + \frac{r}{p_1}, \quad \frac{1}{q} = \frac{1-r}{q_0} + \frac{r}{q_1}$$

for  $0 \leq r \leq 1$ . We also assume

$$(21) \quad \|T_{iy} f\|_{L^{q_0}(x^{2\alpha+1})} \leq A_0(y) \|f\|_{L^{p_0}(x^{2\alpha+1})}$$

and

$$(22) \quad \|T_{1+iy}f\|_{L^{q_1}(x^{2\alpha+1})} \leq A_1(y)\|f\|_{L^{p_1}(x^{2\alpha+1})}$$

for any simple function  $f$ . Finally, suppose

$$\log |A_i(y)| \leq Ae^{a|y|}, \quad a < \pi, \quad i = 0, 1.$$

Then, we may conclude

$$(23) \quad \|T_r f\|_{L^q(x^{2\alpha+1})} \leq A_r \|f\|_{L^p(x^{2\alpha+1})}$$

where

$$\log A_r \leq \int_{\mathbb{R}} \omega(1-r, y) \log A_0(y) dy + \int_{\mathbb{R}} \omega(r, y) \log A_1(y) dy$$

and

$$\omega(1-r, y) = \frac{\tan(\pi r/2)}{2[\tan^2(\pi r/2) + \tanh^2(\pi y/2)] \cos^2(\pi y/2)}.$$

*Proof.* See [17] or [19, Ch. V, § 4].  $\square$

*Remark 3.* Note that, if the family  $\{T_z\}$  depends on a parameter  $k$ , and the estimates (21) and (22) are independent of  $k$ , we can conclude that the boundedness (23) will be uniform in  $k$ .

The proof of Theorem 3 (the uniform boundedness of  $M_{\alpha,k}^\delta$  in the range  $0 < \delta \leq \delta_0$ ) will use an analytic family of operators related with  $M_{\alpha,k}^\delta$ .

Let us consider  $\delta(z) = (1-z)\delta_0 + \varepsilon$  with  $\varepsilon > 0$ ,  $0 \leq \operatorname{Re}(z) \leq 1$  and  $\delta_0 = (2\alpha+1)/2$ . We will take the family of operators

$$(24) \quad M_{\alpha,k}^{\delta(z)}(f, x) = x^{-\alpha} \int_0^\infty f(t) K_{\alpha+k}^{\delta(z)}(x, t) t^{\alpha+1} dt, \quad k \in \mathbb{N},$$

where the kernel  $K_{\alpha+k}^{\delta(z)}$  is as in (17) with  $\delta$  changed by  $\delta(z)$ . This definition of  $M_{\alpha,k}^{\delta(z)}$  is valid for simple functions in  $(0, \infty)$ .

Bessel functions of complex order  $\lambda = \nu + i\mu$  satisfy

$$J_\lambda(t) = \frac{(t/2)^\lambda}{\Gamma(1/2)\Gamma(\lambda+1/2)} \int_0^1 (1-s^2)^{\lambda-1/2} \cos(st) ds, \quad \nu > -1/2$$

(it is just a particular case of (16)), and the estimates

$$(25) \quad |J_{\nu+i\mu}(t)| \leq C_\nu e^{\pi|\mu|} t^{-1/2}, \quad t \geq 1, \quad \nu \geq 0,$$

$$(26) \quad |J_{\nu+i\mu}(t)| \leq C_\nu e^{\pi|\mu|} t^\nu, \quad t > 0, \quad \nu \geq 0.$$

Then, it is not difficult to check that, for any  $k$ , the family of operators (24) is analytic and of admissible growth. The details can be seen in [25, § 3].

Now, let us prove the uniform boundedness for the operators  $M_{\alpha,k}^{\delta(z)}$  in  $\operatorname{Re}(z) = 0$  and  $\operatorname{Re}(z) = 1$  for some values of  $p$ . Lemma 2 will establish the boundedness for  $\operatorname{Re}(z) = 0$ , and Lemma 3 for  $\operatorname{Re}(z) = 1$ .

**Lemma 2.** *Let  $\alpha \geq 0$  and  $1 \leq p \leq \infty$ . Consider  $M_{\alpha,k}^{\delta(z)}$  the family of operators given by (24), where  $\delta(z) = (1-z)\delta_0 + \varepsilon$  with  $\varepsilon > 0$ ,  $0 \leq \operatorname{Re}(z) \leq 1$  and  $\delta_0 = (2\alpha+1)/2$ . Then, the inequality*

$$\|M_{\alpha,k}^{\delta(iy)} f\|_{L^p(x^{2\alpha+1})} \leq A_0(y) \|f\|_{L^p(x^{2\alpha+1})}, \quad k \in \mathbb{N},$$

holds, with  $A_0(y) = C_{\alpha,p} e^{\pi|\delta_0 y|}$ .

*Proof.* First, note that Lemma 1 can be reproduced with the change  $\delta$  by  $\delta(iy)$ . Then, the proof of this lemma is similar to the one of Theorem 2, by applying again the convolution structure. But, this time, we use the estimation

$$\|\mathcal{J}_\alpha^{\delta(iy)}(x)\|_{L^1(x^{2\alpha+1})} \leq C_\alpha e^{\pi|\delta_0 y|},$$

that follows from the bounds (25) and (26) for (15) (with  $\delta(iy)$ ).  $\square$

Now, the boundedness for  $M_{\alpha,k}^{\delta(1+iy)}$ :

**Lemma 3.** *Let  $\alpha \geq 0$  and  $1 \leq p \leq \infty$ . Consider  $M_{\alpha,k}^{\delta(z)}$  the family of operators given by (24), where  $\delta(z) = (1-z)\delta_0 + \varepsilon$  with  $\varepsilon > 0$ ,  $0 \leq \operatorname{Re}(z) \leq 1$  and  $\delta_0 = (2\alpha+1)/2$ . If*

$$(27) \quad \frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1},$$

then the inequality

$$\|M_{\alpha,k}^{\delta(1+iy)} f\|_{L^p(x^{2\alpha+1})} \leq A_1(y) \|f\|_{L^p(x^{2\alpha+1})}, \quad k \in \mathbb{N},$$

holds with  $A_1(y) = C_{\alpha,p} \left(1 + \frac{|\delta_0 y|}{\varepsilon}\right)$ .

*Proof.* Von Lommel's formula

$$sJ_\nu(xs)J_\nu(ts) = \frac{1}{x^2 - t^2} \frac{d}{ds} \{stJ_\nu(xs)J'_\nu(ts) - sxJ'_\nu(xs)J_\nu(ts)\}$$

and an integration by parts in (8) (with  $\delta$  changed by  $\delta(1+iy)$ ) give

$$K_{\alpha+k}^{\delta(1+iy)}(x, t) = 2\delta(1+iy) \int_0^1 s(1-s^2)^{\delta(1+iy)-1} \mathcal{K}_{\alpha+k}^s(x, t) ds,$$

where

$$(28) \quad \mathcal{K}_\nu^s(x, t) = \frac{stJ_\nu(xs)J'_\nu(ts) - sxJ'_\nu(xs)J_\nu(ts)}{x^2 - t^2}.$$

Also, let us denote

$$(29) \quad \mathcal{T}_{\alpha,k}^s(f, x) = x^{-\alpha} \int_0^\infty f(t) \mathcal{K}_{\alpha+k}^s(x, t) t^{\alpha+1} dt.$$

In this way, by using Fubini's theorem, we can write

$$M_{\alpha,k}^{\delta(1+iy)}(f, x) = 2\delta(1+iy) \int_0^1 s(1-s^2)^{\delta(1+iy)-1} \mathcal{T}_{\alpha,k}^s(f, x) ds.$$

Taking into account that  $\delta(1+iy) = \varepsilon - iy\delta_0$  and by applying Minkowsky's integral inequality, we have

$$\begin{aligned} \|M_{\alpha,k}^{\delta(1+iy)} f\|_{L^p(x^{2\alpha+1})} &\leq C(\varepsilon + |\delta_0 y|) \left\| \int_0^1 2s(1-s^2)^{\varepsilon - iy\delta_0 - 1} \mathcal{T}_{\alpha,k}^s(f, x) ds \right\|_{L^p(x^{2\alpha+1})} \\ &\leq C(\varepsilon + |\delta_0 y|) \int_0^1 2s(1-s^2)^{\varepsilon-1} \|\mathcal{T}_{\alpha,k}^s(f, x)\|_{L^p(x^{2\alpha+1})} ds. \end{aligned}$$

Now, we claim that

$$\|\mathcal{T}_{\alpha,k}^s f\|_{L^p(x^{2\alpha+1})} \leq C_{\alpha,p} \|f\|_{L^p(x^{2\alpha+1})}$$

holds for every  $s$  for  $p$  verifying (27). We have  $\mathcal{T}_{\alpha,k}^s(f(\cdot), x) = \mathcal{T}_{\alpha,k}^1(f(\frac{\cdot}{s}), sx)$ , and then it is enough to prove the inequality

$$(30) \quad \|\mathcal{T}_{\alpha,k}^1 f\|_{L^p(x^{2\alpha+1})} \leq C_{\alpha,p} \|f\|_{L^p(x^{2\alpha+1})}.$$

But comparing (28) and (29) with (11), it is clear that

$$\mathcal{T}_{\alpha,k}^1 f = M_{\alpha,k} f.$$

Then, by applying the uniform boundedness of Theorem 1, we get (30).

Finally, using that

$$(\varepsilon + |\delta_0 y|) \int_0^1 2s(1-s^2)^{\varepsilon-1} ds \leq \left(1 + \frac{|\delta_0 y|}{\varepsilon}\right),$$

we conclude

$$\|M_{\alpha,k}^{\delta(1+iy)} f\|_{L^p(x^{2\alpha+1})} \leq C_{\alpha,p} \left(1 + \frac{|\delta_0 y|}{\varepsilon}\right) \|f\|_{L^p(x^{2\alpha+1})}.$$

□

Now,

*Proof of Theorem 3.* We will restrict our attention to  $1 \leq p \leq 2$ . The other values for  $p$  can be obtained by using a duality argument, because

$$\int_0^\infty f(x) M_{\alpha,k}^\delta(g(t), x) x^{2\alpha+1} dx = \int_0^\infty M_{\alpha,k}^\delta(f(x), t) g(t) t^{2\alpha+1} dt.$$

To prove that (9) is a necessary condition, let us take a function  $f \in S^+$  such that  $\mathcal{H}_\alpha(f, x) = 1$  if  $x \in (0, 1)$ . Thus, by using the definition (5) and Sonine's formula (16), we get that

$$\begin{aligned} M_{\alpha,0}^\delta(f, x) &= \mathcal{H}_\alpha((1-t^2)_+^\delta \mathcal{H}_\alpha f, x) = \mathcal{H}_\alpha((1-t^2)_+^\delta, x) \\ &= \int_0^1 \frac{J_\alpha(xt)}{(xt)^\alpha} (1-t^2)^\delta t^{2\alpha+1} dt = 2^\delta \Gamma(\delta+1) \frac{J_{\alpha+\delta+1}(x)}{x^{\alpha+\delta+1}}. \end{aligned}$$

Now, by applying the well-known asymptotic estimate (see [24, § 7.21 (1), p. 199])

$$J_\nu(x) \approx \left(\frac{2}{\pi}\right)^{1/2} x^{-1/2} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}), \quad x \rightarrow \infty,$$

it is clear that  $M_{\alpha,0}^\delta f \notin L^p(x^{2\alpha+1})$  if  $p \leq \frac{4(\alpha+1)}{2\alpha+2\delta+3}$ .

Finally, let us prove that (9) is also sufficient. Let us take the analytic family given by (24), with  $\delta(z) = (1-z)\delta_0 + \varepsilon$ ,  $\delta_0 = (2\alpha+1)/2$  and  $\varepsilon > 0$ , and take  $p_0 = q_0 = 1$ ,  $p_1 = q_1 = 4(\alpha+1)/(2\alpha+3) + \varepsilon$ . Now, let us use Lemmas 2 and 3 for  $p_0$  and  $p_1$ . Then,

$$\|M_{\alpha,k}^{\delta(iy)} f\|_{L^{p_0}(x^{2\alpha+1})} \leq A_0(y) \|f\|_{L^{p_0}(x^{2\alpha+1})},$$

with  $A_0(y) = C_{\alpha,p_0} e^{\pi|\delta_0 y|}$ , and

$$\|M_{\alpha,k}^{\delta(1+iy)} f\|_{L^{p_1}(x^{2\alpha+1})} \leq A_1(y) \|f\|_{L^{p_1}(x^{2\alpha+1})},$$

with  $A_1(y) = C_{\alpha,p_1} \left(1 + \frac{|\delta_0 y|}{\varepsilon}\right)$ . In this way, Theorem 4 ensures

$$(31) \quad \|M_{\alpha,k}^{\delta(t)} f\|_{L^p(x^{2\alpha+1})} \leq C \|f\|_{L^p(x^{2\alpha+1})}$$

for those values for  $p$  which satisfy

$$\frac{1}{p} = (1-r) + \frac{r}{p_1}, \quad 0 \leq r \leq 1.$$

By using  $\delta = \delta(r) = (1-r)\delta_0 + \varepsilon$ , we have  $r = 1 - \frac{\delta-\varepsilon}{\delta_0}$ , so the result holds for those values for  $p$  which satisfy

$$\frac{1}{p} = \left(\frac{\delta-\varepsilon}{\delta_0}\right)\left(1 - \frac{1}{p_1}\right) + \frac{1}{p_1}, \quad \varepsilon \leq \delta \leq \varepsilon + \delta_0.$$

Taking an arbitrarily small  $\varepsilon$ , we see that, if  $0 < \delta < \delta_0$ , the boundedness (31) holds for those values of  $p$  which satisfy  $\frac{1}{p} < \frac{2\alpha+3+2\delta}{4(\alpha+1)}$ . This completes the proof.  $\square$

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