SOLVING DUAL INTEGRAL EQUATIONS ON LEBESGUE SPACES

ÓSCAR CIAURRI, JOSÉ J. GUADALUPE, MARIO PÉREZ,
AND JUAN L. VARONA

Abstract. We study dual integral equations associated with Hankel transforms, that is, dual integral equations of Titchmarsh's type. We reformulate these equations giving a better description in terms of continuous operators on L^p spaces, and we solve them in these spaces. The solution is given both as an operator described in terms of integrals and as a series $\sum_{n=0}^{\infty} c_n J_{\mu+2n+1}$ which converges in the L^p-norm and almost everywhere, where J_ν denotes the Bessel function of order ν. Finally, we study the uniqueness of the solution.

1. Introduction

In some physical problems related with potential and electromagnetic or acoustic radiation theory, sometimes the unknown function satisfies an integral equation over part of the range $(0, \infty)$ and a different integral equation over the rest of the range. These equations are known as dual integral equations. An important case is the so-called dual integral equations of Titchmarsh's type:

$$(1) \qquad \begin{cases} \displaystyle\int_0^\infty t^\beta f(t) J_\alpha(xt)\, dt = g(x) & \text{if } 0 < x < 1, \\ \displaystyle\int_0^\infty f(t) J_\alpha(xt)\, dt = 0 & \text{if } x > 1, \end{cases}$$

where J_α stands for the Bessel function of order α (see [18] or [3, Ch. VII]), g is a given function and f is the unknown function. For a function h,

$$\int_0^\infty h(t) J_\alpha(xt)(xt)^{1/2}\, dt, \quad x > 0,$$

Date: PUBLISHED IN *Studia Math.* **142** (2000), no. 2, 253–267.

2000 *Mathematics Subject Classification.* Primary 45F10; Secondary 42C10.

Key words and phrases. Dual integral equations, Bessel functions, Fourier series, Hankel transform.

Research supported by grants of the DGES and UR.

On April 1, 2000, while this paper was being revised, José J. Guadalupe unexpectedly died. We still can not say how deep is our sorrow.

is usually known as the Hankel transform of h; so, the second equation in (1) means that the Hankel transform of $t^{-1/2}f(t)$ is supported on $[0,1]$, and the first one imposes a condition on the Hankel transform of $t^{\beta-1/2}f(t)$.

There are different methods to solve these equations, most of them only formal. For instance, they can be solved by using Mellin transforms or some other integral transforms. Also, they can be reduced to Fredholm integral equations. Usually, these methods allow to find the solution f as an explicit expression with integrals; we can see some of them in the books [14, p. 337], [10, § 12, p. 65], [7, § 5.11] and [3, p. 76]. Another method consists of solving the equation as a series $\sum_{n=0}^{\infty} c_n J_{\mu+2n+1}$; see [15] and [16], the first one with a large bibliography. But, as long as the authors know, it is only studied as a formal method.

In this paper we pursuit a rigorous approach to solve dual integral equations. We reformulate (1) so as to obtain a better description in terms of operators on L^p spaces, and we find the solution in these spaces. Also, we identify the solution as a Fourier-Neumann series whose L^p and almost everywhere convergence is studied.

The paper is organised as follows: in section 2 we state the dual integral equation in a more convenient form and define some associated operators. Section 3 collects some properties of Bessel functions and Jacobi polynomials. We describe a solution to the dual integral equation in section 4, and section 5 is devoted to the uniqueness of the solution. Sections 6 and 7 contain some of the proofs.

Throughout this paper, unless otherwise stated, we will use C to denote a positive constant independent of f (and all other variables), which can assume different values in different occurrences.

Also, for any function g defined on $[0,1]$, the extension given by $g(x)=0$ at each $x>1$ will be denoted by $\chi_{[0,1]}g$, with a small abuse of notation. Strictly speaking, $\chi_{[0,1]}$ could be understood either as a characteristic function or as an operator taking functions defined on $[0,1]$ to functions defined on $[0,\infty)$.

2. The dual integral equation

Let us define, for $\alpha > -1$, the integral operator $\mathcal{H}_\alpha$ as

$$\mathcal{H}_\alpha(f,x) = \frac{x^{-\alpha/2}}{2}\int_0^\infty f(t)J_\alpha(\sqrt{xt})t^{\alpha/2}\,dt, \quad x>0,$$

for suitably integrable functions f. For instance, $\mathcal{H}_\alpha$ is an isomorphism from the Schwartz class

$$S^+ = \{f \in C^\infty((0,\infty)) : \forall k,j \geq 0,\ |t^k f^{(j)}(t)| < C_{k,j}\}$$

onto itself and $\mathcal{H}_\alpha^2$ is the identity map. This operator is a modified Hankel transform. For $\alpha \geq -1/2$, $1 \leq p \leq 2$, and $1/p + 1/p' = 1$, $\mathcal{H}_\alpha$ extends to a bounded operator from $L^p([0,\infty), x^\alpha\, dx)$ into $L^{p'}([0,\infty), x^\alpha\, dx)$, i.e.

$$\|\mathcal{H}_\alpha f\|_{L^{p'}([0,\infty),x^\alpha dx)} \leq C\|f\|_{L^p([0,\infty),x^\alpha dx)}, \quad f \in L^p([0,\infty), x^\alpha\, dx).$$

However, the Hankel transform does not extend to $L^p([0,\infty), x^\alpha\, dx)$ if $2 < p$ (see [2, 12, 17]).

Another operator will be used: the multiplier of the Hankel transform associated to $\chi_{[0,1]}$, that is, the operator M_α given by $\mathcal{H}_\alpha(M_\alpha f) = \chi_{[0,1]}\mathcal{H}_\alpha f$. This multiplier plays an important role in the study of orthogonal Fourier expansions (see [17] in connection with Fourier-Neumann series, and [11] for Laguerre series).

Herz's classical result determines the range of p such that M_α is a well defined, bounded operator from $L^p([0,\infty), x^\alpha\, dx)$ into itself ([5]; see also [11, 17]):

Proposition 2.1. *Let $\alpha \geq -1/2$ and $1 < p < \infty$. Then*

$$\|M_\alpha f\|_{L^p([0,\infty),x^\alpha dx)} \leq C\|f\|_{L^p([0,\infty),x^\alpha dx)} \iff \tfrac{4(\alpha+1)}{2\alpha+3} < p < \tfrac{4(\alpha+1)}{2\alpha+1}.$$

For more general results on Hankel multipliers, see [12] and the references therein.

In a dense subset of $L^p([0,\infty), x^\alpha\, dx)$ (for instance, S^+)

$$M_\alpha f = \mathcal{H}_\alpha(\chi_{[0,1]}\mathcal{H}_\alpha f)$$

and $\mathcal{H}_\alpha^2 = \mathrm{Id}$. Whenever $\mathcal{H}_\alpha$ is well defined, it follows that $\mathcal{H}_\alpha f$ is supported on $[0,1]$ if and only if $M_\alpha f = f$.

Now, let us reformulate the dual integral equations. With a simple change of notation, we can write (1) as

$$(2) \qquad \begin{cases} \dfrac{x^{-\alpha/2}}{2}\displaystyle\int_0^\infty t^\beta f(t) J_\alpha(\sqrt{xt}) t^{\alpha/2}\, dt = g(x) & \text{if } 0 < x < 1, \\ \dfrac{x^{-\alpha/2}}{2}\displaystyle\int_0^\infty f(t) J_\alpha(\sqrt{xt}) t^{\alpha/2}\, dt = 0 & \text{if } x > 1. \end{cases}$$

The second equation in (2) means that $\operatorname{supp}(\mathcal{H}_\alpha f) \subseteq [0,1]$; in other words, $M_\alpha f = f$, provided that f belongs to a convenient L^p space.

The first equation in (2) can be read as $\mathcal{H}_\alpha(t^\beta f)\chi_{[0,1]} = \chi_{[0,1]}g$. Under certain conditions, $\mathcal{H}_\alpha$ is an inversible operator. Then, we obtain the equivalent equation $M_\alpha(t^\beta f, x) = \mathcal{H}_\alpha(\chi_{[0,1]}g, x)$. It will be convenient to multiply both sides by $x^{-\beta}$, so we get $x^{-\beta}M_\alpha(t^\beta f, x) = x^{-\beta}\mathcal{H}_\alpha(\chi_{[0,1]}g, x)$.

To sum up, we are interested in solving in $L^p([0,\infty), x^\alpha\, dx)$ the equation

$$(3) \qquad \begin{cases} x^{-\beta}M_\alpha(t^\beta f, x) = x^{-\beta}\mathcal{H}_\alpha(\chi_{[0,1]}g, x), \\ M_\alpha f = f. \end{cases}$$

In a strict sense, (2) and (3), are not exactly equivalent if we do not assume that the functions belong to a convenient L^p space. However, it is interesting to note that, for any practical physical application, the interpretation of a dual integral equation and its solution as in (2) is equivalent to its interpretation as in (3).

Together with $\mathcal{H}_\alpha$ and $M_\alpha f = \mathcal{H}_\alpha(\chi_{[0,1]}\mathcal{H}_\alpha f)$, let us consider the operators $M_{\alpha,\beta}$ and $\mathcal{H}_{\alpha,\beta}$ given by

$$\begin{aligned} M_{\alpha,\beta} f &= x^{-\beta} M_\alpha(t^\beta f), \\ \mathcal{H}_{\alpha,\beta} g &= x^{-\beta}\mathcal{H}_\alpha(\chi_{[0,1]} g). \end{aligned}$$

With this notation, the dual integral equation (3) can be written as

$$\left\{ \begin{array}{l} M_{\alpha,\beta} f = \mathcal{H}_{\alpha,\beta} g, \\ M_\alpha f = f. \end{array} \right. \tag{4}$$

Those operators are well defined, for instance, if $f \in S^+$ and $g \in C^\infty([0,1])$. We see below that $M_{\alpha,\beta}$ is bounded with the $L^p([0,\infty), x^\alpha\,dx)$-norm, under some conditions on α, β, and p. Therefore, it extends to a bounded operator on $L^p([0,\infty), x^\alpha\,dx)$. With a similar argument, $\mathcal{H}_{\alpha,\beta}$ extends to a bounded operator from $L^p([0,1], x^\alpha\,dx)$ into $L^p([0,\infty), x^\alpha\,dx)$.

Proposition 2.2. *Let $\alpha \ge -1/2$, $\beta \ge 0$ and $1 < p < \infty$. Then*

$$\|M_{\alpha,\beta} f\|_{L^p([0,\infty),x^\alpha dx)} \le C\|f\|_{L^p([0,\infty),x^\alpha dx)} \iff \tfrac{4(\alpha+1)}{2\alpha+4\beta+3} < p < \tfrac{4(\alpha+1)}{2\alpha+4\beta+1}.$$

Proof. We give only a sketch of the proof. It follows the proof for M_α in [17]. Actually, this is a particular case of weighted versions of Herz's classical result (with power weights).

Let us take $p_0 = \frac{4(\alpha+1)}{2\alpha+4\beta+3}$ and $p_1 = \frac{4(\alpha+1)}{2\alpha+4\beta+1}$. For each $f \in S^+$, Fubini's theorem applies to $M_{\alpha,\beta} f = x^{-\beta}\mathcal{H}_\alpha(\chi_{[0,1]}\mathcal{H}_\alpha(t^\beta f))$, then Lommel's formula

$$\int_0^1 J_\alpha(yt)J_\alpha(yx)y\,dy = \frac{1}{t^2-x^2}(tJ_{\alpha+1}(t)J_\alpha(x) - xJ_\alpha(t)J_{\alpha+1}(x))$$

gives

$$\begin{aligned} M_{\alpha,\beta}(f,x) &= \tfrac{1}{2}x^{-\alpha/2-\beta+1/2}J_{\alpha+1}(x^{1/2})H(t^{\alpha/2+\beta}J_\alpha(t^{1/2})f(t),x) \\ &\quad - \tfrac{1}{2}x^{-\alpha/2-\beta}J_\alpha(x^{1/2})H(t^{\alpha/2+\beta+1/2}J_{\alpha+1}(t^{1/2})f(t),x) \\ &= W_1(f,x) - W_2(f,x), \end{aligned}$$

where H is the Hilbert transform $H(f,x) = \int_0^\infty \frac{f(t)}{x-t}\,dt$. The Hilbert transform is a bounded operator from $L^p([0,\infty), x^\lambda\,dx)$ into itself if and only if $-1 < \lambda < p-1$. Fix $1 < p < \infty$; then, by using the bound $|J_\alpha(x)| \le$

$Cx^{-1/2}$, it is easy to check that W_1 and W_2 are bounded operators on $L^p([0,\infty), x^\alpha\, dx)$ if

$$p_0 < p < \tfrac{4(\alpha+1)}{2\alpha+4\beta-1}$$

(disregard the right hand side inequality if $2\alpha+4\beta-1 \le 0$) and

$$\tfrac{4(\alpha+1)}{2\alpha+4\beta+5} < p < p_1,$$

respectively. Then, $M_{\alpha,\beta}$ is bounded if $p_0 < p < p_1$.

In fact, $p_0 < p < p_1$ is a necessary condition for the boundedness of $M_{\alpha,\beta}$. By interpolation, we only need to observe that $M_{\alpha,\beta}$ is not bounded for $p = p_0$ (if $p_0 > 1$) and $p = p_1$. If $p = p_0 > 1$, W_2 is bounded; however, more precise estimates for the Bessel functions near infinity and a clever election of f show that W_1 is not bounded. Then, $M_{\alpha,\beta}$ is not bounded. The case of $p = p_1$ is analogous. $\square$

Regarding the Hankel transform $\mathcal{H}_\alpha$, we have the following theorem of Rooney ([9, p. 1100], [6], after a change of notation):

Theorem 2.3 (Rooney). *Let $\alpha > -1$, $1 < p \le q < \infty$, $\max\{\frac{1}{p}, 1-\frac{1}{q}\} \le \nu < \alpha + \frac{3}{2}$. Then*

$$\Big(\int_0^\infty |x^{-\nu/2+\alpha/2+3/4}\mathcal{H}_\alpha(h,x)|^q \,\frac{dx}{x}\Big)^{1/q} \le C\Big(\int_0^\infty |x^{\nu/2+\alpha/2+1/4}h(x)|^p\,\frac{dx}{x}\Big)^{1/p}.$$

The boundedness of $\mathcal{H}_{\alpha,\beta}$ follows as a consequence:

Proposition 2.4. *Let $\alpha \ge -1/2$, $\beta \ge 0$, $1 < p < \infty$ and assume*

$$\frac{2(\alpha+1)}{\alpha+\beta+1} \le p < \frac{\alpha+1}{\beta}.$$

Then $\|\mathcal{H}_{\alpha,\beta}g\|_{L^p([0,\infty),x^\alpha dx)} \le C\|g\|_{L^p([0,1],x^\alpha dx)}$.

Proof. Take $\nu = 2\beta + \alpha + \frac{3}{2} - \frac{2(\alpha+1)}{p}$ and $p = q$. It is easy to see that we can apply Theorem 2.3 and get

$$\begin{aligned}
\|\mathcal{H}_{\alpha,\beta}g\|_{L^p([0,\infty),x^\alpha dx)} &= \|x^{-\beta+(\alpha+1)/p}\mathcal{H}_\alpha(\chi_{[0,1]}g)\|_{L^p([0,\infty),\frac{dx}{x})} \\
&\le C\|x^{\beta+(\alpha+1)(1-1/p)}\chi_{[0,1]}g\|_{L^p([0,\infty),\frac{dx}{x})} \\
&= C\|x^{\beta+(\alpha+1)(1-2/p)}\chi_{[0,1]}g\|_{L^p([0,\infty),x^\alpha dx)} \\
&\le C\|g\|_{L^p([0,1],x^\alpha dx)},
\end{aligned}$$

where the last inequality follows from $\beta + (\alpha+1)(1-2/p) \ge 0$. $\square$

Then, our dual integral equation (4) is well posed and the question we try to solve is the following: given any $g \in L^p([0,1], x^\alpha\, dx)$, is there a (unique) solution $f \in L^p([0,\infty), x^\alpha\, dx)$?

3. Bessel functions and Jacobi polynomials

If $\alpha > -1$, the Bessel functions satisfy the orthogonality relation

$$\int_0^\infty J_{\alpha+2n+1}(x) J_{\alpha+2m+1}(x)\,\frac{dx}{x} = \frac{\delta_{nm}}{2(2n+\alpha+1)}, \qquad n,m = 0,1,2,\dots$$

After a change of variable, the system $\{j_n^\alpha\}_{n=0}^\infty$ given by

$$j_n^\alpha(x) = \sqrt{\alpha+2n+1}\,J_{\alpha+2n+1}(\sqrt{x})x^{-\alpha/2-1/2}$$

is orthonormal on $L^2([0,\infty), x^\alpha\,dx)$. There is a tight relation between Bessel functions and Jacobi polynomials $P_n^{(\alpha,\beta)}$ and the following lemma is relevant for our purposes; the first part was proved in [1] and the second part will be proved in section 6. Of course, these formulas hold in Lebesgue spaces, that is, almost everywhere.

Lemma 3.1. *Let $\alpha, \beta > -1$ with $\alpha+\beta > -1$. Then*

$$\mathcal{H}_\alpha(j_n^{\alpha+\beta}, x) = 2^{-\beta}\tfrac{\sqrt{\alpha+\beta+2n+1}\,\Gamma(n+1)}{\Gamma(\beta+n+1)}(1-x)^\beta P_n^{(\alpha,\beta)}(1-2x)\chi_{[0,1]}(x). \tag{5}$$

Assume further $\beta < 1$. Then

$$\chi_{[0,1]}(x)\mathcal{H}_\alpha(t^\beta j_n^{\alpha+\beta}, x) = 2^\beta\tfrac{\sqrt{\alpha+\beta+2n+1}\,\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)}P_n^{(\alpha,\beta)}(1-2x)\chi_{[0,1]}(x). \tag{6}$$

In particular, $\operatorname{supp}(\mathcal{H}_\alpha(j_n^{\alpha+\beta})) \subseteq [0,1]$. However, note that (6) refers only to the Hankel transform of $t^\beta j_n^{\alpha+\beta}$ at $x \in [0,1]$; nothing is claimed for $x > 1$.

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$ of order α, β (see [3, Ch. X] or [13, Ch. IV]) are orthogonal on $[-1,1]$ with respect to the weight $(1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$.

After a change of variable, the system $\{P_n^{(\alpha,\beta)}(1-2x)\}_{n=0}^\infty$ is orthogonal on $[0,1]$ with respect to the weight $x^\alpha(1-x)^\beta$, $\alpha,\beta > -1$. To be precise, the orthogonality relation for these polynomials is

$$\int_0^1 P_n^{(\alpha,\beta)}(1-2x)P_m^{(\alpha,\beta)}(1-2x)x^\alpha(1-x)^\beta\,dx = h_n^{(\alpha,\beta)}\delta_{nm}$$

with

$$h_n^{(\alpha,\beta)} = \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)n!}\,.$$

Let us take

$$p_n^{(\alpha,\beta)}(x) = (h_n^{(\alpha,\beta)})^{-1/2}P_n^{(\alpha,\beta)}(1-2x), \quad n = 0,1,2,\dots \tag{7}$$

This is an orthonormal system on $[0,1]$ with respect to the weight $w(x) = x^\alpha(1-x)^\beta$. For any suitable function g defined on $[0,1]$, its Fourier-Jacobi series is the formal expansion

$$g \sim \sum_{n=0}^\infty a_n(g)p_n^{(\alpha,\beta)}, \quad a_n(g) = \int_0^1 g(x)p_n^{(\alpha,\beta)}(x)x^\alpha(1-x)^\beta\,dx.$$

The following result by Muckenhoupt gives conditions for the uniform boundedness and the mean convergence of $S_n g$ (actually, both are equivalent, by the Banach-Steinhaus theorem):

Theorem 3.2 (Muckenhoupt [8]). *Assume that $\alpha > -1$, $\beta > -1$, $1 < p < \infty$ and let $S_n g$ denote the nth partial sum of the Jacobi polynomial series for g with parameters α and β. Assume that*

$$\left| a + (\alpha+1)\Big(\frac{1}{p} - \frac{1}{2}\Big) \right| < \min\left\{ \frac{1}{4}, \frac{\alpha+1}{2} \right\},$$
$$\left| b + (\beta+1)\Big(\frac{1}{p} - \frac{1}{2}\Big) \right| < \min\left\{ \frac{1}{4}, \frac{\beta+1}{2} \right\}.$$

Then, there exists a constant C such that

$$\|x^a(1-x)^b S_n g\|_{L^p([0,1],x^\alpha(1-x)^\beta\,dx)} \le C\|x^a(1-x)^b g\|_{L^p([0,1],x^\alpha(1-x)^\beta\,dx)}$$

for every $n \in \mathbb{N}$, and

$$\lim_n \|x^a(1-x)^b(S_n g - g)\|_{L^p([0,1],x^\alpha(1-x)^\beta\,dx)} = 0$$

for every g with $\|x^a(1-x)^b g\|_{L^p([0,1],x^\alpha(1-x)^\beta\,dx)} < \infty$.

For our purposes, we will only need the following:

Corollary 3.3. *Let $\alpha \ge -1/2$, $\beta \ge 0$, $1 < p < \infty$ and*

$$\max\left\{ \frac{4(\alpha+1)}{2\alpha+3}, \frac{4}{2\beta+3} \right\} < p < \min\left\{ \frac{4(\alpha+1)}{2\alpha+1}, \frac{4}{2\beta+1} \right\}.$$

Then,

$$\lim_{n\to\infty} S_n g = g$$

in the $L^p([0,1], x^\alpha\,dx)$-norm, for any $g \in L^p([0,1], x^\alpha\,dx)$.

Proof. Take $a = 0$ and $b = -\beta/p$ in the previous result. □

The scheme we use to solve the dual equation (4) goes as follows: expand g as a Fourier-Jacobi series, that is, $g = \sum_{n=0}^\infty a_n p_n^{(\alpha,\beta)}$; then, the solution is $f = \sum_{n=0}^\infty b_n j_n^{\alpha+\beta}$, where b_n is explicitely given in terms of a_n and the series converges both in L^p and almost everywhere.

Series of the form $\sum_{n=0}^\infty c_n J_{\mu+n}$ are usually known as Neumann series. Thus, we are describing the solution of the dual integral equation as a Fourier-Neumann series.

The operator that takes g into f will be proved to be bounded on L^p. It can also be written in terms of integral operators.

4. Main results: The solution of the equation

In this section we introduce the operator $L_{\alpha,\beta}$ and state some of its properties. This operator solves the dual integral equation (4).

Let g a suitable function on $[0,1]$. We define $L_{\alpha,\beta}g$ by

$$L_{\alpha,\beta}(g,x) = \frac{1}{2^{\beta+1}\Gamma(\beta)} \int_0^1 \frac{J_{\alpha+\beta}(\sqrt{xt})}{(xt)^{\alpha/2+\beta/2}} \int_0^t g(u)(t-u)^{\beta-1}u^\alpha \, du \, dt, \quad x > 0,$$

if $\beta > 0$ and $L_{\alpha,0}g = \mathcal{H}_\alpha(\chi_{[0,1]}g)$. Our first result states that $L_{\alpha,\beta}$ is a bounded operator from $L^p([0,1], x^\alpha\, dx)$ into $L^p([0,\infty), x^\alpha\, dx)$:

Theorem 4.1. *Let $\alpha \geq -1/2$, $\beta \geq 0$, $1 < p < \infty$ and*

$$\frac{2(2\alpha+3)}{2(\alpha+\beta)+3} \leq p < \infty.$$

Then

$$\|L_{\alpha,\beta}g\|_{L^p([0,\infty),x^\alpha dx)} \leq C\|g\|_{L^p([0,1],x^\alpha dx)}, \quad g \in L^p([0,1], x^\alpha\, dx).$$

In what follows, we write $P^\uparrow \leq Q$ with the meaning $P < Q$. In this way, we have

$$\max\{A, B^\uparrow\} \leq M \iff A \leq M \text{ and } B < M.$$

This will keep the notation a bit shorter.

Corollary 4.2. *Let $\alpha \geq -1/2$, $\beta \geq 0$, $1 < p < \infty$ and assume*

$$\max\left\{\frac{2(2\alpha+3)}{2(\alpha+\beta)+3}, \left(\frac{4(\alpha+1)}{2\alpha+3}\right)^\uparrow\right\} \leq p < \min\left\{\frac{4(\alpha+1)}{2\alpha+1}, \frac{4}{2\beta+1}\right\}.$$

Then, for any $g \in L^p([0,1], x^\alpha\, dx)$, we have

$$g = \sum_{n=0}^\infty a_n(g) p_n^{(\alpha,\beta)}, \quad a_n(g) = \int_0^1 g(x)p_n^{(\alpha,\beta)}(x)x^\alpha(1-x)^\beta\, dx$$

in the $L^p([0,1], x^\alpha\, dx)$-norm and

$$L_{\alpha,\beta}g = \sum_{n=0}^\infty b_n j_n^{\alpha+\beta}, \quad b_n = 2^{-\beta}\tfrac{\Gamma(\alpha+n+1)^{1/2}(n!)^{1/2}}{\Gamma(\alpha+\beta+n+1)^{1/2}\Gamma(\beta+n+1)^{1/2}} a_n(g)$$

in the $L^p([0,\infty), x^\alpha\, dx)$-norm and almost everywhere.

For the proof of Theorem 4.1 and Corollary 4.2, see section 7.

Before going on, let us write Lemma 3.1 in terms of M_α, $M_{\alpha,\beta}$, and $\mathcal{H}_{\alpha,\beta}$. It is clear from (5) that $\mathcal{H}_\alpha j_n^{\alpha+\beta}$ is supported on $[0,1]$, so that

$$M_\alpha(j_n^{\alpha+\beta}) = j_n^{\alpha+\beta}. \tag{8}$$

And, taking (7) into account, (6) reads as

$$M_{\alpha,\beta}(j_n^{\alpha+\beta}) = x^{-\beta} M_\alpha(t^\beta j_n^{\alpha+\beta}) = x^{-\beta}\mathcal{H}_\alpha(\chi_{[0,1]}\mathcal{H}_\alpha(t^\beta j_n^{\alpha+\beta})) \tag{9}$$
$$= x^{-\beta}\mathcal{H}_\alpha(\chi_{[0,1]} d_n p_n^{(\alpha,\beta)}) = d_n \mathcal{H}_{\alpha,\beta} p_n^{(\alpha,\beta)}$$

with $d_n = 2^\beta \frac{\Gamma(\alpha+\beta+n+1)^{1/2}\Gamma(\beta+n+1)^{1/2}}{\Gamma(\alpha+n+1)^{1/2}(n!)^{1/2}}$.

Our main result is the following:

Theorem 4.3. *Let $\alpha \geq -1/2$, $0 \leq \beta < 1$, $1 < p < \infty$ and*

$$\max\left\{\frac{2(2\alpha+3)}{2(\alpha+\beta)+3}, \left(\frac{4(\alpha+1)}{2\alpha+3}\right)^{\uparrow}\right\} \leq p < \min\left\{\frac{4(\alpha+1)}{2\alpha+4\beta+1}, \frac{4}{2\beta+1}\right\}.$$

For each $g \in L^p([0,1], x^\alpha\,dx)$, $f = L_{\alpha,\beta} g$ is a solution in $L^p([0,\infty), x^\alpha\,dx)$ of the dual integral equation

$$\begin{cases} M_{\alpha,\beta} f = \mathcal{H}_{\alpha,\beta} g, \\ M_\alpha f = f. \end{cases}$$

Proof. Let $g \in L^p([0,1], x^\alpha\,dx)$ and $f = L_{\alpha,\beta} g$. It is easy to see that we can apply Propositions 2.1, 2.2 and 2.4. Since $L_{\alpha,\beta}$ is bounded (by Theorem 4.1), Corollary 4.2 and (9) give

$$M_{\alpha,\beta} f = \lim_{n\to\infty} M_{\alpha,\beta}\Big(\sum_{k=0}^n b_k j_k^{\alpha+\beta}\Big)$$
$$= \lim_{n\to\infty} \mathcal{H}_{\alpha,\beta}\Big(\sum_{k=0}^n b_k d_k p_k^{(\alpha,\beta)}\Big) = \mathcal{H}_{\alpha,\beta} g,$$

while Corollary 4.2 and (8) yield

$$M_\alpha f = \lim_{n\to\infty} M_\alpha\Big(\sum_{k=0}^n b_k j_k^{\alpha+\beta}\Big) = \lim_{n\to\infty}\sum_{k=0}^n b_k j_k^{\alpha+\beta} = f. \qquad \square$$

5. Uniqueness of the solution

Let us consider the L^p subspaces

$$B_{p,\alpha,\beta} = \overline{\operatorname{span}}\{j_n^{\alpha+\beta}(x)\}_{n=0}^\infty \quad \text{(closure in } L^p([0,\infty), x^\alpha\,dx)),$$
$$E_{p,\alpha} = \{f \in L^p([0,\infty), x^\alpha\,dx) : M_\alpha f = f\}.$$

The following results about the mean convergence of Fourier-Neumann series were proved in [1]:

Theorem 5.1. *Let $\alpha > -1$, $\alpha+\beta > -1$, $4/3 < p < 4$, and*

$$\max\left\{-\frac{\alpha+\beta+1}{2}, -\frac{1}{4}\right\} < (\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right) + \frac{\beta}{2} < \frac{1}{4}.$$

Then, for any $f \in B_{p,\alpha,\beta}$ there exists a unique expansion

$$f = \sum_{n=0}^{\infty} b_n(f) j_n^{\alpha+\beta}$$

which holds in the $L^p([0,\infty), x^\alpha\,dx)$-norm. This expansion also holds almost everywhere.

Theorem 5.2. *Let $\alpha \ge -1/2$, $\beta > -1/2$, $4/3 < p$ with*

$$\frac{-1}{4} < (\alpha+1)\left(\frac{1}{2} - \frac{1}{p}\right) < \frac{1}{4}.$$

If $p < 2$, assume further

$$\frac{-1}{4} < (\alpha+1)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{|\beta|}{2}.$$

Then $B_{p,\alpha,\beta} = E_{p,\alpha}$.

Using these results, we can prove

Theorem 5.3. *Let $\alpha \ge -1/2$, $0 \le \beta < 1$, $1 < p < \infty$ and*

$$\max\left\{\frac{2(2\alpha+3)}{2(\alpha+\beta)+3}, \left(\frac{4(\alpha+1)}{2\alpha-2\beta+3}\right)^{\uparrow}\right\} \le p < \min\left\{\frac{4(\alpha+1)}{2\alpha+4\beta+1}, \frac{4}{2\beta+1}\right\}.$$

Then, $f = L_{\alpha,\beta}g$ is the unique solution in $L^p([0,\infty), x^\alpha\,dx)$ of the dual equation

$$\begin{cases} M_{\alpha,\beta}f = \mathcal{H}_{\alpha,\beta}g, \\ M_\alpha f = f. \end{cases}$$

Proof. It is not difficult to check that, under the hypothesis of this theorem, we can deduce the ones of Theorems 5.1 and 5.2 and Theorem 4.3. For instance, $\frac{4}{3} < p$ follows from

$$\frac{4}{3} < \max\left\{\frac{2(2\alpha+3)}{2(\alpha+\beta)+3}, \frac{4(\alpha+1)}{2\alpha-2\beta+3}\right\}.$$

Indeed, if this inequality failed we would have

$$\frac{2(2\alpha+3)}{2(\alpha+\beta)+3} \le \frac{4}{3}, \qquad \frac{4(\alpha+1)}{2\alpha-2\beta+3} \le \frac{4}{3},$$

which yield $2\alpha+3 \le 4\beta$ and $\alpha+2\beta \le 0$, respectively. Then $\alpha \le -3/4$, which contradicts $\alpha \ge -1/2$.

According to Theorem 4.3, $L_{\alpha,\beta}g$ is a solution of the dual equation. Let us see that it is unique. Let f be a solution, that is, $f \in L^p([0,\infty), x^\alpha\,dx)$,

$M_\alpha f = f$ and $M_{\alpha,\beta} f = \mathcal{H}_{\alpha,\beta} g$. In particular, $f \in E_{p,\alpha}$. By Theorems 5.2 and 5.1, we can expand f as a Fourier-Neumann series

$$f = \sum_{n=0}^{\infty} b_n(f) j_n^{\alpha+\beta}$$

which converges in the $L^p([0,\infty), x^\alpha\, dx)$-norm. Proving that each $b_n(f)$ is uniquely determined will suffice.

From (9) and the fact that $M_{\alpha,\beta}$ is a continuous (bounded) operator in L^p, we get

$$\mathcal{H}_{\alpha,\beta} g = M_{\alpha,\beta} f = \sum_{n=0}^{\infty} b_n(f) d_n \mathcal{H}_{\alpha,\beta} p_n^{(\alpha,\beta)}.$$

Our assumptions on α, β, and p, togehter with the estimates

$$J_\alpha(x) = O(x^\alpha), \quad x \to 0^+, \qquad \text{and} \qquad J_\alpha(x) = O(x^{-1/2}), \quad x \to \infty,$$

yield $x^\beta j_k^{\alpha+\beta} \in L^{p'}([0,\infty), x^\alpha\, dx)$, where $1/p + 1/p' = 1$. Hence, the map $h \mapsto \int_0^\infty x^\beta j_k^{\alpha+\beta}(x) h(x) x^\alpha\, dx$ is a continuous operator from $L^p([0,\infty), x^\alpha\, dx)$ into $\mathbb{R}$. Then,

$$\begin{aligned}\int_0^\infty & x^\beta j_k^{\alpha+\beta}(x) \mathcal{H}_{\alpha,\beta}(g, x) x^\alpha\, dx \\ &= \sum_{n=0}^{\infty} b_n(f) d_n \int_0^\infty x^\beta j_k^{\alpha+\beta}(x) \mathcal{H}_{\alpha,\beta}(p_n^{(\alpha,\beta)}, x) x^\alpha\, dx.\end{aligned}$$

Now, recall the multiplication formula for the Hankel transform, which is valid for $h_1, h_2 \in L^2([0,\infty), x^\alpha\, dx)$:

$$\int_0^\infty h_1(x) \mathcal{H}_\alpha(h_2, x) x^\alpha\, dx = \int_0^\infty \mathcal{H}_\alpha(h_1, x) h_2(x) x^\alpha\, dx.$$

Indeed, in S^+ this follows from Fubini's theorem, then it extends to the whole $L^2([0,\infty), x^\alpha\, dx)$ by continuity.

Thus, the definition of $\mathcal{H}_{\alpha,\beta}$, together with (5) and the orthogonality of Jacobi polynomials, yield

$$\begin{aligned}\int_0^\infty x^\beta j_k^{\alpha+\beta}(x) \mathcal{H}_{\alpha,\beta}(p_n^{(\alpha,\beta)}, x) x^\alpha\, dx &= \int_0^\infty j_k^{\alpha+\beta}(x) \mathcal{H}_\alpha(\chi_{[0,1]} p_n^{(\alpha,\beta)}, x) x^\alpha\, dx \\ &= r_k \int_0^1 (1-x)^\beta p_k^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x) x^\alpha\, dx = r_k \delta_{kn}\end{aligned}$$

with a constant $r_k \neq 0$ (actually, $r_k = 1/d_k$, where d_k comes from (9), as before). Therefore,

$$\int_0^\infty x^\beta j_n^{\alpha+\beta}(x) \mathcal{H}_{\alpha,\beta} g(x) x^\alpha\, dx = b_n(f). \qquad \square$$

6. Proof of Lemma 3.1

As we already mentioned, the first part of Lemma 3.1 was proved in [1], so we only prove now the second part. Let ${}_2F_1$ denote, as usual, the hypergeometric function. We use the formula

$$\int_0^\infty t^{-\lambda} J_\mu(at) J_\nu(bt)\,dt$$
$$= \frac{2^{-\lambda}\Gamma(\frac{1}{2}(\mu+\nu-\lambda+1))b^\nu}{a^{\nu-\lambda+1}\Gamma(\nu+1)\Gamma(\frac{1}{2}(\lambda+\mu-\nu+1))}\,{}_2F_1\left(\tfrac{\mu+\nu-\lambda+1}{2}, \tfrac{\nu-\lambda-\mu+1}{2}; \nu+1; \tfrac{b^2}{a^2}\right), \quad 0 < b < a,$$

valid when $\mu+\nu-\lambda > -1$ and $\lambda > -1$ (see [4, 8.11 (9), p. 48] or [18, 13.4 (2), p. 401]). Take $a = 1$ and $x = b^2$, with parameters $\lambda = -\beta$, $\mu = \alpha+\beta+2n+1$ and $\nu = \alpha$. Then, for $\beta < 1$, $\alpha+\beta > -1$, and $0 < x < 1$ we get, after a change of variable,

$$\frac{x^{-\alpha/2}}{2}\int_0^\infty y^\beta j_n^{\alpha+\beta}(y) J_\alpha(\sqrt{xy}) y^{\alpha/2}\,dy$$
$$= \frac{2^\beta\sqrt{\alpha+\beta+2n+1}\,\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)}\,{}_2F_1(\alpha+\beta+n+1, -n; \alpha+1; x).$$

Taking into account that

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)}\,{}_2F_1(\alpha+\beta+n+1, -n; \alpha+1; \tfrac{1-x}{2})$$

whenever $\alpha, \beta > -1$ and $-1 < x < 1$, it follows

$$\frac{x^{-\alpha/2}}{2}\int_0^\infty y^\beta j_n^{\alpha+\beta}(y) J_\alpha(\sqrt{xy}) y^{\alpha/2}\,dy$$
$$= \frac{2^\beta\sqrt{\alpha+\beta+2n+1}\,\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)} P_n^{(\alpha,\beta)}(1-2x),$$

if $x \in (0,1)$. We have not finished yet, because this integral must be understood as an improper Riemann integral, not a Lebesgue integral. In other words, this means

$$\lim_{R\to\infty} \mathcal{H}_\alpha(t^\beta j_n^{\alpha+\beta}\chi_{[0,R]}, x) = \frac{2^\beta\sqrt{\alpha+\beta+2n+1}\,\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)} P_n^{(\alpha,\beta)}(1-2x),$$

pointwisely on $(0,1)$. Now, $\mathcal{H}_\alpha$ is a bounded operator, so that

$$\lim_{R\to\infty} \mathcal{H}_\alpha(t^\beta j_n^{\alpha+\beta}\chi_{[0,R]}) = \mathcal{H}_\alpha(t^\beta j_n^{\alpha+\beta})$$

in the L^p-norm. It follows that

$$\mathcal{H}_\alpha(t^\beta j_n^{\alpha+\beta}, x) = \frac{2^\beta\sqrt{\alpha+\beta+2n+1}\,\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)} P_n^{(\alpha,\beta)}(1-2x),$$

almost everywhere on $[0,1]$.

7. Proof of Theorem 4.1 and Corollary 4.2

Lemma 7.1. *Let $\alpha \ge -1/2$, $\beta \ge 0$, and $1 < p < \infty$, with $\frac{2(2\alpha+3)}{2(\alpha+\beta)+3} \le p$. Then*

$$\|\mathcal{H}_{\alpha+\beta}(\chi_{[0,1]}h)\|_{L^p([0,\infty),x^\alpha\,dx)} \le C\|h\|_{L^p([0,1],x^\alpha\,dx)}, \quad h \in L^p([0,1],x^\alpha\,dx).$$

Proof. Take $\nu = \alpha+\beta+\frac{3}{2}-2\frac{\alpha+1}{p}$. Then

$$\|\mathcal{H}_{\alpha+\beta}(\chi_{[0,1]}h)\|_{L^p([0,\infty),x^\alpha\,dx)} = \|x^{-\nu/2+(\alpha+\beta)/2+3/4}\mathcal{H}_{\alpha+\beta}(\chi_{[0,1]}h)\|_{L^p([0,\infty),\frac{dx}{x})}.$$

With our assumptions on α, β, and p we can apply Theorem 2.3 and get

$$\begin{aligned}\|\mathcal{H}_{\alpha+\beta}(\chi_{[0,1]}h)\|_{L^p([0,\infty),x^\alpha\,dx)} &\le C\|x^{\nu/2+(\alpha+\beta)/2+1/4}\chi_{[0,1]}h\|_{L^p([0,\infty),\frac{dx}{x})}\\ &= C\|x^{\alpha+\beta+1-2(\alpha+1)/p}h\|_{L^p([0,1],x^\alpha\,dx)}.\end{aligned}$$

The easy observation that $\alpha+\beta+1-2\frac{\alpha+1}{p} \ge 0$ finishes the proof. □

Proof of Theorem 4.1. Lemma 7.1 proves Theorem 4.1 in the case $\beta = 0$, since $L_{\alpha,0}g = \mathcal{H}_\alpha(\chi_{[0,1]}g)$. Now, observe that $L_{\alpha,\beta}g = 2^{-\beta}\mathcal{H}_{\alpha+\beta}(\chi_{[0,1]}I_{\alpha,\beta}g)$ if $\beta > 0$, where

$$I_{\alpha,\beta}(g,x) = \frac{x^{-(\alpha+\beta)}}{\Gamma(\beta)}\int_0^x g(t)(x-t)^{\beta-1}t^\alpha\,dt, \quad 0 < x < 1$$

is the Erdélyi-Kober operator. It is well known that this operator is bounded in $L^p([0,1],x^\alpha\,dx)$ if $\alpha > -1$, $\beta > 0$, and $1 < p < \infty$. Indeed, after a change of variable we obtain

$$I_{\alpha,\beta}(g,x) = \frac{1}{\Gamma(\beta)}\int_0^1 (1-z)^{\beta-1}z^\alpha g(xz)\,dz$$

and, by Minkowski's integral inequality,

$$\begin{aligned}\|I_{\alpha,\beta}g\|_{L^p([0,1],x^\alpha dx)} &\le \frac{1}{\Gamma(\beta)}\int_0^1 (1-z)^{\beta-1}z^\alpha\|g(xz)\|_{L^p([0,1],x^\alpha dx)}\,dz\\ &\le \|g\|_{L^p([0,1],x^\alpha dx)}\frac{1}{\Gamma(\beta)}\int_0^1 (1-z)^{\beta-1}z^{\alpha-(\alpha+1)/p}\,dz\\ &= C\|g\|_{L^p([0,1],x^\alpha dx)},\end{aligned}$$

where we have used that $\|g(xz)\|_{L^p([0,1],x^\alpha dx)} \le z^{-(\alpha+1)/p}\|g\|_{L^p([0,1],x^\alpha dx)}$ and

$$\int_0^1 (1-z)^{\beta-1}z^{\alpha-(\alpha+1)/p}\,dz = B(\beta,(\alpha+1)(1-1/p)) < \infty.$$

Thus, $I_{\alpha,\beta}$ is bounded and again Lemma 7.1 proves the theorem. □

Proof of Corollary 4.2. Let $g \in L^p([0,1], x^\alpha\,dx)$. Under our assumptions on α, β, and p, it is easy to check that we can apply Corollary 3.3. Therefore,

$$g = \sum_{n=0}^{\infty} a_n(g) p_n^{(\alpha,\beta)}, \quad a_n(g) = \int_0^1 g(x) p_n^{(\alpha,\beta)}(x) x^\alpha (1-x)^\beta\,dx,$$

in the $L^p([0,1], x^\alpha\,dx)$-norm. By Theorem 4.1, $L_{\alpha,\beta}$ is a continuous (i.e. bounded) operator from $L^p([0,1], x^\alpha\,dx)$ into $L^p([0,\infty), x^\alpha\,dx)$. Then,

$$L_{\alpha,\beta} g = \sum_{n=0}^{\infty} a_n(g) L_{\alpha,\beta} p_n^{(\alpha,\beta)},$$

where the convergence holds in the L^p-norm. Now, consider the following formula (see [4, 13.1 (43), p. 191]):

$$I_{\alpha,\beta}(P_n^{(\alpha,\beta)}(1-2t), x) = \tfrac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\beta+n+1)} P_n^{(\alpha+\beta,0)}(1-2x).$$

Lemma 3.1 (with parameters $\alpha+\beta$ and 0 instead of α and β, respectively) gives $\mathcal{H}_{\alpha+\beta}(j_n^{\alpha+\beta}, x) = \sqrt{\alpha+\beta+2n+1}\, P_n^{(\alpha+\beta,0)}(1-2x)\chi_{[0,1]}(x)$, so that

$$\mathcal{H}_{\alpha+\beta}(\chi_{[0,1]} P_n^{(\alpha+\beta,0)}(1-2t)) = (\alpha+\beta+2n+1)^{-1/2} j_n^{\alpha+\beta} \tag{10}$$

(since $\mathcal{H}_{\alpha+\beta}^2 = \mathrm{Id}$ in L^2). Thus,

$$L_{\alpha,\beta}(P_n^{(\alpha,\beta)}(1-2t)) = 2^{-\beta} \frac{\Gamma(\alpha+n+1)}{\sqrt{\alpha+\beta+2n+1}\,\Gamma(\alpha+\beta+n+1)} j_n^{\alpha+\beta} \tag{11}$$

if $\beta > 0$. In the case $\beta = 0$, (10) and $L_{\alpha,0} g = \mathcal{H}_\alpha(\chi_{[0,1]} g)$ give (11), as well. In terms of the normalised polynomials $p_n^{(\alpha,\beta)}$, this means

$$L_{\alpha,\beta} p_n^{(\alpha,\beta)} = 2^{-\beta} \tfrac{\Gamma(\alpha+n+1)^{1/2} (n!)^{1/2}}{\Gamma(\alpha+\beta+n+1)^{1/2} \Gamma(\beta+n+1)^{1/2}} j_n^{\alpha+\beta},$$

so that

$$L_{\alpha,\beta} g = \sum_{n=0}^{\infty} a_n(g) 2^{-\beta} \tfrac{\Gamma(\alpha+n+1)^{1/2} (n!)^{1/2}}{\Gamma(\alpha+\beta+n+1)^{1/2} \Gamma(\beta+n+1)^{1/2}} j_n^{\alpha+\beta}$$

in the L^p-norm and, by Theorem 5.1, almost everywhere. □

References

[1] Ó. Ciaurri, J. J. Guadalupe, M. Pérez and J. L. Varona, *Mean and almost everywhere convergence of Fourier-Neumann series*, J. Math. Anal. Appl. 236 (1999), 125–147.

[2] A. J. Durán, *On Hankel transform*, Proc. Amer. Math. Soc. 110 (1990), 417–424.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Higher Transcendental Functions,* Vol. II, McGraw-Hill, New York, 1953.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Tables of Integral Transforms,* Vol. II, McGraw-Hill, New York, 1954.

[5] C. S. Herz, *On the mean inversion of Fourier and Hankel transforms*, Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 996–999.

[6] P. Heywood and P. G. Rooney, *A weighted norm inequality for the Hankel transformation*, Proc. Roy. Soc. Edinburgh Sect. A 99 (1984), 45–50.
[7] H. Hochstadt, *Integral equations,* Wiley, New York, 1973.
[8] B. Muckenhoupt, *Mean convergence of Jacobi series*, Proc. Amer. Math. Soc. 23 (1969), 306–310.
[9] P. G. Rooney, *A technique for studying the boundedness and extendability of certain types of operators*, Canad. J. Math. 25 (1973), 1090–1102.
[10] I. N. Sneddon, *Fourier Transforms,* McGraw-Hill, New York, 1951. Republication: Dover, New York, 1995.
[11] K. Stempak, *Transplanting maximal inequalities between Laguerre and Hankel multipliers*, Monatsh. Math. 122 (1996), 187–197.
[12] K. Stempak and W. Trebels, *Hankel multipliers and transplantation operators*, Studia Math. 126 (1997), 51–66.
[13] G. Szegő, *Orthogonal Polynomials* (4th edition), Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, RI, 1975.
[14] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals,* Oxford, New York, 1937.
[15] C. J. Tranter, *Integral Transforms in Mathematical Physics* (3th edition), Methuen, London, 1966.
[16] C. J. Tranter, *Bessel Functions with Some Physical Applications,* English Univ. Press, London, 1968.
[17] J. L. Varona, *Fourier series of functions whose Hankel transform is supported on* $[0,1]$, Constr. Approx. 10 (1994), 65–75.
[18] G. N. Watson, *A Treatise on the Theory of Bessel Functions,* Cambridge Univ. Press, 1966.

Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio J. L. Vives, Calle Luis de Ulloa s/n, 26004 Logroño, Spain
E-mail address: `osciaurr@dmc.unirioja.es`

Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio J. L. Vives, Calle Luis de Ulloa s/n, 26004 Logroño, Spain
E-mail address: `joguadal@dmc.unirioja.es`

Departamento de Matemáticas, Universidad de Zaragoza, Edificio de Matemáticas, Ciudad Universitaria s/n, 50009 Zaragoza, Spain
E-mail address: `mperez@posta.unizar.es`

Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio J. L. Vives, Calle Luis de Ulloa s/n, 26004 Logroño, Spain
E-mail address: `jvarona@dmc.unirioja.es`
URL: `http://www.unirioja.es/dptos/dmc/jvarona/welcome.html`