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Mean and Almost Everywhere Convergence of Fourier–Neumann Series

Óscar Ciaurri* and José J. Guadalupe†

Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio J. L. Vives, Calle Luis de Ulloa s/n, 26004 Logroño, Spain
E-mail: osciaurr@dmc.unirioja.es, joguadal@dmc.unirioja.es

and

Mario Pérez†

Departamento de Matemáticas, Universidad de Zaragoza, Edificio de Matemáticas, Ciudad Universitaria s/n, 50009 Zaragoza, Spain
E-mail: mperez@posta.unizar.es

and

Juan L. Varona‡

Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio J. L. Vives, Calle Luis de Ulloa, s/n, 26004 Logroño, Spain
E-mail: jvarona@dmc.unirioja.es

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Let J_μ denote the Bessel function of order μ. The functions $x^{-\alpha/2-\beta/2-1/2}J_{\alpha+\beta+2n+1}(x^{1/2})$, $n = 0, 1, 2, \dots$, form an orthogonal system in $L^2((0,\infty), x^{\alpha+\beta}\,dx)$ when $\alpha + \beta > -1$. In this paper we analyze the range of p, α, and β for which the Fourier series with respect to this system converges in the $L^p((0,\infty), x^\alpha\,dx)$-norm. Also, we describe the space in which the span of the system is dense and we show some of its properties. Finally, we study the almost everywhere convergence of the Fourier series for functions in such spaces. © 1999 Academic Press

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‡ URL: http://www.unirioja.es/dptos/dmc/jvarona/welcome.html

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1. INTRODUCTION AND NOTATION

Let J_μ be the Bessel function of order μ. For $\alpha > -1$, the formula

$$\int_0^\infty J_{\alpha+2n+1}(x) J_{\alpha+2m+1}(x) \frac{dx}{x} = \frac{\delta_{nm}}{2(2n+\alpha+1)}, \qquad n, m = 0, 1, 2, \ldots$$

(see [15, Chap. XIII, 13.41 (7), p. 404; and 15, Chap. XIII, 13.42 (1), p. 405]), provides an orthonormal system $\{j_n^\alpha\}_{n=0}^\infty$ in $L^2((0,\infty), x^\alpha\, dx)$ [$L^2(x^\alpha)$, from now on], given by

$$j_n^\alpha(x) = \sqrt{\alpha + 2n + 1}\, J_{\alpha+2n+1}(\sqrt{x})\, x^{-\alpha/2-1/2}, \qquad n = 0, 1, 2, \ldots.$$

For each suitable function f, let $S_n f$ be the nth partial sum of its Fourier series with respect to the system $\{j_n^\alpha\}_{n=0}^\infty$. Series of this kind are a particular case of series $\sum_{n\geq 0} a_n J_{\alpha+n}$, which are usually called Neumann series, so that we refer to $S_n f$ as a Fourier–Neumann series. In [14], one of the authors studied the mean convergence in $L^p(x^\alpha)$ of these Fourier series. In this context, some operators and spaces were introduced. In this paper we extend these results and also study the almost everywhere convergence.

For $\alpha > -1$, let us define the integral operator $\mathscr{H}_\alpha$ by

$$\mathscr{H}_\alpha(f, x) = \frac{x^{-\alpha/2}}{2} \int_0^\infty f(t) J_\alpha(\sqrt{xt})\, t^{\alpha/2}\, dt, \qquad x > 0,$$

for suitable functions f. This is a modified Hankel transform: the (non-modified) Hankel transform is the integral operator with kernel $J_\alpha(xt)(xt)^{1/2}$ and unweighted Lebesgue measure. See [3, 12, 8] for some modified and non-modified Hankel transforms. In the case $\alpha \geq -\frac{1}{2}$, the Hankel transform satisfies

$$\|\mathscr{H}_\alpha f\|_{L^\infty(x^\alpha)} \leq C \|f\|_{L^1(x^\alpha)}, \qquad f \in L^1(x^\alpha),$$

with some constant C independent of f. Moreover, $\mathscr{H}_\alpha$ can be defined in $L^2(x^\alpha)$ satisfying $\int_0^\infty (\mathscr{H}_\alpha f) g x^\alpha\, dx = \int_0^\infty (\mathscr{H}_\alpha g) f x^\alpha\, dx$, $\mathscr{H}_\alpha^2 = \mathrm{Id}$, and $\|\mathscr{H}_\alpha f\|_{L^2(x^\alpha)} = \|f\|_{L^2(x^\alpha)}$. From these results and interpolation we obtain

$$\|\mathscr{H}_\alpha f\|_{L^q(x^\alpha)} \leq C \|f\|_{L^p(x^\alpha)}, \qquad f \in L^p(x^\alpha),$$

for $1 \leq p \leq 2$, where q denotes, here and in the rest of the paper, the conjugate of p, that is, $1/p + 1/q = 1$.

The Hankel transform of the function j_n^α is

$$\mathscr{H}_\alpha(j_n^\alpha, x) = \sqrt{\alpha + 2n + 1}\, P_n^{(\alpha,0)}(1 - 2x)\,\chi_{[0,1]}(x),$$

where $P_n^{(\alpha,\beta)}(x)$ is the nth Jacobi polynomial of order (α, β); see, for instance, [5, Chap. 8.11, (5), p. 47] (a thorough description of Jacobi polynomials can be found in [4, Chap. X; 13]).

Remark. There is a delicacy with this formula. Actually, $\mathscr{H}_\alpha$ was defined, as a first step, as a Lebesgue integral for suitably integrable functions. Then, $\mathscr{H}_\alpha$ is extended to L^p spaces where the integral representation is no longer valid for some functions. Now, the integrals from [5, Chap. 8.11, (5), p. 47] are improper Riemann integrals. Hence, the proper understanding of those integrals should be

$$\begin{aligned}\lim_{N\to\infty} \frac{x^{-\alpha/2}}{2} \int_0^N j_n^\alpha(t) J_\alpha(\sqrt{xt})\, t^{\alpha/2}\, dt \\ = \sqrt{\alpha + 2n + 1}\, P_n^{(\alpha,0)}(1 - 2x)\,\chi_{[0,1]}(x).\end{aligned}$$

Since $j_n^\alpha \chi_{[0,N]}$ is an integrable function, the integral form of $\mathscr{H}_\alpha$ is valid here and we can conclude that

$$\lim_{N\to\infty} \mathscr{H}_\alpha\big(j_n^\alpha \chi_{[0,N]}, x\big) = \sqrt{\alpha + 2n + 1}\, P_n^{(\alpha,0)}(1 - 2x)\,\chi_{[0,1]}(x),$$

where the limit holds in the almost everywhere sense. Finally, the L^p boundedness of the operator $\mathscr{H}_\alpha$ for $4(\alpha + 1)/(2\alpha + 3) < p < 2$ and the fact that $\lim_{N\to\infty} j_n^\alpha \chi_{[0,N]} = j_n^\alpha$ in L^p yields

$$\mathscr{H}_\alpha(j_n^\alpha, x) = \sqrt{\alpha + 2n + 1}\, P_n^{(\alpha,0)}(1 - 2x)\,\chi_{[0,1]}(x)$$

in L^p. Similar comments apply to Lemma 3 below.

Since the Hankel transform of j_n^α is supported on $[0, 1]$, not every function $f \in L^p(x^\alpha)$, $1 < p \le 2$, can be approximated in norm by its Fourier series $S_n f$. As a first approach, any such function should, at least, have its Hankel transform supported on $[0, 1]$. But we also deal with spaces $L^p(x^\alpha)$, $p > 2$ where $\mathscr{H}_\alpha$ is not defined and so, we need to describe the functions that we want to approximate in a different, but, in some sense, similar way.

The main tool here is M_α, the multiplier for the Hankel transform. For $\alpha \ge -\frac{1}{2}$ and $-\frac{1}{4} < (\alpha + 1)(\frac{1}{2} - \frac{1}{p}) < \frac{1}{4}$, M_α is a bounded operator from $L^p(x^\alpha)$ into itself (this is known as Herz's theorem, see [7]). Also,

$$\mathscr{H}_\alpha(M_\alpha f) = \mathscr{H}_\alpha(f)\,\chi_{[0,1]},$$

for $f \in L^p(x^\alpha) \cap L^2(x^\alpha)$, $M_\alpha^2 f = M_\alpha f$ for $f \in L^p(x^\alpha)$ and

$$\int_0^\infty f(x) M_\alpha(g, x) x^\alpha \, dx = \int_0^\infty g(x) M_\alpha(f, x) x^\alpha \, dx \tag{1}$$

for $f \in L^p(x^\alpha)$ and $g \in L^q(x^\alpha)$.

DEFINITION 1. For each α and p with $\alpha \geq -\frac{1}{2}$ and $-\frac{1}{4} < (\alpha+1)(\frac{1}{2} - \frac{1}{p}) < \frac{1}{4}$, let us define the $L^p(x^\alpha)$ subspace

$$E_{p,\alpha} = \{ f \in L^p(x^\alpha) : M_\alpha f = f \} = M_\alpha(L^p(x^\alpha)).$$

It is clear that, for $f \in E_{p,\alpha} \cap L^2(x^\alpha)$, the Hankel transform of f is supported on $[0,1]$ and so these spaces are suitable for our purposes. The spaces $E_{p,\alpha}$ have some interesting properties: For $s < r$, $E_{s,\alpha} \subset E_{r,\alpha}$ and the inclusion is continuous and dense. Besides, the dual space is $(E_{p,\alpha})' = E_{q,\alpha}$.

Let us also consider, for each $\alpha \geq -1$ and each suitable p (we go into the details later), the $L^p(x^\alpha)$ subspace

$$B_{p,\alpha} = \overline{\operatorname{span}}\{ j_n^\alpha(x) \}_{n=0}^\infty \qquad (\text{closure in } L^p(x^\alpha)).$$

In [14], one of us showed that $S_n f \to f$ in the $L^p(x^\alpha)$-norm for any $f \in B_{p,\alpha}$, if $\alpha \geq -\frac{1}{2}$ and

$$\max\left\{ \frac{4}{3}, \frac{4(\alpha+1)}{2\alpha+3} \right\} < p < \min\left\{ 4, \frac{4(\alpha+1)}{2\alpha+1} \right\};$$

moreover, for this range of p, we showed that $B_{p,\alpha} = E_{p,\alpha}$. Therefore, $\{j_n^\alpha\}_{n=0}^\infty$ is a basis for the space $E_{p,\alpha}$. By the way, notice that for $\alpha \geq -\frac{1}{2}$,

$$\frac{4(\alpha+1)}{2\alpha+3} < p \quad \Leftrightarrow \quad -\frac{1}{4} < (\alpha+1)\left(\frac{1}{2} - \frac{1}{p}\right),$$

$$p < \frac{4(\alpha+1)}{2\alpha+1} \quad \Leftrightarrow \quad (\alpha+1)\left(\frac{1}{2} - \frac{1}{p}\right) < \frac{1}{4}.$$

Our purpose in this paper is to improve and extend these convergence results, and show additional properties of the $E_{p,\alpha}$ spaces. In particular, we find some conditions on α, β, and p under which the functions $j_n^{\alpha+\beta}$ $(n = 0, 1, 2, \dots)$ are a basis for $E_{p,\alpha}$. For instance, β can be taken so that $\alpha + \beta$ is half an integer, which makes the functions $j_n^{\alpha+\beta}$ better known. The almost everywhere convergence of $S_n f$ is studied, as well.

Also, we can interpret the convergence in the following way: changing the parameters, we take $\{j_n^{\lambda}\}_{n=0}^{\infty}$, which is orthogonal in $L^2(x^{\lambda})$, and we study the convergence in $L^p(x^{\mu})$. This is a typical solution in the study of mean convergence of Fourier series. For instance, in the case of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, Pollard [10] studied the convergence in the natural space $L^p((-1,1),(1-x)^{\alpha}(1+x)^{\beta})$ and, later, Muckenhoupt [9] described the behavior in $L^p((-1,1),(1-x)^{a}(1+x)^{b})$. Similar situations occur with other orthogonal systems (Laguerre, Hermite, Freud weights, Bessel and Dini).

We are interested in the approximation of functions in $L^p(x^{\alpha})$ by Fourier series in the system $\{j_n^{\alpha+\beta}\}_{n=0}^{\infty}$. So, our first target is to determine the range of p, α, and β for which $j_n^{\alpha+\beta} \in L^p(x^{\alpha})$ for all $n \in \mathbb{N}$. We do this in Section 2.

In Section 3 we state some of the main results of this paper: the uniform boundedness and convergence of the partial sum operator of Fourier–Neumann series. The proofs are given in Sections 6 and 7. The mean convergence can only hold for functions in the closure of the linear combinations of the functions $j_n^{\alpha+\beta}$. In Section 4 this space is shown to coincide with $E_{p,\alpha}$ under some conditions on p, α, and β. Some applications are given in Section 5.

Throughout this paper, unless otherwise stated, we use C (or C_1) to denote a positive constant independent of n (and all other variables), which can assume different values in different occurrences. Also, in what follows, $a_n \sim b_n$, for $a_n, b_n > 0$, means $C \le a_n/b_n \le C_1$.

2. THE SPACES $B_{p,\alpha,\beta}$

We use here the well-known estimates (see [4; 15, Chap. III, 3.1 (8), p. 40; 15, Chap. VII, 7.21 (1), p. 199]):

$$J_{\mu}(x) = \frac{x^{\mu}}{2^{\mu}\Gamma(\mu+1)} + O(x^{\mu+2}), \qquad x \to 0+, \tag{2}$$

and

$$J_{\mu}(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x - \frac{\mu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1})\right], \qquad x \to \infty. \tag{3}$$

LEMMA 1. *Let $a > -1$, $1 < p < \infty$. Then, $j_n^a \in L^p(x^b)$ for all $n = 0, 1, 2, \ldots$ if and only if $b > -1$ and $-\frac{1}{4} < (b+1)(\frac{1}{2} - \frac{1}{p}) + \frac{a-b}{2}$. Fur-*

thermore, in this case,

$$\|j_n^a\|_{L^p(x^b)} \sim \begin{cases} n^{-(a+1)+2(b+1)/p}, & \text{if } p < 4, \\ n^{-(2a-b+1)/2}(\log n)^{1/4}, & \text{if } p = 4, \\ n^{-(5/6+a)+(6b+4)/(3p)}, & \text{if } p > 4, \end{cases}$$

Proof. Inequalities $b > -1$ and $-\frac{1}{4} < (b+1)(\frac{1}{2} - \frac{1}{p}) + \frac{a-b}{2}$ follow from (2) and (3). Then, estimates such as (12) below (see [1, 2]) show that $\|j_n^a\|_{L^p(x^b)}$ is bounded above by a constant times the right-hand side. The lower bound follows from more precise estimates for the Bessel functions, as shown in [1, 2]. For a similar expression, see [11]. ∎

As a consequence, the following definition makes sense.

DEFINITION 2. For each α, β, and p with $\alpha > -1$, $\alpha + \beta > -1$, $1 < p < \infty$, and

$$-\frac{1}{4} < (\alpha+1)\left(\frac{1}{2} - \frac{1}{p}\right) + \frac{\beta}{2},$$

let us define

$$B_{p,\alpha,\beta} = \overline{\operatorname{span}}\{j_n^{\alpha+\beta}(x)\}_{n=0}^{\infty} \qquad (\text{closure in } L^p(x^\alpha)).$$

Note that we assume $\alpha > -1$ in the definition of $B_{p,\alpha,\beta}$; however, we require $\alpha \geq -\frac{1}{2}$ for $E_{p,\alpha}$. Actually, the boundedness of M_α can be studied also for $\alpha > -1$, so that the definition of $E_{p,\alpha}$ can be extended to the whole range $\alpha > -1$. But in the case $\alpha < -\frac{1}{2}$, the $\mathscr{H}_\alpha$ transform does not have as good properties as in the case $\alpha \geq -\frac{1}{2}$. As a consequence, the spaces $E_{p,\alpha}$ do not behave for $\alpha < -\frac{1}{2}$ like for $\alpha \geq -\frac{1}{2}$. Thus, some of the results in this paper are established for $\alpha > -1$, but we require $\alpha \geq -\frac{1}{2}$ when $E_{p,\alpha}$ appears.

The following lemma proves that $B_{p,\alpha} \subset B_{p,\alpha,\beta}$, under some conditions on α, β, and p.

LEMMA 2. *Let* $\alpha > -1$, $\alpha + \beta > -1$, *and* $1 < p < 4$ *such that*

$$\frac{-1}{4} < (\alpha+1)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{|\beta|}{2}.$$

Then,

$$j_n^\alpha = \sum_{k=n}^{\infty} a_{n,k} j_k^{\alpha+\beta}, \tag{4}$$

pointwise and in $L^p(x^\alpha)$, *where*

$$a_{n,k} = \frac{2^\beta \sqrt{\alpha+2n+1}\,\sqrt{\alpha+\beta+2k+1}\,\Gamma(1-\beta)\Gamma(\alpha+\beta+k+n+1)}{\Gamma(1+k-n)\Gamma(1-\beta-k+n)\Gamma(\alpha+k+n+2)}. \tag{5}$$

Remark. If $\beta \in \mathbb{N}$, then $\Gamma(1-\beta)/\Gamma(1-\beta-k+n)$ should be replaced by $-\beta(-\beta-1)(-\beta-2)\cdots(1-\beta-k+n)$ in formula (5).

Proof. The pointwise convergence and (5) follow from [15, Chap. V, 5.21 (1), p. 139] (conditions $\alpha > -1$ and $\alpha+\beta > -1$ are required). Strictly speaking, condition $\beta \notin \mathbb{N}$ should also be assumed, following [15]. But this is only a formal requirement to get $a_{n,k}$ in the form of (5).

For the L^p convergence, we need only prove that the series converges: that the sum is precisely j_n^α then follows from the fact that this holds in the almost everywhere sense.

If β is an integer, then there are only finitely many $a_{n,k} \neq 0$ and the series in (4) is a finite sum. If β is not an integer, Stirling's formula for the gamma function gives, for each fixed n,

$$|a_{n,k}| \sim k^{2\beta-3/2}, \qquad k \to \infty.$$

Also, from Lemma 1, $p < 4$ and $\frac{-1}{4} < (\alpha+1)(\frac{1}{2}-\frac{1}{p}) + \frac{\beta}{2}$, we have

$$\|j_k^{\alpha+\beta}\|_{L^p(x^\alpha)} \sim k^{-(\alpha+\beta+1)+2(\alpha+1)/p}.$$

These estimates and $\frac{-1}{4} < (\alpha+1)(\frac{1}{2}-\frac{1}{p}) - \frac{\beta}{2}$ prove that

$$\sum_{k=n}^{\infty} |a_{n,k}|\, \|j_k^{\alpha+\beta}\|_{L^p(x^\alpha)} < \infty.$$

∎

3. UNIFORM BOUNDEDNESS AND CONVERGENCE OF FOURIER–NEUMANN SERIES

Let us consider the partial sums of the Fourier series with respect to the system $\{j_n^{\alpha+\beta}\}_{n=0}^\infty$:

$$S_n(f,x) = \sum_{k=0}^{n} c_k(f)\, j_k^{\alpha+\beta}(x), \qquad c_k(f) = \int_0^\infty f(t)\, j_k^{\alpha+\beta}(t)\, t^{\alpha+\beta}\, dt.$$

We are interested in the study of the uniform boundedness of the partial sum operators

$$S_n \colon L^p(x^\alpha) \to L^p(x^\alpha).$$

Our result is

THEOREM 1. *Let* $\alpha > -1$, $\alpha + \beta > -1$, *and* $1 < p < \infty$. *There exists a constant* $C > 0$ *such that*

$$\|S_n f\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}, \qquad f \in L^p(x^\alpha),\, n \in \mathbb{N},$$

if and only if $\frac{4}{3} < p < 4$ *and*

$$\begin{aligned} -\frac{\alpha+\beta+1}{2} &< (\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{\beta}{2}, \\ -\frac{1}{4} &< (\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{\beta}{2} < \frac{1}{4}. \end{aligned} \tag{6}$$

Proof. See Section 6. ∎

COROLLARY 2. *Let* $\alpha > -1$, $\alpha + \beta > -1$, $\frac{4}{3} < p < 4$, *and*

$$\begin{aligned} -\frac{\alpha+\beta+1}{2} &< (\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{\beta}{2}, \\ -\frac{1}{4} &< (\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{\beta}{2} < \frac{1}{4}. \end{aligned}$$

Then, $S_n f \to f$ *in* $L^p(x^\alpha)$ *for all* $f \in B_{p,\alpha,\beta}$.

Proof. $B_{p,\alpha,\beta}$ is the closure in $L^p(x^\alpha)$ of the orthogonal system, so this is just a standard consequence of Theorem 1. ∎

Regarding the almost everywhere convergence of Fourier−Neumann series, we have

THEOREM 3. *Let* $\alpha > -1$, $\alpha + \beta > -1$, $\frac{4}{3} < p < 4$, *and*

$$\begin{aligned} -\frac{\alpha+\beta+1}{2} &< (\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{\beta}{2}, \\ -\frac{1}{4} &< (\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{\beta}{2} < \frac{1}{4}. \end{aligned}$$

Then, $S_n f \to f$ *almost everywhere for any* $f \in B_{p,\alpha,\beta}$.

Proof. See Section 7. ∎

4. THE HANKEL TRANSFORM OF ORDER α FOR $j_n^{\alpha+\beta}$ AND THE SPACES $B_{p,\alpha,\beta}$ AND $E_{p,\alpha}$

Theorem 1 and Corollary 2 are more interesting if we can describe the space $B_{p,\alpha,\beta}$. In this section, we find some conditions under which $B_{p,\alpha,\beta}$ and $E_{p,\alpha}$ coincide.

As we pointed out, the Hankel transform of order α for j_n^α can be written in terms of the nth Jacobi polynomial of order $(\alpha, 0)$. It is not difficult to obtain $\mathscr{H}_\alpha(j_n^{\alpha+\beta})$ from known results about integrals of products of Bessel functions that can be expressed in terms of hypergeometric ${}_2F_1$ functions. But the relation between $\mathscr{H}_\alpha(j_n^{\alpha+\beta})$ and the Jacobi polynomials of order (α, β) is not easily found in the literature. For instance, it does not appear in the standard references [4, 15, 5]. For the sake of completeness, in this section we obtain $\mathscr{H}_\alpha(j_n^{\alpha+\beta})$ explicitly in terms of $P_n^{(\alpha,\beta)}$.

LEMMA 3. *For $\alpha, \beta > -1$ with $\alpha + \beta > -1$,*

$$\mathscr{H}_\alpha\left(j_n^{\alpha+\beta}, x\right) = 2^{-\beta} \frac{\sqrt{\alpha+\beta+2n+1}\,\Gamma(n+1)}{\Gamma(\beta+n+1)}(1-x)^\beta \times P_n^{(\alpha,\beta)}(1-2x)\,\chi_{[0,1]}(x).$$

In particular, $\operatorname{supp}(\mathscr{H}_\alpha(j_n^{\alpha+\beta})) \subseteq [0,1]$.

Proof. We use the formula

$$\int_0^\infty t^{-\lambda} J_\mu(at) J_\nu(bt)\,dt = \frac{b^\nu a^{\lambda-\nu-1}\Gamma\left(\frac{\mu+\nu-\lambda+1}{2}\right)}{2^\lambda \Gamma(\nu+1)\Gamma\left(\frac{\lambda+\mu-\nu+1}{2}\right)} \times {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; \frac{b^2}{a^2}\right), \quad (7)$$

valid when $0 < b < a$, $\mu + \nu - \lambda > -1$, and $\lambda > -1$; here, ${}_2F_1$ denotes the hypergeometric function (see [5, Chap. 8.11, (9), p. 48; 15, Chap. XIII, 13.4 (2), p. 401]).

Taking $a = 1$ and $x = b^2$ in (7), and making the corresponding changes of variable and parameters ($\nu = \alpha$, $\mu = \alpha + \beta + 2n + 1$, $\lambda = \beta$) we get

$$\mathscr{H}_\alpha\left(j_n^{\alpha+\beta}, x\right) = \frac{\sqrt{2n + \alpha + \beta + 1}\,\Gamma(\alpha + n + 1)}{2^\beta \Gamma(\beta + n + 1)\Gamma(\alpha + 1)} \times {}_2F_1(\alpha + n + 1, -n - \beta; \alpha + 1; x),$$

which is valid for $\alpha > -1$ and $\beta > -1$ in the interval $0 < x < 1$. Now, we have

$$\begin{aligned} &{}_2F_1(\alpha + n + 1, -n - \beta; \alpha + 1; x) \\ &\quad = (1 - x)^\beta {}_2F_1(-n, \alpha + \beta + n + 1; \alpha + 1; x), \end{aligned}$$

where $\alpha, \beta > -1$, $n = 0, 1, 2, \ldots$, and

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + 1)} {}_2F_1\left(-n, \alpha + \beta + n + 1; \alpha + 1; \frac{1 - x}{2}\right), \qquad \alpha, \beta > -1.$$

Therefore,

$$\mathscr{H}_\alpha\left(j_n^{\alpha+\beta}, x\right) = \frac{\sqrt{\alpha + \beta + 2n + 1}\,\Gamma(n + 1)}{2^\beta \Gamma(\beta + n + 1)} (1 - x)^\beta P_n^{(\alpha,\beta)}(1 - 2x), \qquad x \in (0, 1).$$

Now, let us calculate $\mathscr{H}_\alpha(j_n^{\alpha+\beta}, x)$ for $x > 1$. To do that, let us take $x = a^2$, $b = 1$, $\nu = \alpha + \beta + 2n + 1$, $\mu = \alpha$, and $\lambda = \beta$ in (7). In this way, $\frac{1}{2}(\lambda + \mu - \nu + 1) = 0, -1, -2, \ldots,$ so the coefficient $1/\Gamma(\frac{1}{2}(\lambda + \mu - \nu + 1))$ vanishes and we get $\mathscr{H}_\alpha(j_n^{\alpha+\beta}, x) = 0$. ∎

THEOREM 4. *Let* $\alpha \geq -\frac{1}{2}$, $\beta > -\frac{1}{2}$, $\frac{4}{3} < p$, *with*

$$-\frac{1}{4} < (\alpha + 1)\left(\frac{1}{2} - \frac{1}{p}\right) < \frac{1}{4}.$$

If $p < 2$, *assume further*

$$-\frac{1}{4} < (\alpha + 1)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{|\beta|}{2}.$$

Then $B_{p,\alpha,\beta} = E_{p,\alpha}$.

Proof. *Case* $p = 2$. The spaces $B_{2,\alpha,\beta}$ and $E_{2,\alpha}$ are well defined. Also $M_\alpha(j_n^{\alpha+\beta}) = j_n^{\alpha+\beta}$. In other words, $B_{2,\alpha,\beta} \subset E_{2,\alpha}$. If they were not equal, by the Hahn–Banach theorem there should exist some $T \in (E_{2,\alpha})'$, $T \neq 0$, such that $T(j_n^{\alpha+\beta}) = 0\ \forall n$. But $(E_{2,\alpha})' = E_{2,\alpha}$, so there exists $\varphi \in E_{2,\alpha}$, $\varphi \neq 0$, such that $\int_0^\infty \varphi j_n^{\alpha+\beta} x^\alpha\, dx = 0$ for every n. Then

$$0 = \int_0^\infty \varphi j_n^{\alpha+\beta} x^\alpha\, dx = \int_0^\infty (\mathscr{H}_\alpha \varphi)\big(\mathscr{H}_\alpha j_n^{\alpha+\beta}\big) x^\alpha\, dx$$

$$= k_n \int_0^1 (\mathscr{H}_\alpha \varphi) P_n^{(\alpha,\beta)}(1-2x)(1-x)^\beta x^\alpha\, dx$$

for every nonnegative integer n. Now, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are a complete orthogonal system with respect to the measure $(1-x)^\alpha(1+x)^\beta\, dx$ on $(-1,1)$. A change of variable proves that the polynomials $P_n^{(\alpha,\beta)}(1-2x)$ are a complete orthogonal system with respect to the measure $(1-x)^\beta x^\alpha\, dx$ on $(0,1)$. Thus, $\mathscr{H}_\alpha \varphi = 0$ on $(0,1)$. Since $\varphi \in E_{2,\alpha}$, we also have $\mathscr{H}_\alpha \varphi = 0$ on $(1,\infty)$. Therefore, $\mathscr{H}_\alpha \varphi = 0$ and we arrive at the contradiction $\varphi = 0$.

Case $p > 2$. Note that α, β, and p meet the requirements of Definition 2. Also, by the preceding case, we have $j_n^{\alpha+\beta} \in E_{2,\alpha} \subset E_{p,\alpha}$. Thus, $B_{p,\alpha,\beta} \subset E_{p,\alpha}$.

Now, let $f \in E_{p,\alpha}$. Given $\varepsilon > 0$, there exists a function $g \in L^2(x^\alpha) \cap L^p(x^\alpha)$ such that $\|f - g\|_{L^p(x^\alpha)} < \varepsilon$. let $h = M_\alpha g$; then $h \in L^2(x^\alpha) \cap L^p(x^\alpha)$ and $M_\alpha h = h$, so that $h \in E_{2,\alpha} \cap E_{p,\alpha} = B_{2,\alpha,\beta} \cap E_{p,\alpha}$. Since M_α is continuous, $\|f - h\|_{L^p(x^\alpha)} = \|M_\alpha f - M_\alpha g\|_{L^p(x^\alpha)} < C\varepsilon$. As $h \in B_{2,\alpha,\beta}$, there exists $h' \in \operatorname{span}\{j_n^{\alpha+\beta}\}_{n=0}^\infty$ such that $\|h - h'\|_{L^2(x^\alpha)} < \varepsilon$. The inclusion $E_{2,\alpha} \subset E_{p,\alpha}$ gives $\|h - h'\|_{L^p(x^\alpha)} < C_1\varepsilon$, so that, by the triangle inequality, $\|f - h'\|_{L^p(x^\alpha)} < C_2\varepsilon$. This gives the inclusion $E_{p,\alpha} \subset B_{p,\alpha,\beta}$.

Case $p < 2$. By Lemmas 1 and 3, $j_n^{\alpha+\beta} \in L^2(x^\alpha)$ and $\mathscr{H}_\alpha(j_n^{\alpha+\beta})$ is supported on $[0,1]$, so that $M_\alpha j_n^{\alpha+\beta} = j_n^{\alpha+\beta}$. Since $j_n^{\alpha+\beta} \in L^p(x^\alpha)$ by Lemma 1, it follows that $j_n^{\alpha+\beta} \in E_{p,\alpha}$. Therefore, $B_{p,\alpha,\beta} \subset E_{p,\alpha}$.

The equality follows if we prove that the only operator $T \in (E_{p,\alpha})'$ such that $T(f) = 0$ for all $f \in B_{p,\alpha,\beta}$ is $T = 0$. For such an operator, we have $T(j_n^\alpha) = 0$ for every $n \geq 0$, since $j_n^\alpha \in B_{p,\alpha,\beta}$ by Lemma 2. On the other hand, by the duality $(E_{p,\alpha})' = E_{q,\alpha}$, where $1/p + 1/q = 1$, there exists some $\varphi \in E_{q,\alpha}$ such that

$$T(f) = \int_0^\infty \varphi f x^\alpha\, dx, \qquad f \in E_{p,\alpha}.$$

In particular,

$$\int_0^\infty \varphi j_n^\alpha x^\alpha \, dx = 0, \qquad n \geq 0. \tag{8}$$

Under the present conditions on p and α, the preceding case gives $B_{q,\alpha,0} = E_{q,\alpha}$, so that $\varphi \in B_{q,\alpha,0}$. This, together with (8) and Corollary 2, gives $\varphi = 0$. ∎

5. APPLICATIONS

Some properties of the spaces $E_{p,\alpha}$ can be obtained from Theorem 4. Two examples are given here, after this preliminary result.

COROLLARY 5. *Let* $\alpha \geq -\frac{1}{2}$, $\beta = 0$, *and* $\frac{4}{3} < p < 4$ *verifying*

$$-\frac{1}{4} < (\alpha + 1)\left(\frac{1}{2} - \frac{1}{p}\right) < \frac{1}{4}.$$

Then, $S_n f \to M_\alpha f$ *in* $L^p(x^\alpha)$ *and almost everywhere for all* $f \in L^p(x^\alpha)$.

Proof. Let $f \in L^p(x^\alpha)$, and so $M_\alpha f \in E_{p,\alpha}$. Then, by Theorems 4 and 3, $S_n(M_\alpha f) \to M_\alpha f$ in $L^p(x^\alpha)$ and almost everywhere. So, we only need to show that $S_n(M_\alpha f) = S_n(f)$, and this is clear because, by (1),

$$\int_0^\infty (M_\alpha f) j_n^{\alpha+\beta} x^\alpha \, dx = \int_0^\infty f\left(M_\alpha j_n^{\alpha+\beta}\right) x^\alpha \, dx = \int_0^\infty f j_n^{\alpha+\beta} x^\alpha \, dx.$$

∎

COROLLARY 6. *Let* $\alpha \geq -\frac{1}{2}, -\frac{1}{2} < \beta < 1$, $\frac{4}{3} < p < 4$ *with* $\alpha + \beta \geq -\frac{1}{2}$,

$$-\frac{1}{4} < (\alpha + \beta + 1)\left(\frac{1}{2} - \frac{1}{p}\right) < \frac{1}{4},$$

$$\max\left\{-\frac{1}{4}, -\frac{1}{4} - \frac{\beta}{2}, -\frac{\alpha+1}{2} - \beta\right\} < (\alpha + 1)\left(\frac{1}{2} - \frac{1}{p}\right) < \min\left\{\frac{1}{4}, \frac{1}{4} - \frac{\beta}{2}\right\}.$$

If $p < 2$, *assume further* $\beta < \frac{1}{2}$ *and* $-\frac{1}{4} + \frac{\beta}{2} < (\alpha + 1)(\frac{1}{2} - \frac{1}{p})$.

Then, $E_{p,\alpha} \cap L^p(x^{\alpha+\beta}) \subset E_{p,\alpha+\beta}$.

Proof. Let $f \in E_{p,\alpha} \cap L^p(x^{\alpha+\beta})$. By Theorems 4 and 3, $S_n f \to f$ almost everywhere. Since $f \in L^p(x^{\alpha+\beta})$, Corollary 5 (with $\alpha+\beta$ instead of α) gives $S_n f \to M_{\alpha+\beta} f$ almost everywhere. Then, $f = M_{\alpha+\beta} f$ almost everywhere, that is, $f \in E_{p,\alpha+\beta}$. ∎

COROLLARY 7. *Let* $\alpha \geq -\frac{1}{2}$, $\beta \in (-\frac{1}{2}, \frac{1}{2})$, $\frac{4}{3} < p < 4$ *with* $\alpha + \beta \geq -\frac{1}{2}$,

$$\max\left\{-\frac{1}{4}, -\frac{1}{4} - \frac{\beta}{2}, -\frac{\alpha+1}{2} - \beta\right\} < (\alpha+1)\left(\frac{1}{2} - \frac{1}{p}\right) < \min\left\{\frac{1}{4}, \frac{1}{4} - \frac{\beta}{2}\right\},$$

$$\max\left\{-\frac{1}{4}, -\frac{1}{4} + \frac{\beta}{2}, -\frac{\alpha+1}{2} + \frac{\beta}{2}\right\} < (\alpha+\beta+1)\left(\frac{1}{2} - \frac{1}{p}\right) < \min\left\{\frac{1}{4}, \frac{1}{4} + \frac{\beta}{2}\right\}.$$

If $p < 2$, *assume further* $-\frac{1}{4} + \frac{\beta}{2} < (\alpha+1)(\frac{1}{2} - \frac{1}{p})$ *and* $-\frac{1}{4} - \frac{\beta}{2} < (\alpha + \beta + 1)(\frac{1}{2} - \frac{1}{p})$.

Then, $E_{p,\alpha} \cap L^p(x^{\alpha+\beta}) = E_{p,\alpha+\beta} \cap L^p(x^{\alpha})$.

Proof. The inclusion "$\subseteq$" is clear by Corollary 6. The inclusion "$\supseteq$" follows also by Corollary 6 with $\alpha+\beta$ instead of α, and β instead of $-\beta$. ∎

Theorem 4 gives different bases for $E_{p,\alpha}$ for different values of β. It seems interesting to obtain the expressions for the change of basis between $\{j_n^{\alpha}\}_{n=0}^{\infty}$ and $\{j_n^{\alpha+\beta}\}_{n=0}^{\infty}$ in $E_{p,\alpha}$.

COROLLARY 8. *Let* $\alpha \geq -\frac{1}{2}$, $-\frac{1}{2} < \beta < 1$, $\frac{4}{3} < p < 4$ *with*

$$\max\left\{-\frac{1}{4}, -\frac{1}{4} - \frac{\beta}{2}, -\frac{\alpha+1}{2} - \beta\right\} < (\alpha+1)\left(\frac{1}{2} - \frac{1}{p}\right) < \min\left\{\frac{1}{4}, \frac{1}{4} - \frac{\beta}{2}\right\}.$$

If $p < 2$, *assume further* $-\frac{1}{4} + \frac{\beta}{2} < (\alpha + 1)(\frac{1}{2} - \frac{1}{p})$. *Then, both* $\{j_n^{\alpha}\}_{n=0}^{\infty}$ *and* $\{j_n^{\alpha+\beta}\}_{n=0}^{\infty}$ *are bases of the space* $E_{p,\alpha}$ *and the change of basis is given by*

$$j_n^{\alpha} = \sum_{k=n}^{\infty} a_{n,k} j_k^{\alpha+\beta}, \qquad j_n^{\alpha+\beta} = \sum_{k=n}^{\infty} b_{n,k} j_k^{\alpha},$$

where

$$a_{n,k} = \frac{2^{\beta}\sqrt{\alpha+2n+1}\,\sqrt{\alpha+\beta+2k+1}\,\Gamma(1-\beta)\Gamma(\alpha+\beta+k+n+1)}{\Gamma(1+k-n)\Gamma(1-\beta-k+n)\Gamma(\alpha+k+n+2)},$$

$$b_{n,k} = \frac{2^{-\beta}\sqrt{\alpha+\beta+2n+1}\,\sqrt{\alpha+2k+1}\,\Gamma(1+\beta)\Gamma(\alpha+k+n+1)}{\Gamma(1+k-n)\Gamma(1+\beta-k+n)\Gamma(\alpha+\beta+k+n+2)}. \tag{9}$$

Proof. By Theorem 4, $B_{p,\alpha,0} = E_{p,\alpha}$, so that $j_n^{\alpha} \in E_{p,\alpha}$. Also, by Theorem 4, $E_{p,\alpha} = B_{p,\alpha,\beta}$. Then, Corollary 2 gives $S_N j_n^{\alpha} \to j_n^{\alpha}$ in $L^p(x^{\alpha})$ as $N \to \infty$, that is,

$$j_n^{\alpha} = \sum_{k=0}^{\infty} a_{n,k} j_k^{\alpha+\beta},$$

in $L^p(x^{\alpha})$, where

$$a_{n,k} = \int_0^{\infty} j_n^{\alpha} j_k^{\alpha+\beta} x^{\alpha+\beta}\, dx.$$

In a similar way, $j_n^{\alpha+\beta} \in B_{p,\alpha,\beta} = B_{p,\alpha,0}$ and $j_n^{\alpha+\beta} = \sum_{k=0}^{\infty} b_{n,k} j_k^{\alpha}$ in $L^p(x^{\alpha})$, where

$$b_{n,k} = \int_0^{\infty} j_n^{\alpha+\beta} j_n^{\alpha} x^{\alpha}\, dx.$$

Finally, [15, Chap. XIII, 13.41 (2), p. 403] gives $a_{n,k} = b_{n,k} = 0$ for $k < n$ and (9) for $k \geq n$. ∎

Similar expressions for the change of basis between different bases $\{j_n^{\alpha+\beta}\}_{n=0}^{\infty}$ and $\{j_n^{\alpha+\beta'}\}_{n=0}^{\infty}$ in $E_{p,\alpha}$ can be obtained. Details are left to the reader.

6. PROOF OF THEOREM 1

6.1. *Necessary conditions*

Let us begin by showing that conditions (6) are necessary for the uniform boundedness in Theorem 1. Assume S_n is uniformly bounded.

Then the operators given by

$$T_n(f,x) = S_n(f,x) - S_{n-1}(f,x) = j_n^{\alpha+\beta}(x)\int_0^\infty f(t)\, j_n^{\alpha+\beta}(t)\, t^{\alpha+\beta}\, dt$$

are uniformly bounded as well, i.e.,

$$\|T_n f\|_{L^p(x^\alpha)} = \left|\int_0^\infty f(t)\, j_n^{\alpha+\beta}(t)\, t^{\alpha+\beta}\, dt\right| \|j_n^{\alpha+\beta}\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)},$$

with a constant C independent of n and f. By duality, this means

$$\|t^\beta j_n^{\alpha+\beta}\|_{L^q(t^\alpha)} \|j_n^{\alpha+\beta}\|_{L^p(x^\alpha)} \le C,$$

where $1/p + 1/q = 1$. Taking n fixed (it suffices $n = 0$) and applying the first part of Lemma 1 gives (6). Now, provided (6) holds, the norm estimates of Lemma 1 give

$$\|j_n^{\alpha+\beta}\|_{L^p(x^\alpha)} \sim \begin{cases} n^{-(\alpha+\beta+1)+2(\alpha+1)/p}, & \text{if } p < 4, \\ n^{-(\alpha+2\beta+1)/2}(\log n)^{1/4}, & \text{if } p = 4, \\ n^{-(5/6+\alpha+\beta)+(6\alpha+4)/(3p)}, & \text{if } p > 4, \end{cases}$$

and

$$\|x^\beta j_n^{\alpha+\beta}\|_{L^q(x^\alpha)} \sim \begin{cases} n^{-(\alpha-\beta+1)+2(\alpha+1)/q}, & \text{if } q < 4 \left(\text{i.e., } p > \tfrac{4}{3}\right), \\ n^{-(\alpha-2\beta+1)/2}(\log n)^{1/4}, & \text{if } q = 4 \left(\text{i.e., } p = \tfrac{4}{3}\right), \\ n^{-(5/6+\alpha-\beta)+(6\alpha+4)/(3q)}, & \text{if } q > 4 \left(\text{i.e., } p < \tfrac{4}{3}\right). \end{cases}$$

This implies $\frac{4}{3} < p < 4$.

6.2. *Sufficient conditions*

Let us now assume $\frac{4}{3} < p < 4$ and (6), and prove the uniform boundedness of the partial sum operators S_n. They can be written as

$$S_n(f,x) = \int_0^\infty f(t) K_n(x,t)\, t^{\alpha+\beta}\, dt,$$

where

$$K_n(x,t) = \sum_{k=0}^{n} j_k^{\alpha+\beta}(x)\, j_k^{\alpha+\beta}(t).$$

The next lemma gives a suitable decomposition of the kernel $K_n(x,t)$ associated to S_n. For a similar formula, with a different proof, see [14].

LEMMA 4. *Let $n \in \mathbb{N}$ and $\lambda > -1$. Then*

$$\sum_{k=0}^{n} 2(\lambda + 2k + 1) J_{\lambda+2k+1}(x) J_{\lambda+2k+1}(t)$$

$$= \frac{xt}{x^2 - t^2} \big[xJ_{\lambda+1}(x) J_\lambda(t) - tJ_\lambda(x) J_{\lambda+1}(t)$$

$$+ xJ'_{\lambda+2n+2}(x) J_{\lambda+2n+2}(t) - tJ_{\lambda+2n+2}(x) J'_{\lambda+2n+2}(t) \big].$$

Proof. Using the equality $J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)$ (see [15, Chap. III, 3.2, p. 45]) to express $J_{\mu-1}$ and $J_{\mu+2}$ in terms of J_μ and $J_{\mu+1}$ yields the formula

$$\frac{xt}{x^2 - t^2} \big[xJ_\mu(x) J_{\mu-1}(t) - tJ_{\mu-1}(x) J_\mu(t)$$

$$- xJ_{\mu+2}(x) J_{\mu+1}(t) + tJ_{\mu+1}(x) J_{\mu+2}(t) \big]$$

$$= 2\mu J_\mu(x) J_\mu(t).$$

This now gives

$$\sum_{k=0}^{n} 2(\lambda + 2k + 1) J_{\lambda+2k+1}(x) J_{\lambda+2k+1}(t)$$

$$= \frac{xt}{x^2 - t^2} \big[xJ_{\lambda+1}(x) J_\lambda(t) - tJ_\lambda(x) J_{\lambda+1}(t)$$

$$- xJ_{\lambda+2n+3}(x) J_{\lambda+2n+2}(t) + tJ_{\lambda+2n+2}(x) J_{\lambda+2n+3}(t) \big].$$

Finally, use the formula $zJ_{\nu+1}(z) = \nu J_\nu(z) - zJ'_\nu(z)$ (see [15, Chap. III, 3.2, p. 45]) to take out $J_{\lambda+2n+3}$. ∎

From the definition, we have

$$S_n(f,x) = x^{-\alpha/2-\beta/2-1/2}$$

$$\times \int_0^\infty \left(\sum_{k=0}^{n} (\alpha + \beta + 2k + 1) J_{\alpha+\beta+2k+1}(\sqrt{x}) J_{\alpha+\beta+2k+1}(\sqrt{t}) \right)$$

$$\times t^{\alpha/2+\beta/2-1/2} f(t)\, dt$$

so that Lemma 4 with $\lambda = \alpha + \beta$ leads to

$$S_n f = W_1 f - W_2 f + W_{3,n} f - W_{4,n} f,$$

where

$$W_1(f,x) = \tfrac{1}{2} x^{-\alpha/2-\beta/2+1/2} J_{\alpha+\beta+1}(x^{1/2}) H\big(t^{\alpha/2+\beta/2} J_{\alpha+\beta}(t^{1/2}) f(t), x\big),$$

$$W_2(f,x) = \tfrac{1}{2} x^{-\alpha/2-\beta/2} J_{\alpha+\beta}(x^{1/2}) H\big(t^{\alpha/2+\beta/2+1/2} J_{\alpha+\beta+1}(t^{1/2}) f(t), x\big),$$

$$W_{3,n}(f,x) = \tfrac{1}{2} x^{-\alpha/2-\beta/2+1/2} J'_{\nu}(x^{1/2}) H\big(t^{\alpha/2+\beta/2} J_{\nu}(t^{1/2}) f(t), x\big),$$

$$W_{4,n}(f,x) = \tfrac{1}{2} x^{-\alpha/2-\beta/2} J_{\nu}(x^{1/2}) H\big(t^{\alpha/2+\beta/2+1/2} J'_{\nu}(t^{1/2}) f(t), x\big),$$

and $\nu = \lambda + 2n + 2 = \alpha + \beta + 2n + 2$. Here, H denotes the Hilbert transform on $(0,\infty)$, which is defined by

$$H(g,x) = \int_0^\infty \frac{g(t)}{x-t}\,dt$$

(the integral must be considered as a principal value).

Thus, we can conclude that the partial sum operators S_n are uniformly bounded if we can prove that the operators W_1, W_2 are bounded and the operators $W_{3,n}, W_{4,n}$ are uniformly bounded for $n \geq 0$. We use good estimates for the Bessel functions and the A_p theory of weights to prove the boundedness of the Hilbert transform.

Let us start with the bounds for the Bessel functions and their derivatives.

From the estimates (2) and (3) it follows that, for $\mu > -1$,

$$|J_\mu(x)| \leq C_\mu x^\mu, \qquad x \in (0,1], \tag{10}$$

$$|J_\mu(x)| \leq C_\mu x^{-1/2}, \qquad x \in [1,\infty), \tag{11}$$

with a C_μ constant depending on μ.

Moreover, we need bounds for the Bessel functions $J_{\alpha+\beta+2n+2}$ (and their derivatives) with constants independent of n. So, we make use of the bounds

$$|J_\nu(x)| \leq C x^{-1/4} (|x-\nu| + \nu^{1/3})^{-1/4}, \qquad x \in (0,\infty), \tag{12}$$

$$|J'_\nu(x)| \leq C x^{-3/4} (|x-\nu| + \nu^{1/3})^{1/4}, \qquad x \in (0,\infty), \tag{13}$$

with some constant C independent of ν. They follow from those of [1, 2], for instance, and were already used in [14].

6.3. *Some results on Hilbert transforms and A_p theory*

To analyze the boundedness of the Hilbert transform, some notation and previous results are necessary. As usual, for $1 < p < \infty$ we write $q = p/(p-1)$, i.e., $1/p + 1/q = 1$. A weight is a nonnegative Lebesgue-measurable function on $(0,\infty)$. The class $A_p(0,\infty)$ [A_p, for short] consists of those weights w such that, for every subinterval $I \subseteq (0,\infty)$,

$$\left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{-q/p}\right)^{p/q} \le C,$$

where C is a positive constant independent of I, and $|I|$ denotes the length of I. The A_p constant of w is the least constant C verifying this inequality and is denoted by $A_p(w)$. We refer the reader to [6] for further details on A_p classes.

Fix $1 < p < \infty$; then the Hilbert transform H is a bounded linear operator on $L^p(w)$, for any weight $w \in A_p$. The norm of $H\colon L^p(w) \to L^p(w)$ and the A_p constant of w depend only on each other, in the sense that given some constant C which verifies the A_p condition for w, another constant C_1 depending only on C can be chosen so that $\|H\| \le C_1$, and vice versa. Therefore, for a sequence $\{w_n\}_{n\in\mathbb{N}}$ uniformly in A_p, i.e., with some constant C verifying the A_p condition for every w_n, the Hilbert transform is uniformly bounded on $L^p(w_n)$, $n \in \mathbb{N}$.

Let us see some auxiliary results related with A_p weights:

LEMMA 5. *Let u, v, w be weights on $(0,\infty)$ and γ be a positive constant. Then*

(a) *$w(x) \in A_p$ if and only if $w(\gamma x) \in A_p$; both weights have the same A_p constant.*

(b) *$w \in A_p$ if and only if $\gamma w \in A_p$; both weights have also the same A_p constant.*

(c) *If $u, v \in A_p$, then $u + v \in A_p$ and $A_p(u+v) \le A_p(u) + A_p(v)$.*

(d) *If $u, v \in A_p$ and $1/w = 1/u + 1/v$, then $w \in A_p$ and $A_p(w) \le C[A_p(u) + A_p(v)]$.*

Proof. Parts (a) and (b) are trivial. Part (c) follows easily from the inequality

$$\left(\frac{1}{|I|}\int_I (u+v)^{-q/p}\right)^{p/q} \le \min\left\{\left(\frac{1}{|I|}\int_I u^{-q/p}\right)^{p/q}, \left(\frac{1}{|I|}\int_I v^{-q/p}\right)^{p/q}\right\}.$$

Part (d) is a consequence of (c) and the fact that $u \in A_p \Leftrightarrow u^{-q/p} \in A_q$, with $A_q(u^{-q/p}) = [A_p(u)]^{q/p}$. ∎

The proof of the next lemma is not difficult, but cumbersome, so we omit it. For the weight in (c), observe that $x^r|x^{1/2}-1|^2 \sim x^r$ near 0, $x^r|x^{1/2}-1|^s \sim |x-1|^s$ near 1 and $x^r|x^{1/2}-1|^s \sim x^{r+s/2}$ near ∞, whence the three conditions follow.

LEMMA 6. *Let $r, s \in \mathbb{R}$. We have*

(a) $x^r \in A_p \Leftrightarrow -1 < r < p-1$.

(b) *Set* $\Phi(x) = x^r$ *if* $x \in (0,1)$ *and* $\Phi(x) = x^s$ *if* $x \in (1,\infty)$. *Then,* $\Phi \in A_p$ *if and only if* $-1 < r < p-1$ *and* $-1 < s < p-1$.

(c) $x^r|x^{1/2}-1|^s \in A_p \Leftrightarrow -1 < r < p-1$, $-1 < s < p-1$, *and* $-1 < r + s/2 < p-1$.

To simplify the notation, in the rest of this section we write $\lambda = \alpha + \beta$ and $\nu = \alpha + \beta + 2n + 2$.

6.4. *Boundedness of the operators W_1 and W_2*

From the definition, it follows that

$$\|W_1 f\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)},$$

if and only if

$$\|Hg\|_{L^p(x^{\alpha-\lambda p/2+p/2}|J_{\lambda+1}(x^{1/2})|^p)} \le C\|g\|_{L^p(x^{\alpha-\lambda p/2}|J_\lambda(x^{1/2})|^{-p})}.$$

Proving that there is a weight $\Phi \in A_p$ with

$$Cx^{\alpha-\lambda p/2+p/2}\left|J_{\lambda+1}(x^{1/2})\right|^p \le \Phi(x) \le C_1 x^{\alpha-\lambda p/2}\left|J_\lambda(x^{1/2})\right|^{-p} \quad (14)$$

are enough. According to the bounds (10) and (11), we have

$$x^{\alpha-\lambda p/2+p/2}\left|J_{\lambda+1}(x^{1/2})\right|^p \le \begin{cases} Cx^{\alpha+p}, & \text{if } x \in (0,1), \\ Cx^{\alpha-\lambda p/2+p/4}, & \text{if } x \in (1,\infty), \end{cases}$$

$$x^{\alpha-\lambda p/2}\left|J_\lambda(x^{1/2})\right|^{-p} \ge \begin{cases} Cx^{\alpha-\lambda p}, & \text{if } x \in (0,1), \\ Cx^{\alpha-\lambda p/2+p/4}, & \text{if } x \in (1,\infty). \end{cases}$$

Let us try

$$\Phi(x) = \begin{cases} x^r, & \text{if } x \in (0,1), \\ x^{\alpha-\lambda p/2+p/4}, & \text{if } x \in (1,\infty). \end{cases}$$

By (b) in Lemma 6, conditions (14) and $\Phi \in A_p$ hold if

$$\begin{gathered} \alpha - \lambda p \le r \le \alpha + p, \\ -1 < r < p - 1, \\ -1 < \alpha - \lambda p/2 + p/4 < p - 1. \end{gathered} \tag{15}$$

The third line is equivalent to

$$\frac{-1}{4} < (\alpha + 1)\left(\frac{1}{2} - \frac{1}{p}\right) + \frac{\beta}{2} < \frac{3}{4},$$

which follows from (6). For the inequalities in (15) involving r, it suffices to show that

$$\max\{-1, \alpha - \lambda p\} < \min\{p - 1, \alpha + p\}.$$

This follows from $\alpha > -1$, $\alpha + \beta > -1$, $1 < p < \infty$, and (6), as well. The case of W_2 is entirely similar.

6.5. *Uniform boundedness of the operators* $W_{3,n}$

Here,

$$\|W_{3,n} f\|_{L^p(x^\alpha)} \le C \|f\|_{L^p(x^\alpha)},$$

if and only if

$$\|Hg\|_{L^p(x^{\alpha - \lambda p/2 + p/2} |J'_\nu(x^{1/2})|^p)} \le C \|g\|_{L^p(x^{\alpha - \lambda p/2} |J_\nu(x^{1/2})|^{-p})}.$$

From the bounds (13) and (12),

$$x^{\alpha - \lambda p/2 + p/2} \left| J'_\nu(x^{1/2}) \right|^p \le C x^{\alpha - \lambda p/2 + p/8} \left[|x^{1/2} - \nu| + \nu^{1/3} \right]^{p/4},$$

$$x^{\alpha - \lambda p/2} \left| J_\nu(x^{1/2}) \right|^{-p} \ge C x^{\alpha - \lambda p/2 + p/8} \left[|x^{1/2} - \nu| + \nu^{1/3} \right]^{p/4}.$$

It suffices to prove that $\varphi_\nu \in A_p$ uniformly in n (recall that $\nu = \alpha + \beta + 2n + 2$), with

$$\varphi_\nu(x) = x^{\alpha - \lambda p/2 + p/8} \left[|x^{1/2} - \nu| + \nu^{1/3} \right]^{p/4}.$$

From Lemma 5, we have

$$\begin{aligned} \varphi_\nu(x) \in A_p \text{ unif.} \quad &\Leftrightarrow \quad \varphi_\nu(\nu^2 x) \in A_p \text{ unif.} \\ &\Leftrightarrow \quad x^{\alpha - \lambda p/2 + p/8} \left[|x^{1/2} - 1| + \nu^{-2/3} \right]^{p/4} \in A_p \text{ unif.} \\ &\Leftrightarrow \quad x^{\alpha - \lambda p/2 + p/8} |x^{1/2} - 1|^{p/4} \\ &\qquad + \nu^{-p/6} x^{\alpha - \lambda p/2 + p/8} \in A_p \text{ unif.}, \end{aligned}$$

where the last equivalence follows from

$$\left[|x^{1/2}-1|+\nu^{-2/3}\right]^{p/4}\sim|x^{1/2}-1|^{p/4}+\nu^{-p/6}.$$

Now, Lemmas 5 and 6 prove that those weights belong to A_p uniformly.

6.6. *Uniform boundedness of the operators* $W_{4,n}$

Finally,

$$\|W_{4,n}f\|_{L^p(x^\alpha)}\le C\|f\|_{L^p(x^\alpha)},$$

if and only if

$$\|Hg\|_{L^p(x^{\alpha-\lambda p/2}|J_\nu(x^{1/2})|^p)}\le C\|g\|_{L^p(x^{\alpha-\lambda p/2-p/2}|J'_\nu(x^{1/2})|^{-p})}.$$

Also,

$$x^{\alpha-\lambda p/2}\left|J_\nu(x^{1/2})\right|^p\le Cx^{\alpha-\lambda p/2-p/8}\left[|x^{1/2}-\nu|+\nu^{1/3}\right]^{-p/4},$$
$$x^{\alpha-\lambda p/2-p/2}\left|J'_\nu(x^{1/2})\right|^{-p}\ge Cx^{\alpha-\lambda p/2-p/8}\left[|x^{1/2}-\nu|+\nu^{1/3}\right]^{-p/4},$$

so let us put

$$\psi_\nu(x)=x^{\alpha-\lambda p/2-p/8}\left[|x^{1/2}-\nu|+\nu^{1/3}\right]^{-p/4},$$

and show that $\psi_\nu\in A_p$ uniformly in n. Indeed,

$$\begin{aligned}\psi_\nu(x)\in A_p\text{ unif.}\quad&\Leftrightarrow\quad\psi_\nu(\nu^2x)\in A_p\text{ unif.}\\&\Leftrightarrow\quad x^{\alpha-\lambda p/2-p/8}\left[|x^{1/2}-1|+\nu^{-2/3}\right]^{-p/4}\in A_p\text{ unif.},\end{aligned}$$

and

$$\begin{aligned}&\left(x^{\alpha-\lambda p/2-p/8}\left[|x^{1/2}-1|+\nu^{-2/3}\right]^{-p/4}\right)^{-1}\\&\quad\sim x^{-\alpha+\lambda p/2+p/8}\left[|x^{1/2}-1|^{p/4}+\nu^{-p/6}\right]\\&\quad=\left[x^{\alpha-\lambda p/2-p/8}|x^{1/2}-1|^{-p/4}\right]^{-1}+\left[\nu^{p/6}x^{\alpha-\lambda p/2-p/8}\right]^{-1},\end{aligned}$$

so that Lemmas 5 and 6 lead to the desired conclusion. The proof of Theorem 1 is now complete.

7. PROOF OF THEOREM 3

We only need to prove that $S_n(f,x)$ converges to some $g(x)$ almost everywhere. This, together with Corollary 2, gives $g = f$ almost everywhere. Now, recall that $S_n f = \sum_{k=0}^{n} c_k(f) j_k^{\alpha+\beta}$, where

$$c_k(f) = \int_0^\infty f(t) j_k^{\alpha+\beta}(t) t^{\alpha+\beta}\, dt.$$

It follows from Lemma 1 that $x^\beta j_n^{\alpha+\beta} \in L^q(x^\alpha)$; moreover, $\|x^\beta j_n^{\alpha+\beta}\|_{L^q(x^\alpha)} \le Cn^\delta$ for some constant $\delta = \delta(p, \alpha, \beta)$. Thus,

$$|c_n(f)| \le \|f\|_{L^p(x^\alpha)} \|x^\beta j_n^{\alpha+\beta}\|_{L^q(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)} n^\delta.$$

Now, according to [15, Chap. III, 3.31 (1), p. 49] we have

$$|J_\nu(x)| \le \frac{2^{-\nu} x^\nu}{\Gamma(\nu+1)}, \qquad \nu > -\frac{1}{2}.$$

Therefore,

$$\begin{aligned} |j_n^{\alpha-\beta}(x)| &= \sqrt{\alpha+\beta+2n+1}\, \left|J_{\alpha+\beta+2n+1}(\sqrt{x})\right| x^{-(\alpha+\beta+1)/2} \\ &\le \frac{\sqrt{\alpha+\beta+2n+1}\, 2^{-(\alpha+\beta+2n+1)} x^n}{\Gamma(\alpha+\beta+2n+2)}, \end{aligned}$$

so that

$$|c_n(f) j_n^{\alpha+\beta}(x)| \le C\|f\|_{L^p(x^\alpha)} \frac{n^{\delta+1/2}(x/4)^n}{\Gamma(\alpha+\beta+2n+2)}$$

and the series $\sum_{n=0}^{\infty} c_n(f) j_n^{\alpha+\beta}(x)$ converges.

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REFERENCES

1. J. A. Barceló and A. Córdoba, Band-limited functions: L^p-convergence, *Trans. Amer. Math. Soc.* **313** (1989), 655–669.
2. A. Córdoba, The disc multiplier, *Duke Math. J.* **58** (1989), 21–27.
3. A. J. Durán, On Hankel transform, *Proc. Amer. Math. Soc.* **110** (1990), 417–424.

4. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, "Higher Transcendental Functions," Vol. 2, McGraw-Hill, New York, 1953.
5. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, "Tables of Integral Transforms," Vol. 2, McGraw-Hill, New York, 1954.
6. J. García-Cuerva and J. L. Rubio de Francia, "Weighted Norm Inequalities and Related Topics," Mathematical Studies, Vol. 116, North-Holland, Amsterdam, 1985.
7. C. S. Herz, On the mean inversion of Fourier and Hankel transforms, *Proc. Nat. Acad. Sci. U.S.A.* **40** (1954), 996–999.
8. J. M. Méndez and M. M. Socas, A pair of generalized Hankel–Clifford transformations and their applications, *J. Math. Anal. Appl.* **154** (1991), 543–557.
9. B. Muckenhoupt, Mean convergence of Jacobi series, *Proc. Amer. Math. Soc.* **23** (1969), 306–310.
10. H. Pollard, The mean convergence of orthogonal series. III, *Duke Math. J.* **16** (1949), 189–191.
11. K. Stempak, A weighted uniform L^p estimate of Bessel functions: A note on a paper of K. Guo: "A uniform L^p estimate of Bessel functions and distributions supported on S^{n-1}," *Proc. Amer. Math. Soc.*, to appear.
12. K. Stempak and W. Trebels, Hankel multipliers and transplanting operators, *Studia Math.* **126** (1997), 51–66.
13. G. Szegö, "Orthogonal Polynomials," 4th ed., American Mathematical Society Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, RI, 1975.
14. J. L. Varona, Fourier series of functions whose Hankel transform is supported on $[0, 1]$, *Constr. Approx.* **10** (1994), 65–75.
15. G. N. Watson, "A Treatise on the Theory of Bessel Functions," 2nd ed., Cambridge Univ. Press, Cambridge, U.K., 1944.