COMMUTATORS AND ANALYTIC DEPENDENCE OF FOURIER-BESSEL SERIES ON $(0,\infty)$

José J. Guadalupe, Mario Pérez and Juan L. Varona

Abstract. In this paper we study the boundedness of the commutators $[b, S_n]$ where b is a BMO function and S_n denotes the n-th partial sum of the Fourier-Bessel series on $(0,\infty)$. Perturbing the measure by $\exp(2b)$ we obtain that certain operators related to S_n depend analytically on the functional parameter b.

0. Introduction.

Let J_α be the Bessel function of order $\alpha > -1$. The formula

$$\int_0^\infty J_{\alpha+2n+1}(x) J_{\alpha+2m+1}(x)\,\frac{dx}{x} = \begin{cases} 0, & \text{if } n \neq m \\ 2^{-1}(\alpha+2n+1)^{-1}, & \text{if } n = m \end{cases}$$

(see [14, XIII.13.41 (7), p. 404] and [14, XIII.13.42 (1), p. 405]) provides an orthonormal system $(j_n^\alpha)_{n\ge 0}$ in $L^2((0,\infty), x^\alpha\,dx)$ [$L^2(x^\alpha)$, from now on], given by

$$j_n^\alpha(x) = \sqrt{\alpha+2n+1}\,J_{\alpha+2n+1}(\sqrt{x})\,x^{-\alpha/2-1/2}.$$

In this paper we consider the Fourier expansion associated with this orthonormal system, which is usually referred to as the Fourier-Bessel series on $(0,\infty)$. For any suitable function f and any $n \ge 0$, the n-th partial sum of this expansion is given by

$$S_n f = \sum_{k=0}^n c_k(f) j_k^\alpha, \qquad c_k(f) = \int_0^\infty f(t) j_k^\alpha(t) t^\alpha\,dt.$$

We also consider the commutator of the Fourier-Bessel series on $(0,\infty)$ and the multiplication operator associated to a BMO function; this is defined, for any given $b \in \mathrm{BMO}$ and $n \ge 0$, as

$$[b, S_n]f = bS_n(f) - S_n(bf).$$

In the case $\alpha \ge -1/2$, one of the authors proved in [13] that the Fourier-Bessel series is bounded in $L^p(x^\alpha)$, i.e., there exists some constant $C > 0$ (depending on α and p) such that for every $n \ge 0$ and every $f \in L^p(x^\alpha)$,

$$\|S_n f\|_{L^p(x^\alpha)} \le C \|f\|_{L^p(x^\alpha)},$$

1991 *Mathematics Subject Classification.* Primary 42C10.
Key words and phrases. Fourier-Bessel series, commutators, BMO, A_p weights.
Research supported by DGICYT and by UR.
THIS PAPER HAS BEEN PUBLISHED IN: *Canad. Math. Bull.* **42** (1999), 198–208.

if and only if $\max\{4/3, 4(\alpha+1)/(2\alpha+3)\} < p < \min\{4, 4(\alpha+1)/(2\alpha+1)\}$. In Theorem 1 we will extend this result to the case $\alpha > -1$ and prove the corresponding inequality for the commutator $[b, S_n]$, $b \in \mathrm{BMO}$.

Regarding the commutator $[b, S_n]$, results of this type are of independent interest and have been widely studied for many classical operators; see [2, 10, 11, 12, 4], for instance.

In our case, the commutator $[b, S_n]$ is closely related to the problem of perturbating the orthonormal system. Given an orthonormal system $(\varphi_n)_{n\ge 0}$ in some $L^2(\nu)$ space and a suitable function b (in some sense close to 0), the classical Gram-Schmidt procedure can be applied to $(\varphi_n)_{n\ge 0}$ so as to obtain a new orthonormal system in $L^2(e^{2b}d\nu)$, which we will refer to as a perturbated system. In this natural way a mapping can be defined that associates a perturbated system (and a perturbated orthogonal expansion) to each (small) function b. For different compact perturbations of orthogonal polynomial systems and further references, see [7, 9, 1].

Let us take the system $(j_n^\alpha)_{n\ge 0}$ in $L^2(x^\alpha)$ as our starting point. Let $\mathbf{S}_n(b)$ stand for the n-th partial sum operator of the Fourier series associated to the perturbed measure $e^{2b}x^\alpha\,dx$ in the aforementioned way. Once the boundedness properties of $S_n = \mathbf{S}_n(0)$ have been established, it is interesting to study the mapping $b \mapsto \mathbf{S}_n(b)$. This is not, however, a convenient setting, since each perturbed series $\mathbf{S}_n(b)$ acts on a different space $L^2(e^{2b}x^\alpha)$. Instead, we can consider the operators

$$V_n(b) = e^b \mathbf{S}_n(b) e^{-b}.$$

Now, each $V_n(b)$ acts on $L^2(x^\alpha)$ and its norm coincides with the operator norm of $\mathbf{S}_n(b)$ acting on $L^2(e^{2b}x^\alpha)$. The problem is further simplified if we take the operators

$$T_n(b) = e^b \mathbf{S}_n(0) e^{-b},$$

i.e., $T_n(b)f = e^b S_n(e^{-b}f)$. Indeed, it has been proved in [3] that the family $(V_n(b))_{n\ge 0}$ depends analytically on b belonging to a neighbourhood of 0 in the complexification of BMO whenever the family $(T_n(b))_{n\ge 0}$ does too.

We will prove in Theorem 2 that the family of operators $(T_n(b))_{n\ge 0}$ acting on $L^2(x^\alpha)$ is uniformly bounded for b belonging to some neighbourhood of 0 in the complexification of BMO. As a consequence (see [3, Propositions 2.1 and 2.3]), the operator-valued mappings $(T_n)_{n\ge 0}$ are uniformly analytic in a neighbourhood of 0 in the complexification of BMO and so are $(V_n)_{n\ge 0}$.

Now, the connection between $[b, S_n]$ and the perturbated Fourier series comes via the Gâteaux differential of T_n at 0 in the direction b:

$$\frac{d}{dz} T_n(zb)\big|_{z=0} = [b, S_n].$$

In this way, the uniform analyticity of T_n in a neighbourhood of 0 gives the L^2-boundedness of $[b, S_n]$.

1. Main results.

If b is a locally Lebesgue-integrable function on $(0,\infty)$, the mean of b on an interval $I \subseteq (0,\infty)$ is

$$b_I = \frac{1}{|I|}\int_I b(x)\,dx.$$

The function b is said to have bounded mean oscillation on $(0,\infty)$ if

$$\|b\|_{\mathrm{BMO}} = \sup_I \frac{1}{|I|}\int_I |b(x)-b_I|\,dx$$

is finite, where the supremum is taken over all the intervals $I \subseteq (0,\infty)$. The space BMO of real-valued functions (modulo constants) having bounded mean oscillation on $(0,\infty)$ is a real Banach space with $\|\cdot\|_{\mathrm{BMO}}$ as its norm.

Theorem 1. *Let $1<p<\infty$, $-1<\alpha$ such that*

$$\begin{cases} 4/3<p<4, & \text{if } -1<\alpha<0;\\ \dfrac{4(\alpha+1)}{2\alpha+3}<p<\dfrac{4(\alpha+1)}{2\alpha+1}, & \text{if } 0\le\alpha. \end{cases}$$

(a) *There exists some constant $C>0$ such that, for every $f\in L^p(x^\alpha)$ and $n\ge 0$,*

$$\|S_n f\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}.$$

(b) *If $b\in\mathrm{BMO}$, then there exists some constant $C>0$ such that, for every $f\in L^p(x^\alpha)$ and $n\ge 0$,*

$$\|[S_n,b]f\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}.$$

Throughout this paper, we will denote by C a positive constant which is independent of n and f, but may be different in each occurrence, even within the same formula.

Theorem 2. *Let $1<p<\infty$, $-1<\alpha$ such that*

$$\begin{cases} 4/3<p<4, & \text{if } -1<\alpha<0;\\ \dfrac{4(\alpha+1)}{2\alpha+3}<p<\dfrac{4(\alpha+1)}{2\alpha+1}, & \text{if } 0\le\alpha. \end{cases}$$

Then there exist some $C,\delta>0$ such that, for all $b\in\mathrm{BMO}$ with $\|b\|_{\mathrm{BMO}}<\delta$,

$$\sup_n \|T_n(b)\|_{L^p(x^\alpha)\to L^p(x^\alpha)} \le C.$$

The next corollary is just a consequence of Theorem 2 and [3, Prop. 2.3].

Corollary. *The sequences of operators $(T_n(b))_{n\ge0}$ and $(V_n(b))_{n\ge0}$, acting on the space $L^2(x^\alpha)$, are uniformly analytic in a neighbourhood of* 0 *in the complexification of* BMO.

Some notation and previous results will be necessary. For $1<p<\infty$, we write $p'=p/(p-1)$, i.e., $1/p+1/p'=1$. A weight is a nonnegative Lebesgue-measurable function on $(0,\infty)$. The class $A_p(0,\infty)$ [A_p, for short] consists of those pairs of weights (u,v) such that, for every subinterval $I\subseteq(0,\infty)$,

$$\frac{1}{|I|}\int_I u\left(\frac{1}{|I|}\int_I v^{-p'/p}\right)^{p/p'} \le C,$$

where C is a positive constant independent of I, and $|I|$ denotes the length of I. The A_p constant of (u,v) is the least constant C verifying this inequality and will be denoted by $A_p(u,v)$. A single weight w is said to belong to A_p if $(w,w) \in A_p$; in this case we denote the constant by $A_p(w)$. We refer the reader to [6] for further details on A_p classes.

The Hilbert transform on $(0,\infty)$ will be denoted by H. Fix $1 < p < \infty$; then H is a bounded linear operator on $L^p(w)$, for any weight $w \in A_p$. The norm of $H\colon L^p(w) \longrightarrow L^p(w)$ and the A_p constant of w depend only one on another, in the sense that given some constant C which verifies the A_p condition for w, another constant C_1 depending only on C can be chosen so that $\|H\| \le C_1$, and viceversa. Therefore, for a sequence $(w_n)_{n\in\mathbb{N}}$ uniformly in A_p, i.e., with some constant C verifying the A_p condition for every w_n, the Hilbert transform is uniformly bounded on $L^p(w_n)$, $n \in \mathbb{N}$. We refer the reader again to [6] for further details.

Also, if (u,v) is a pair of weights such that $C_1 u \le w \le C_2 v$ for some $w \in A_p$, we deduce that H is a bounded operator from $L^p(v)$ into $L^p(u)$. The existence of such a weight w is equivalent to $(u^\delta, v^\delta) \in A_p$ for some $\delta > 1$ (see [8]). For short, this is written as $(u,v) \in A_p^\delta$.

Analogous results hold also with the commutator $[b,H]$, for any $b \in \mathrm{BMO}$ (see [2], for instance). Namely, given $b \in \mathrm{BMO}$ and $w \in A_p$, $[b,H]$ is a bounded operator on $L^p(w)$ with a norm that depends only on the BMO-norm of b and the A_p constant of w, in the sense above.

2. Proofs.

Let us start with some auxiliary results:

Lemma 1. *Let u, v, w be weights on $(0,+\infty)$, $\lambda > 0$.*

(a) *$w(x) \in A_p$ if and only if $w(\lambda x) \in A_p$; both weights have the same A_p constant.*
(b) *$w \in A_p$ if and only if $\lambda w \in A_p$; both weights have also the same A_p constant.*
(c) *If $u, v \in A_p$, then $u+v \in A_p$ and $A_p(u+v) \le A_p(u) + A_p(v)$.*
(d) *If $u, v \in A_p$ and $1/w = 1/u + 1/v$, then $w \in A_p$ and $A_p(w) \le C[A_p(u) + A_p(v)]$.*

Proof. Parts (a) and (b) are trivial. Part (c) follows easily from the inequality

$$\left(\frac{1}{|I|}\int_I (u+v)^{-p'/p}\right)^{p/p'} \le \min\left\{ \left(\frac{1}{|I|}\int_I u^{-p'/p}\right)^{p/p'}, \left(\frac{1}{|I|}\int_I v^{-p'/p}\right)^{p/p'} \right\}.$$

Part (d) is a consequence of (c) and the fact that $u \in A_p \iff u^{-p'/p} \in A_{p'}$, with $A_{p'}(u^{-p'/p}) = [A_p(u)]^{p'/p}$. □

The proof of the next lemma is not difficult, but cumbersome, so we omit it. For the weight in (c), observe that $x^r|x^{1/2}-1|^s \sim x^r$ near 0, $x^r|x^{1/2}-1|^s \sim |x-1|^s$ near 1 and $x^r|x^{1/2}-1|^s \sim x^{r+s/2}$ near ∞, whence the three conditions follow.

Lemma 2. *Let $r, s \in \mathbb{R}$.*

(a) *$x^r \in A_p \iff -1 < r < p-1$.*
(b) *Set $\Phi(x) = x^r$ if $x \in (0,1)$ and $\Phi(x) = x^s$ if $x \in (1,\infty)$. Then, $\Phi \in A_p$ if and only if $-1 < r < p-1$ and $-1 < s < p-1$.*
(c) *$x^r|x^{1/2}-1|^s \in A_p \iff -1 < r < p-1$, $-1 < s < p-1$ and $-1 < r + s/2 < p-1$.*

Lemma 3. *Let $n \in \mathbb{N}$, $\alpha > -1$. Then*

$$\begin{aligned}\sum_{k=0}^{n} & 2(\alpha+2k+1)J_{\alpha+2k+1}(x)J_{\alpha+2k+1}(t) \\ &= \frac{xt}{x^2-t^2}\,[xJ_{\alpha+1}(x)J_\alpha(t) - tJ_\alpha(x)J_{\alpha+1}(t) \\ &\qquad + xJ'_{\alpha+2n+2}(x)J_{\alpha+2n+2}(t) - tJ_{\alpha+2n+2}(x)J'_{\alpha+2n+2}(t)].\end{aligned}$$

Proof. Using the equality $J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z)$ (see [14, III.3.2, p. 45]) to express $J_{\mu-1}$ and $J_{\mu+2}$ in terms of J_μ and $J_{\mu+1}$ proves the formula

$$\begin{aligned}\frac{xt}{x^2-t^2}\,[xJ_\mu(x)J_{\mu-1}(t) - tJ_{\mu-1}(x)J_\mu(t) - xJ_{\mu+2}(x)J_{\mu+1}(t) + tJ_{\mu+1}(x)J_{\mu+2}(t)] \\ = 2\mu J_\mu(x)J_\mu(t).\end{aligned}$$

This gives now

$$\begin{aligned}\sum_{k=0}^{n} & 2(\alpha+2k+1)J_{\alpha+2k+1}(x)J_{\alpha+2k+1}(t) \\ &= \frac{xt}{x^2-t^2}\,[xJ_{\alpha+1}(x)J_\alpha(t) - tJ_\alpha(x)J_{\alpha+1}(t) \\ &\qquad - xJ_{\alpha+2n+3}(x)J_{\alpha+2n+2}(t) + tJ_{\alpha+2n+2}(x)J_{\alpha+2n+3}(t)].\end{aligned}$$

Finally, use the formula $zJ_{\nu+1}(z) = \nu J_\nu(z) - zJ'_\nu(z)$ (see [14, III.3.2, p. 45]) to take out $J_{\alpha+2n+3}$. □

Proof of Theorem 1. From the definition,

$$S_nf(x) = x^{-\frac{\alpha}{2}-\frac{1}{2}}\int_0^\infty \left[\sum_{k=0}^{n}(\alpha+2k+1)J_{\alpha+2k+1}(x^{\frac{1}{2}})J_{\alpha+2k+1}(t^{\frac{1}{2}})\right] t^{\frac{\alpha}{2}-\frac{1}{2}}f(t)\,dt$$

so that Lemma 3 leads to

$$S_nf = W_1f - W_2f + W_{3,n}f - W_{4,n}f,$$

where

$$\begin{aligned}W_1f(x) &= 2^{-1}x^{-\alpha/2+1/2}J_{\alpha+1}(x^{1/2})\,H\big(t^{\alpha/2}J_\alpha(t^{1/2})f(t)\big)(x), \\ W_2f(x) &= 2^{-1}x^{-\alpha/2}J_\alpha(x^{1/2})\,H\big(t^{\alpha/2+1/2}J_{\alpha+1}(t^{1/2})f(t)\big)(x), \\ W_{3,n}f(x) &= 2^{-1}x^{-\alpha/2+1/2}J'_\nu(x^{1/2})\,H\big(t^{\alpha/2}J_\nu(t^{1/2})f(t)\big)(x), \\ W_{4,n}f(x) &= 2^{-1}x^{-\alpha/2}J_\nu(x^{1/2})\,H\big(t^{\alpha/2+1/2}J'_\nu(t^{1/2})f(t)\big)(x)\end{aligned}$$

and $\nu = \alpha + 2n + 2$. Thus, we will show that the operators W_1, W_2 are bounded and the operators $W_{3,n}$, $W_{4,n}$ are uniformly bounded for $n \geq 0$. The proof for the commutator $[b, S_n]$ is the same: just put $[b, H]$ instead of H.

(I) *Boundedness of the operator* W_1*.* From the definition, it follows that

$$\|W_1 f\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}$$

if and only if

$$\|Hg\|_{L^p(x^{\alpha-\alpha p/2+p/2}|J_{\alpha+1}(x^{1/2})|^p)} \le C\|g\|_{L^p(x^{\alpha-\alpha p/2}|J_\alpha(x^{1/2})|^{-p})}.$$

Proving that there is a weight $\Phi \in A_p$ with

$$(1) \qquad Cx^{\alpha-\alpha p/2+p/2}|J_{\alpha+1}(x^{1/2})|^p \le \Phi(x) \le Cx^{\alpha-\alpha p/2}|J_\alpha(x^{1/2})|^{-p}$$

will be enough. According to the bounds

$$\begin{aligned} |J_\alpha(x)| &\le C_\alpha x^\alpha, \quad x \in (0,1), \\ |J_\alpha(x)| &\le C_\alpha x^{-1/2}, \quad x \in (1,\infty) \end{aligned}$$

(see, e.g., [14, III.3.1 (8), p. 40] and [14, VII.7.21 (1), p. 199]), we have

$$x^{\alpha-\alpha p/2+p/2}|J_{\alpha+1}(x^{1/2})|^p \le \begin{cases} Cx^{\alpha+p}, & \text{if } x \in (0,1), \\ Cx^{\alpha-\alpha p/2+p/4}, & \text{if } x \in (1,\infty), \end{cases}$$
$$x^{\alpha-\alpha p/2}|J_\alpha(x^{1/2})|^{-p} \ge \begin{cases} Cx^{\alpha-\alpha p}, & \text{if } x \in (0,1), \\ Cx^{\alpha-\alpha p/2+p/4}, & \text{if } x \in (1,\infty). \end{cases}$$

Let us try

$$\Phi(x) = \begin{cases} x^r, & \text{if } x \in (0,1), \\ x^{\alpha-\alpha p/2+p/4}, & \text{if } x \in (1,\infty). \end{cases}$$

By (b) in Lemma 2, conditions (1) and $\Phi \in A_p$ will hold if

$$\begin{cases} \alpha - \alpha p \le r \le \alpha + p, \\ -1 < r < p - 1, \\ -1 < \alpha - \alpha p/2 + p/4 < p - 1. \end{cases}$$

The third line is equivalent to

$$\frac{2\alpha - 1}{4}\, p < \alpha + 1, \qquad \alpha + 1 < \frac{2\alpha + 3}{4}\, p,$$

and these hold, by the hypothesis. For the inequalities involving r it suffices

$$\max\{-1, \alpha - \alpha p\} < \min\{p - 1, \alpha + p\}.$$

It is easy to check that this also holds, whenever $\alpha > -1$ and $p > 1$.

(II) *Boundedness of the operator* W_2*.* The proof is entirely similar: we have

$$\|W_2 f\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}$$

if and only if

$$\|Hg\|_{L^p(x^{\alpha-\alpha p/2}|J_\alpha(x^{1/2})|^p)} \le C\|g\|_{L^p(x^{\alpha-\alpha p/2-p/2}|J_{\alpha+1}(x^{1/2})|^{-p})}$$

so that we can prove that there is a weight $\Psi \in A_p$ with

$$Cx^{\alpha-\alpha p/2}|J_\alpha(x^{1/2})|^p \le \Psi(x) \le Cx^{\alpha-\alpha p/2-p/2}|J_{\alpha+1}(x^{1/2})|^{-p}. \tag{2}$$

Now we have

$$x^{\alpha-\alpha p/2}|J_\alpha(x^{1/2})|^p \le \begin{cases} Cx^{\alpha}, & \text{if } x\in(0,1), \\ Cx^{\alpha-\alpha p/2-p/4}, & \text{if } x\in(1,\infty), \end{cases}$$

$$x^{\alpha-\alpha p/2-p/2}|J_{\alpha+1}(x^{1/2})|^{-p} \ge \begin{cases} Cx^{\alpha-\alpha p-p}, & \text{if } x\in(0,1), \\ Cx^{\alpha-\alpha p/2-p/4}, & \text{if } x\in(1,\infty). \end{cases}$$

Setting

$$\Psi(x) = \begin{cases} x^r, & \text{if } x\in(0,1), \\ x^{\alpha-\alpha p/2-p/4}, & \text{if } x\in(1,\infty), \end{cases}$$

conditions (2) and $\Psi \in A_p$ will hold if

$$\begin{cases} \alpha-\alpha p-p \le r \le \alpha, \\ -1 < r < p-1, \\ -1 < \alpha-\alpha p/2-p/4 < p-1. \end{cases}$$

The third line is equivalent to

$$\frac{2\alpha+1}{4}\,p < \alpha+1, \qquad \alpha+1 < \frac{2\alpha+5}{4}\,p,$$

and these hold, by the hypothesis. For the inequalities involving r we only need

$$\max\{-1, \alpha-\alpha p-p\} < \min\{p-1, \alpha\}.$$

It is easy to check that this also holds, whenever $\alpha > -1$ and $p > 1$.

(III) *Uniform boundedness of the operators $W_{3,n}$.* Here,

$$\|W_{3,n}f\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}$$

if and only if

$$\|Hg\|_{L^p(x^{\alpha-\alpha p/2+p/2}|J'_\nu(x^{1/2})|^p)} \le C\|g\|_{L^p(x^{\alpha-\alpha p/2}|J_\nu(x^{1/2})|^{-p})}.$$

We make now use of the bounds

$$|J_\nu(x)| \le Cx^{-1/4}\left[|x-\nu|+\nu^{1/3}\right]^{-1/4}, \quad \nu=\alpha+2n+2,\ x\in(0,\infty),$$

$$|J'_\nu(x)| \le Cx^{-3/4}\left[|x-\nu|+\nu^{1/3}\right]^{1/4}, \quad \nu=\alpha+2n+2,\ x\in(0,\infty),$$

with some universal constant C. They follow from those of [5], for instance. Therefore,

$$x^{\alpha-\alpha p/2+p/2}|J'_\nu(x^{1/2})|^p \le Cx^{\alpha-\alpha p/2+p/8}\left[|x^{1/2}-\nu|+\nu^{1/3}\right]^{p/4},$$

$$x^{\alpha-\alpha p/2}|J_\nu(x^{1/2})|^{-p} \ge Cx^{\alpha-\alpha p/2+p/8}\left[|x^{1/2}-\nu|+\nu^{1/3}\right]^{p/4}.$$

It will suffice to prove that $\varphi_\nu \in A_p$ uniformly in n, with

$$\varphi_\nu(x) = x^{\alpha-\alpha p/2+p/8}\left[|x^{1/2}-\nu| + \nu^{1/3}\right]^{p/4}. \tag{3}$$

From Lemma 1, we have

$$\begin{aligned}\varphi_\nu(x) \in A_p \text{ unif.} &\iff \varphi_\nu(\nu^2 x) \in A_p \text{ unif.}\\ &\iff x^{\alpha-\alpha p/2+p/8}\left[|x^{1/2}-1| + \nu^{-2/3}\right]^{p/4} \in A_p \text{ unif.}\\ &\iff x^{\alpha-\alpha p/2+p/8}|x^{1/2}-1|^{p/4} + \nu^{-p/6}x^{\alpha-\alpha p/2+p/8} \in A_p \text{ unif.},\end{aligned}$$

where the last equivalence follows from

$$\left[|x^{1/2}-1| + \nu^{-2/3}\right]^{p/4} \sim |x^{1/2}-1|^{p/4} + \nu^{-p/6},$$

i.e., the ratio of both terms is bounded below and above by two positive constants not depending on n or x. Now, again by Lemma 1, proving that $x^{\alpha-\alpha p/2+p/8} \in A_p$ and $x^{\alpha-\alpha p/2+p/8}|x^{1/2}-1|^{p/4} \in A_p$ will suffice. According to Lemma 2,

$$\begin{aligned}\begin{cases} x^{\alpha-\alpha p/2+p/8} \in A_p \\ x^{\alpha-\alpha p/2+p/8}|x^{1/2}-1|^{p/4} \in A_p \end{cases} &\iff \begin{cases} -1 < \alpha-\alpha p/2+p/8 < p-1 \\ -1 < p/4 < p-1 \\ -1 < \alpha-\alpha p/2+p/4 < p-1 \end{cases}\\ &\iff \begin{cases} -1 < \alpha-\alpha p/2+p/8 \\ 4/3 < p \\ \alpha-\alpha p/2+p/4 < p-1 \end{cases}\\ &\iff \begin{cases} \frac{2\alpha-1/2}{4}\,p < \alpha+1 < \frac{2\alpha+3}{4}\,p \\ 4/3 < p \end{cases}\end{aligned}$$

and these inequalities follow from the initial conditions.

(IV) *Uniform boundedness of the operators $W_{4,n}$.* Finally,

$$\|W_{4,n}f\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}$$

if and only if

$$\|Hg\|_{L^p(x^{\alpha-\alpha p/2}|J_\nu(x^{1/2})|^p)} \le C\|g\|_{L^p(x^{\alpha-\alpha p/2-p/2}|J'_\nu(x^{1/2})|^{-p})}.$$

Also,

$$x^{\alpha-\alpha p/2}|J_\nu(x^{1/2})|^p \le Cx^{\alpha-\alpha p/2-p/8}\left[|x^{1/2}-\nu| + \nu^{1/3}\right]^{-p/4},$$

$$x^{\alpha-\alpha p/2-p/2}|J'_\nu(x^{1/2})|^{-p} \ge Cx^{\alpha-\alpha p/2-p/8}\left[|x^{1/2}-\nu| + \nu^{1/3}\right]^{-p/4}$$

so let us put

$$\psi_\nu(x) = x^{\alpha-\alpha p/2-p/8}\left[|x^{1/2}-\nu| + \nu^{1/3}\right]^{-p/4} \tag{4}$$

and show that $\psi_\nu \in A_p$ uniformly in n. Indeed,

$$\begin{aligned}\psi_\nu(x) \in A_p \text{ unif.} &\iff \psi_\nu(\nu^2 x) \in A_p \text{ unif.}\\ &\iff x^{\alpha-\alpha p/2-p/8}\left[|x^{1/2}-1|+\nu^{-2/3}\right]^{-p/4} \in A_p \text{ unif.}\end{aligned}$$

and

$$\begin{aligned}&\left(x^{\alpha-\alpha p/2-p/8}\left[|x^{1/2}-1|+\nu^{-2/3}\right]^{-p/4}\right)^{-1}\\ &\qquad\sim x^{-\alpha+\alpha p/2+p/8}\left[|x^{1/2}-1|^{p/4}+\nu^{-p/6}\right]\\ &\qquad= \left[x^{\alpha-\alpha p/2-p/8}|x^{1/2}-1|^{-p/4}\right]^{-1}+\left[\nu^{p/6}x^{\alpha-\alpha p/2-p/8}\right]^{-1}\end{aligned}$$

so that proving that $x^{\alpha-\alpha p/2-p/8}|x^{1/2}-1|^{-p/4} \in A_p$ and $x^{\alpha-\alpha p/2-p/8} \in A_p$ will suffice. But

$$\begin{aligned}\begin{cases} x^{\alpha-\alpha p/2-p/8} \in A_p\\ x^{\alpha-\alpha p/2-p/8}|x^{1/2}-1|^{-p/4} \in A_p\end{cases} &\iff \begin{cases} -1<\alpha-\alpha p/2-p/8<p-1\\ -1<-p/4<p-1\\ -1<\alpha-\alpha p/2-p/4<p-1\end{cases}\\ &\iff \begin{cases} \alpha-\alpha p/2-p/8<p-1\\ p<4\\ -1<\alpha-\alpha p/2-p/4\end{cases}\\ &\iff \begin{cases} \frac{2\alpha+1}{4}p<\alpha+1<\frac{2\alpha+9/2}{4}p\\ p<4\end{cases}\end{aligned}$$

and these inequalities hold by the hypothesis. The proof of Theorem 1 is now complete. □

Proof of Theorem 2. For each $n \geq 0$ and $b \in \mathrm{BMO}$, $T_n(b)\colon L^p(x^\alpha) \longrightarrow L^p(x^\alpha)$ is bounded if and only if $S_n\colon L^p(e^{pb}x^\alpha) \longrightarrow L^p(e^{pb}x^\alpha)$ is bounded, and both operators have the same norm. Thus, we can follow the proof of Theorem 1 and conclude that conditions (1), (2), (3) and (4), i.e.,

$$\begin{gathered} Cx^{\alpha-\alpha p/2+p/2}|J_{\alpha+1}(x^{1/2})|^p \leq \Phi(x) \leq Cx^{\alpha-\alpha p/2}|J_\alpha(x^{1/2})|^{-p},\\ Cx^{\alpha-\alpha p/2}|J_\alpha(x^{1/2})|^p \leq \Psi(x) \leq Cx^{\alpha-\alpha p/2-p/2}|J_{\alpha+1}(x^{1/2})|^{-p},\\ \varphi_\nu(x) = x^{\alpha-\alpha p/2+p/8}\left[|x^{1/2}-\nu|+\nu^{1/3}\right]^{p/4},\\ \psi_\nu(x) = x^{\alpha-\alpha p/2-p/8}\left[|x^{1/2}-\nu|+\nu^{1/3}\right]^{-p/4},\end{gathered}$$

are still sufficient, if we require now $e^{pb}\Phi, e^{pb}\Psi, e^{pb}\varphi_\nu, e^{pb}\psi_\nu \in A_p$ uniformly in ν. The proof of Theorem 1, together with next lemma, finish the proof of Theorem 2. □

Lemma 4. *Let $1 < p < \infty$. For each $\phi \in A_p$, there exists some $\delta > 0$ such that $e^{pb}\phi \in A_p$ whenever $b \in \mathrm{BMO}$ with $\|b\|_{\mathrm{BMO}} < \delta$. Moreover, δ and the A_p constant of $e^{pb}\phi$ depend only on the A_p constant of ϕ.*

Remark. Again, statements like "δ depends only on the A_p constant of ϕ" should be understood as: given a constant $C > 0$ which verifies the A_p condition for ϕ, some δ can be chosen depending only on C.

Proof. If $\phi \in A_p$, there exists some $\varepsilon > 1$ such that $\phi^\varepsilon \in A_p$; moreover, ε and the A_p constant of ϕ^ε depend only on the A_p constant of ϕ [6, Theorem IV.2.7, p. 399]. Take now $1/\varepsilon + 1/\varepsilon' = 1$. There exists some $\delta > 0$ such that

$$\|b\|_{\mathrm{BMO}} < \delta \Longrightarrow e^{p\varepsilon' b} \in A_p;$$

here, δ and the A_p constant of $e^{p\varepsilon' b}$ depend only on ε' [6, p. 409]. This, together with $\phi^\varepsilon \in A_p$ and Hölder's inequality, imply $e^{pb}\phi \in A_p$ with an A_p constant depending only on the A_p constant of ϕ. □

References

1. A. I. Aptekarev, A. Branquinho and F. Marcellán, *Toda-type differential equations for the recurrence coefficients of orthogonal polynomials and Freud transformation*, J. Comput. Appl. Math. **78** (1997), 139–160.
2. S. Bloom, *A commutator theorem and weighted* BMO, Trans. Amer. Math. Soc. **292** (1985), 103–122.
3. R. R. Coifman and M. A. M. Murray, *Uniform analyticity of orthogonal projections*, Trans. Amer. Math. Soc. **312** (1989), 779–817.
4. R. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635.
5. A. Córdoba, *The disc multiplier*, Duke Math. J. **58** (1989), 21–29.
6. J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.
7. F. Marcellán, J. S. Dehesa and A. Ronveaux, *On orthogonal polynomials with perturbed recurrence relations*, J. Comput. Appl. Math. **30** (1990), 203–212.
8. C. J. Neugebauer, *Inserting A_p weights*, Proc. Amer. Math. Soc. **87** (1983), 644–648.
9. P. Nevai and W. Van Assche, *Compact perturbations of orthogonal polynomials*, Pacific J. Math. **153** (1992), 163–184.
10. C. Segovia and J. L. Torrea, *Vector-valued commutators and applications*, Indiana Univ. Math. J. **38** (1989), 959–971.
11. ———, *Weighted inequalities for commutators of fractional and singular integrals*, Publicacions Matemàtiques **35** (1991), 209–235.
12. ———, *Higher order commutators for vector-valued Calderón-Zygmund operators*, Trans. Amer. Math. Soc. **336** (1993), 537–556.
13. J. L. Varona, *Fourier series of functions whose Hankel transform is supported on* $[0, 1]$, Constr. Approx. **10** (1994), 65–75.
14. G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge Univ. Press, Cambridge, 1944 (reprinted 1995).

José J. Guadalupe. Dpto. de Matemáticas y Computación. Universidad de La Rioja. 26004 Logroño. Spain.

Mario Pérez. Dpto. de Matemáticas. Universidad de Zaragoza. 50009 Zaragoza. Spain.
E-mail address: mperez@posta.unizar.es

Juan L. Varona. Dpto. de Matemáticas y Computación. Universidad de La Rioja. 26004 Logroño. Spain.
E-mail address: jvarona@dmc.unirioja.es