Constr. Approx. (1996) 12: 341–360

CONSTRUCTIVE APPROXIMATION

Weighted Norm Inequalities for Polynomial Expansions Associated to Some Measures with Mass Points

J. J. Guadalupe, M. Pérez, F. J. Ruiz, and J. L. Varona

Abstract. Fourier series in orthogonal polynomials with respect to a measure ν on $[-1, 1]$ are studied when ν is a linear combination of a generalized Jacobi weight and finitely many Dirac deltas in $[-1, 1]$. We prove some weighted norm inequalities for the partial sum operators S_n, their maximal operator S^*, and the commutator $[M_b, S_n]$, where M_b denotes the operator of pointwise multiplication by $b \in \mathrm{BMO}$. We also prove some norm inequalities for S_n when ν is a sum of a Laguerre weight on $\mathbf{R}^+$ and a positive mass on 0.

Introduction

Let ν be a positive Borel measure on $\mathbf{R}$ with infinitely many points of increase and such that all the moments

$$\int_{\mathbf{R}} x^n \, d\nu \qquad (n = 0, 1, \dots)$$

are finite. For each suitable function f, let $S_n f$ denote the nth partial sum of the Fourier expansion of f with respect to the system of orthogonal polynomials associated to $d\nu$.

The uniform boundedness of the operators S_n: $L^p(d\nu) \to L^p(d\nu)$ $(1 < p < \infty)$ and some weighted versions $uS_n(v^{-1}\cdot)$: $L^p(d\nu) \to L^p(d\nu)$ have been characterized when ν is:

(a) a Jacobi or generalized Jacobi weight on $[-1, 1]$ (see [P], [M1], and [B]); and
(b) a Laguerre weight on $\mathbf{R}^+$ or a Hermite weight on $\mathbf{R}$ (see [AW], [M2], and [M3]).

This uniform boundedness is equivalent, in rather general settings, to the L^p convergence of $S_n f$ to f.

Let us consider for simplicity the unweighted case. For a generalized Jacobi weight, not only the uniform boundedness of the operators S_n has been studied, but also that of the maximal operator S^* defined by

$$S^* f(x) = \sup_n |S_n f(x)|$$

Date received: November 8, 1993. Date revised: October 16, 1995. Communicated by Doron S. Lubinsky.
AMS classification: Primary 42C10.
Key words and phrases: Orthonormal polynomials, Weighted norm inequalities, Fourier series, Maximal operator, Commutator, Jacobi weights, Laguerre weights, Mass points.

(see [B]). For some orthogonal systems which include Jacobi polynomials and Bessel functions, the maximal operator S^* has been considered by Gilbert [G] by means of transference theorems.

Obviously, the boundedness of S^* on $L^p(d\nu)$ implies the uniform boundedness of S_n (and the $L^p(d\nu)$ convergence of $S_n f$ to f). But it also implies, by standard arguments, the ν-a.e. convergence of $S_n f$ to f. For these weights, the typical situation is that the operators $S_n\colon L^p(d\nu) \to L^p(d\nu)$ and even S^* are uniformly bounded if and only if p belongs to some explicitly given open interval (p_0, p_1) (the interval of *mean convergence*). In short, in this case the S_n are said to be of *strong* (p, p)-type.

Then, for the endpoints $p = p_0, p_1$ of the interval of mean convergence it is natural to study the *weak* (p, p)-type, i.e., the uniform boundedness of the operators

$$S_n\colon L^p(d\nu) \to L^{p,\infty}(d\nu),$$

as well as the *restricted weak* (p, p)-type, i.e., the uniform boundedness

$$S_n\colon L^{p,1}(d\nu) \to L^{p,\infty}(d\nu).$$

Here, $L^{p,r}(d\nu)$ stands for the classical Lorentz space of all measurable functions f satisfying

$$\begin{aligned} \|f\|_{L^{p,r}(d\nu)} &= \left(\frac{r}{p}\int_0^\infty [t^{1/p} f^*(t)]^r \frac{dt}{t}\right)^{1/r} < \infty \qquad (1 \le p < \infty,\ 1 \le r < \infty), \\ \|f\|_{L^{p,\infty}(d\nu)} &= \|f\|_{L^p_*(d\nu)} = \sup_{t>0} t^{1/p} f^*(t) < \infty \qquad (1 \le p \le \infty), \end{aligned}$$

where f^* denotes the nonincreasing rearrangement of f. We refer the reader to [SW, Section V.3] for further information on these topics.

For $d\nu = dx$ on $[-1, 1]$ (Fourier–Legendre series; $p_0 = 4/3$, $p_1 = 4$), S. Chanillo [C] proved that the S_n are not of weak (p, p)-type for $p = 4$ but they are of restricted weak (p, p)-type for $p = 4/3$ and $p = 4$. In [GPV1] and [GPV2] these results were established for any Fourier–Jacobi series ($d\nu = (1 - x)^\alpha (1 + x)^\beta\, dx$ on $[-1, 1]$, $\alpha, \beta > -1$) and $p = p_0$, $p = p_1$. The $L^{p,r}$ behavior of S_n was also studied by L. Colzani [Co] for Fourier–Legendre series.

In this paper, we consider these problems for Fourier expansions with respect to measures of the form

$$\nu = \mu + \sum_{i=1}^{k} M_i \delta_{a_i},$$

where $M_i > 0$ $(i = 1, \ldots, k)$, δ_a denotes the Dirac delta on $a \in \mathbf{R}$ and μ is a generalized Jacobi weight or, in some cases, a Laguerre or Hermite weight. In the particular case of a Jacobi weight and two mass points on 1 and -1, the corresponding orthonormal polynomials were studied by Koornwinder in [K] from the point of view of differential equations (see also [Ch], [AE], [Kr], and [Li]). Our method consists of relating the operators S_n to some other operators similar to (and expressible in terms of) the Fourier expansions with respect to μ and polynomial modifications of μ.

This method also applies to the commutator $[M_b, S_n]$, where M_b is the operator of pointwise multiplication by a given function b, i.e., $M_b f = bf$. Given a linear operator

T acting on functions, say $T\colon L^p(d\nu) \to L^p(d\nu)$, and a function b, the commutator of M_b and T is defined by

$$[M_b, T]f = bT(f) - T(bf).$$

The first results on this commutator were obtained by R. R. Coifman, R. Rochberg, and G. Weiss (see [CRW]). They proved that if T is the classical Hilbert transform and $1 < p < \infty$, then $[M_b, T]$ is a bounded operator on $L^p(\mathbf{R})$ if and only if $b \in \mathrm{BMO}(\mathbf{R})$. The boundedness of this commutator has been studied in more general settings by several authors (see, e.g., [BL], [ST1], [ST2], [ST3], and [GHST]).

Let us also mention that the boundedness of the commutator $[M_b, T]$ (or, in our case, $[M_b, S_n]$) for b in some real Banach space B is closely related to the analyticity (in our case, uniform analyticity) of the operator-valued function

$$\begin{aligned} \mathcal{T}\colon \overline{B} &\to \mathcal{L}(L^p(d\nu), L^p(d\nu)),\\ b &\mapsto \mathcal{T}(b) = M_{e^b} T M_{e^{-b}} \end{aligned}$$

in a neighborhood of $0 \in \overline{B}$, where $\overline{B}$ denotes the complexification of B and $\mathcal{L}(L^p(d\nu), L^p(d\nu))$ is the space of bounded linear operators from $L^p(d\nu)$ into itself. In fact, the first Gâteaux differential of $\mathcal{T}$ at 0 in the direction $b \in B$ is

$$\frac{d}{ds}\mathcal{T}(sb)\bigg|_{s=0} = [M_b, T]$$

(see [CM] and [L] for further details). It is via this relationship that R. R. Coifman and M. A. M. Murray proved [CM] the uniform boundedness of the commutator

$$[M_b, S_n]\colon L^2(d\nu) \to L^2(d\nu),$$

when $d\nu$ is a Jacobi weight ($d\nu = (1-x)^\alpha (1+x)^\beta\, dx$ on $[-1, 1]$) with $\alpha, \beta > -1/2$ and $b \in \mathrm{BMO}$.

Here, we prove the uniform boundedness of $[M_b, S_n]$ (as well as a weighted version) in $L^p(d\nu)$, $1 < p < \infty$, where $d\nu$ is a generalized Jacobi weight with possibly a finite collection of Dirac deltas on $[-1, 1]$ and again $b \in \mathrm{BMO}$.

This paper is organized as follows: in Section 1 we present the basic notation and technical results. In Section 2 we consider the maximal operator S^* related to a generalized Jacobi weight function with finitely many Dirac masses on $[-1, 1]$. As a consequence, the L^p and a.e. convergence of the Fourier series follow. For these measures (with some restriction), the commutator $[M_b, S_n]$ is studied in Section 3. Weak and restricted weak boundedness at the endpoint of the interval of mean convergence for Jacobi weights with Dirac masses on $[-1, 1]$ are the subject of Section 4. Finally, in Section 5 we point out how L^p boundedness of Fourier expansions with respect to a Laguerre or Hermite weight with a positive mass on 0 can be established.

1. Notations and Technical Results

Let μ be a positive Borel measure on $\mathbf{R}$ with infinitely many points of increase and such that all the moments

$$\int_{\mathbf{R}} x^n\, d\mu \qquad (n = 0, 1, \dots)$$

are finite. Let $a_i \in \mathbf{R}$ $(i = 1, \dots, k)$ with $a_i \neq a_j$ for $i \neq j$ and assume $\mu(\{a_i\}) = 0$ $(i = 1, \dots, k)$. Let $M_i > 0$ $(i = 1, \dots, k)$ and write

$$\nu = \mu + \sum_{i=1}^{k} M_i \delta_{a_i}, \tag{1}$$

where δ_a denotes a Dirac delta on a:

$$\int_{\mathbf{R}} f \, d\delta_a = f(a).$$

Then, there is a sequence $\{P_n\}_{n \geq 0}$ of polynomials,

$$P_n(x) = k_n x^n + \cdots, \quad k_n > 0, \qquad \deg P_n = n,$$

such that

$$\int_{\mathbf{R}} P_n P_m \, d\nu = \begin{cases} 0 & \text{if } n \neq m; \\ 1 & \text{if } n = m. \end{cases}$$

The nth partial sum operator of the Fourier expansion in terms of P_n is the operator S_n given by

$$S_n f(x) = \int_{\mathbf{R}} L_n(x, y) f(y) \, d\nu(y),$$

where

$$L_n(x, y) = \sum_{j=0}^{n} P_j(x) P_j(y)$$

is the nth kernel relative to the measure $d\nu$. If we denote

$$T_n f(x) = \int_{\mathbf{R}} L_n(x, y) f(y) \, d\mu(y),$$

then, according to (1), we have

$$S_n f(x) = T_n f(x) + \sum_{i=1}^{k} M_i L_n(x, a_i) f(a_i). \tag{2}$$

By a weight function we mean a nonnegative, measurable function. We are interested in finding conditions for the uniform boundedness of the operators

$$u S_n(v^{-1} \cdot) \colon L^p(d\nu) \to L^p(d\nu)$$

where u and v are weights, i.e., for the inequality

$$\|u S_n(v^{-1} f)\|_{L^p(d\nu)} \leq C \|f\|_{L^p(d\nu)}$$

to hold for $n \geq 0$ and $f \in L^p(d\nu)$, and also for the weaker boundedness

$$u S_n(v^{-1} \cdot) \colon L^p(d\nu) \to L^{p,\infty}(d\nu)$$

or

$$u S_n(v^{-1} \cdot) \colon L^{p,1}(d\nu) \to L^{p,\infty}(d\nu).$$

Actually, the last one is equivalent (see [SW, Theorem 3.13]) to

$$\|uS_n(v^{-1}\chi_E)\|_{L^{p,\infty}(d\nu)} \leq C\|\chi_E\|_{L^p(d\nu)}$$

for every measurable set E. In this context, notice that the values $u(a_i)$, $v(a_i)$ are significant here, since $\nu(\{a_i\}) > 0$.

In what follows, given $1 \leq p \leq \infty$ we will denote by p' the conjugate exponent, i.e., $1 \leq p' \leq \infty$, $1/p + 1/p' = 1$. Also, we will take $0 \cdot \infty = 0$ and by C we will mean a constant, not depending on n, f, but possibly different at each occurrence.

Then, we have the following results:

Lemma 1. *With the above notation, let* $1 < p < \infty$, $1 < q < \infty$, $1 \leq r \leq \infty$, *and* $1 \leq s \leq \infty$; *let* u, v *be two weight functions on* $\mathbf{R}$ *with* $u(a_i) < \infty$, $0 < v(a_i)$, $i = 1, \dots, k$. *Then, there exists some constant* $C > 0$ *such that*:

$$\|uS_n(v^{-1}f)\|_{L^{p,r}(d\nu)} \leq C\|f\|_{L^{q,s}(d\nu)} \tag{3}$$

for every $f \in L^{q,s}(d\nu)$, $n \geq 0$, *if and only if there exists* $C > 0$ *such that*:

(a) $\|uT_n(v^{-1}f)\|_{L^{p,r}(d\mu)} \leq C\|f\|_{L^{q,s}(d\mu)}$, $\quad f \in L^{q,s}(d\mu)$, $n \geq 0$;
(b) $u(a_i)\|v^{-1}L_n(x, a_i)\|_{L^{q',s'}(d\mu)} \leq C$, $\quad n \geq 0$, $i = 1, \dots, k$; *and*
(c) $v(a_i)^{-1}\|uL_n(x, a_i)\|_{L^{p,r}(d\mu)} \leq C$, $\quad n \geq 0$, $i = 1, \dots, k$.

We can state a similar result about the maximal operator S^* defined by

$$S^*f(x) = \sup_n |S_n f(x)|.$$

Let us also take

$$T^*f(x) = \sup_n |T_n f(x)|$$

and

$$L^*(x, y) = \sup_n |L_n(x, y)|.$$

Lemma 2. *With the above notation, let* $1 < p < \infty$, $1 < q < \infty$, $1 \leq r \leq \infty$, *and* $1 \leq s \leq \infty$; *let* u, v *be two weight functions on* $\mathbf{R}$ *with* $u(a_i) < \infty$, $0 < v(a_i)$, $i = 1, \dots, k$. *Then, there exists some constant* $C > 0$ *such that*:

$$\|uS^*(v^{-1}f)\|_{L^{p,r}(d\nu)} \leq C\|f\|_{L^{q,s}(d\nu)},$$

for every $f \in L^{q,s}(d\nu)$ *if and only if there exists* $C > 0$ *such that*:

(a) $\|uT^*(v^{-1}f)\|_{L^{p,r}(d\mu)} \leq C\|f\|_{L^{q,s}(d\mu)}$, $\quad f \in L^{q,s}(d\mu)$;
(b) $u(a_i)\|v^{-1}L_n(x, a_i)\|_{L^{q',s'}(d\mu)} \leq C$, $\quad n \geq 0$, $i = 1, \dots, k$; *and*
(c) $v(a_i)^{-1}\|uL^*(x, a_i)\|_{L^{p,r}(d\mu)} \leq C$, $\quad i = 1, \dots, k$.

Finally, we also have the analogous result for the commutator (notice that $b(a_i)$ is significant here, too):

Lemma 3. *With the above notation, let* $1 < p < \infty$, $1 < q < \infty$, $1 \le r \le \infty$, *and* $1 \le s \le \infty$; *let* u, v *be two weight functions on* $\mathbf{R}$ *with* $u(a_i) < \infty$, $0 < v(a_i)$, $i = 1, \dots, k$. *Let* b *be a function on* $\mathbf{R}$ *with* $b(a_i) < \infty$, $i = 1, \dots, k$. *Then, there exists some constant* $C > 0$ *such that*:

$$\|u[M_b, S_n](v^{-1} f)\|_{L^{p,r}(d\nu)} \le C\|f\|_{L^{q,s}(d\nu)}$$

for every $f \in L^{q,s}(d\nu)$, $n \ge 0$, *if and only if there exists* $C > 0$ *such that*:

(a) $\|u[M_b, T_n](v^{-1} f)\|_{L^{p,r}(d\mu)} \le C\|f\|_{L^{q,s}(d\mu)}$, $\quad f \in L^{q,s}(d\mu)$, $n \ge 0$;
(b) $u(a_i)\|v^{-1}L_n(x, a_i)[b(x) - b(a_i)]\|_{L^{q',s'}(d\mu)} \le C$, $\quad n \ge 0$, $i = 1, \dots, k$; *and*
(c) $v(a_i)^{-1}\|uL_n(x, a_i)[b(x) - b(a_i)]\|_{L^{p,r}(d\mu)} \le C$, $\quad n \ge 0$, $i = 1, \dots, k$.

Remark. From the definition, we have $\|\chi_E\|_{L^{p,r}(d\sigma)} = \sigma(E)^{1/p}$ for any measure σ and any measurable set E. In particular, any function is a.e. a characteristic function with respect to a measure of the form $M\delta_a$, thus $\|f\|_{L^{p,r}(M\delta_a)} = M^{1/p}|f(a)|$. As a consequence, we obtain in our case

$$\|f\|_{L^{p,r}(d\nu)} \sim \|f\|_{L^{p,r}(d\mu)} + \sum_{i=1}^{k} M_i^{1/p}|f(a_i)|,$$

where "$\sim$" means that the ratio is bounded above and below by two positive constants. Actually, the M_i can be removed.

Proof of Lemma 1. (I) Suppose (3) holds. Let $f \in L^{q,s}(d\mu)$ and put $g(a_i) = 0$ ($i = 1, \dots, k$), $g(x) = f(x)$ otherwise. Clearly $f = g$ (μ-a.e.), so that from relation (2) and the definition of T_n it follows

$$uS_n(v^{-1}g) = uT_n(v^{-1}g) = uT_n(v^{-1} f).$$

It is also easy to deduce

$$\|g\|_{L^{q,s}(d\nu)} = \|g\|_{L^{q,s}(d\mu)} = \|f\|_{L^{q,s}(d\mu)}$$

(g has the same distribution function with respect to ν and μ and it coincides with the distribution function of f with respect to μ). Therefore, from (3) applied to g, we have

$$\|uT_n(v^{-1} f)\|_{L^{p,r}(d\nu)} \le C\|f\|_{L^{q,s}(d\mu)} \tag{4}$$

for every $f \in L^{q,s}(d\mu)$ and $n \ge 0$.

Now, set $f = \chi_{\{a_i\}}$; then,

$$\begin{aligned} u(x)S_n(v^{-1} f)(x) &= u(x)M_i L_n(x, a_i)v(a_i)^{-1}, \\ \|f\|_{L^{q,s}(d\nu)} &= M_i^{1/q} \end{aligned}$$

(for the last equality, see the previous remark). Therefore, from (3), applied to f, it follows

$$v(a_i)^{-1}\|uL_n(x, a_i)\|_{L^{p,r}(d\nu)} \le C \tag{5}$$

for every $n \ge 0$ and $i = 1, \dots, k$.

(II) Conversely, suppose (4) and (5) hold. Then, for every $f \in L^{q,s}(d\nu)$ we have, from (2),

$$\begin{aligned} \|uS_n(v^{-1}f)\|_{L^{p,r}(d\nu)} &\le \|uT_n(v^{-1}f)\|_{L^{p,r}(d\nu)} \\ &\quad + \sum_{i=1}^{k} M_i \|uL_n(x,a_i)v(a_i)^{-1}f(a_i)\|_{L^{p,r}(d\nu)} \\ &\le C\|f\|_{L^{q,s}(d\mu)} + C\sum_{i=1}^{k} M_i|f(a_i)| \le C\|f\|_{L^{q,s}(d\nu)}, \end{aligned}$$

using the previous remark. That is: (3) is equivalent to (4) and (5). We will see now that (5) is the same as (c) and that (4) is equivalent to (a) and (b).

(III) Fix an $i \in \{1,\ldots,k\}$. Then

$$\|uL_n(x,a_i)\|_{L^{p,r}(d\nu)} \sim \|uL_n(x,a_i)\|_{L^{p,r}(d\mu)} + \sum_{j=1}^{k} M_j^{1/p} u(a_j)|L_n(a_j,a_i)|.$$

Now, from the Cauchy–Schwarz inequality we have

$$|L_n(a_j,a_i)| \le L_n(a_j,a_j)^{1/2} L_n(a_i,a_i)^{1/2}$$

and the sequences $\{L_n(a_j,a_j)\}_{n\ge 0}$ $(j = 1,\ldots,k)$ are bounded, since $\nu(\{a_j\}) > 0$ (see [N, p. 4]). Hence,

$$\|uL_n(x,a_i)\|_{L^{p,r}(d\mu)} \le \|uL_n(x,a_i)\|_{L^{p,r}(d\nu)} \le \|uL_n(x,a_i)\|_{L^{p,r}(d\mu)} + C.$$

This means that (5) is actually equivalent to (c).

(IV) Let us now take condition (4). From the previous remark again,

$$\begin{aligned} \|uT_n(v^{-1}f)\|_{L^{p,r}(d\nu)} &\sim \|uT_n(v^{-1}f)\|_{L^{p,r}(d\mu)} \\ &\quad + \sum_{i=1}^{k} M_i^{1/p} u(a_i)|T_n(v^{-1}f)(a_i)|. \end{aligned}$$

Thus, (4) is equivalent to condition (a), together with

$$u(a_i)|T_n(v^{-1}f)(a_i)| \le C\|f\|_{L^{q,s}(d\mu)}.$$

Having in mind that

$$u(a_i)T_n(v^{-1}f)(a_i) = u(a_i)\int_{\mathbf{R}} v(x)^{-1} L_n(x,a_i) f(x)\,d\mu(x),$$

that means, by duality,

$$u(a_i)\|v(x)^{-1}L_n(x,a_i)\|_{L^{q',s'}(d\mu)} \le C,$$

i.e., condition (b). ∎

The proofs of Lemmas 2 and 3 are essentially the same, so we omit them.

The following result provides sufficient conditions for the uniform boundedness of the operators T_n in terms of the boundedness of the Fourier series corresponding to the measure μ and other related measures. Recalling that

$$\nu = \mu + \sum_{i=1}^{k} M_i \delta_{a_i},$$

we define, for each set $A \subseteq \{a_1, \dots, a_k\}$, the measure

$$d\mu^A(x) = \prod_{a_i \in A} (x - a_i)^2 \, d\mu(x)$$

(for $A = \emptyset$ we just get $d\mu$) and the associated partial sum operators $\widetilde{S}_n^A$. We also define the weight

$$w^A(x) = \prod_{a_i \in A} |x - a_i|^{1-2/p}.$$

With this notation, we have:

Lemma 4. *If for each $A \subseteq \{a_1, \dots, a_k\}$ there exists a constant C such that*

$$\|u w^A \widetilde{S}_n^A([v w^A]^{-1} f)\|_{L^p(d\mu^A)} \le C \|f\|_{L^p(d\mu^A)}$$

for every $n \ge 0$ and $f \in L^p(d\mu^A)$, then there also exists a constant C such that

$$\|u T_n(v^{-1} f)\|_{L^p(d\mu)} \le C \|f\|_{L^p(d\mu)}$$

for $n \ge 0$ and $f \in L^p(d\mu)$.

Proof. Let us denote by $K_n^A(x, y)$ the nth kernel relative to the measure $d\mu^A$. In the case $k = 1$, we have

$$L_n(x, y) = C_n K_n^{\emptyset}(x, y) + (1 - C_n)(x - a_1)(y - a_1) K_{n-1}^{\{a_1\}}(x, y)$$

with $0 < C_n < 1$, $\forall n \in \mathbf{N}$ (see [GPRV3, Proposition 5]). By induction on k, it can be shown that

$$L_n(x, y) = \sum_A C_n^A \left[\prod_{a_i \in A} (x - a_i)(y - a_i) \right] K_{n-|A|}^A(x, y),$$

where the sum is taken over all the subsets $A \subseteq \{a_1, \dots, a_k\}$, $|A|$ is the cardinal of A and for each n

$$\sum_A C_n^A = 1, \qquad 0 < C_n^A < 1, \quad \forall A.$$

From this expression, we deduce

$$T_n(v^{-1} f)(x) = \sum_A C_n^A \left[\prod_{a_i \in A} (x - a_i) \right] \widetilde{S}_{n-|A|}^A \left(\frac{v(y)^{-1} f(y)}{\prod_{a_i \in A} (y - a_i)}, x \right).$$

Thus, for the uniform boundedness of the operators T_n it is enough to have

$$\left\| u(x) \prod_{a_i \in A} (x - a_i) \widetilde{S}_n^A \left(\frac{v(y)^{-1} f(y)}{\prod_{a_i \in A} (y - a_i)}, x \right) \right\|_{L^p(d\mu)} \le C \|f\|_{L^p(d\mu)}$$

for $n \ge 0$ and $f \in L^p(d\mu)$. This is simply our hypothesis, except for a change of notation. ■

Analogous results can be stated about the maximal operator T^* and the commutator.

2. The Maximal Operator S^* for a Generalized Jacobi Weight with Mass Points on $[-1, 1]$

Let $d\mu = w\,dx$, with w a generalized Jacobi weight, that is:

$$w(x) = h(x)(1-x)^{\alpha}(1+x)^{\beta}\prod_{i=1}^{N}|x-t_i|^{\gamma_i}, \qquad x \in [-1, 1],$$

where:

(a) $\alpha, \beta, \gamma_i > -1$, $t_i \in (-1, 1)$, $t_i \neq t_j$ for $i \neq j$;
(b) h is a positive, continuous function on $[-1, 1]$ and $w(h, \delta)\delta^{-1} \in L^1(0, 2)$, $w(h, \delta)$ being the modulus of continuity of h.

Let

$$\nu = \mu + \sum_{i=1}^{k} M_i\delta_{a_i} \quad \text{with } M_i > 0, \qquad a_i \in [-1, 1] \qquad (i = 1, \ldots, k).$$

Let us also take two weights u and v defined on $[-1, 1]$ as follows:

$$u(x) = (1-x)^{a}(1+x)^{b}\prod_{i=1}^{N}|x-t_i|^{g_i}, \qquad \text{if} \quad x \neq a_i\ \forall i; \quad 0 < u(a_i) < \infty;$$

$$v(x) = (1-x)^{A}(1+x)^{B}\prod_{i=1}^{N}|x-t_i|^{G_i}, \qquad \text{if} \quad x \neq a_i\ \forall i; \quad 0 < v(a_i) < \infty, \tag{6}$$

where $a, b, g_i, A, B, G_i \in \mathbf{R}$.

Theorem 5. *Let $1 < p < \infty$. Then, there exists a constant $C > 0$ such that*:

$$\|uS^*(v^{-1}f)\|_{L^p(d\nu)} \leq C\|f\|_{L^p(d\nu)}$$

for every $f \in L^p(d\nu)$ if and only if the inequalities

$$\begin{cases} A + (\alpha+1)(\frac{1}{p} - \frac{1}{2}) < \min\{\frac{1}{4}, \frac{\alpha+1}{2}\}, \\ B + (\beta+1)(\frac{1}{p} - \frac{1}{2}) < \min\{\frac{1}{4}, \frac{\beta+1}{2}\}, \\ G_i + (\gamma_i+1)(\frac{1}{p} - \frac{1}{2}) < \min\{\frac{1}{2}, \frac{\gamma_i+1}{2}\} \qquad (i = 1, \ldots, N), \end{cases} \tag{7}$$

$$\begin{cases} a + (\alpha+1)(\frac{1}{p} - \frac{1}{2}) > -\min\{\frac{1}{4}, \frac{\alpha+1}{2}\}, \\ b + (\beta+1)(\frac{1}{p} - \frac{1}{2}) > -\min\{\frac{1}{4}, \frac{\beta+1}{2}\}, \\ g_i + (\gamma_i+1)(\frac{1}{p} - \frac{1}{2}) > -\min\{\frac{1}{2}, \frac{\gamma_i+1}{2}\} \qquad (i = 1, \ldots, N), \end{cases} \tag{8}$$

and

$$A \leq a, \qquad B \leq b, \qquad G_i \leq g_i \quad (i = 1, \ldots, N) \tag{9}$$

hold.

Remark. This result is true in the case of a generalized Jacobi weight, with no mass points. It was proved by V. M. Badkov (see [B]) for one weight ($u = v$). In the two weight case, the "if" part can be obtained as a consequence, by inserting a suitable weight ρ, $u \le \rho \le v$. Regarding the "only if" part,

$$\|uS^*(v^{-1}f)\|_{L^p(d\nu)} \le C\|f\|_{L^p(d\nu)}$$

implies

$$\|uS_n(v^{-1}f)\|_{L^p(d\nu)} \le C\|f\|_{L^p(d\nu)}$$

and this, in turn, implies (7), (8), and (9) (see [GPRV1]).

Proof of the Theorem. Assume

$$\|uS^*(v^{-1}f)\|_{L^p(d\nu)} \le C\|f\|_{L^p(d\nu)}$$

for $f \in L^p(d\nu)$. Proceeding as in [GPRV1, Theorem 6], we obtain (7), (8), and (9).

Assume now that (7), (8), and (9) hold. According to Lemma 2 with $p = q = r = s$ and the analogue of Lemma 4, we only need to prove the following inequalities:

(a) $\|uw^A(\widetilde{S}^A)^*([vw^A]^{-1}f)\|_{L^p(d\mu^A)} \le C\|f\|_{L^p(d\mu^A)}, \qquad f \in L^p(d\mu^A);$
(b) $\|v^{-1}L_n(x, a_i)\|_{L^{p'}(w)} \le C, \qquad n \ge 0,\ \ i = 1, \dots, k$; and
(c) $\|uL^*(x, a_i)\|_{L^p(w)} \le C, \qquad i = 1, \dots, k.$

Condition (a) refers to the boundedness of the maximal operators related to the measures $d\mu^A$, which are generalized Jacobi weights, with no mass points. In this case, the corresponding inequalities (7), (8) and (9), with the appropriate exponents, imply the boundedness. It is easy to see that they actually hold.

To check inequalities (b) and (c), we can use the following estimates for the kernels $L_n(x, a_i)$ (see [GPRV3]): if $a_i \ne \pm 1$,

$$\begin{aligned} |L_n(x, a_i)| \ \le\ & C(1 - x + n^{-2})^{-(2\alpha+1)/4}(1 + x + n^{-2})^{-(2\beta+1)/4} \\ & \times \prod_{t_j \ne a_i} (|x - t_j| + n^{-1})^{-\gamma_j/2}; \end{aligned} \tag{10}$$

if 1 is a mass point,

$$|L_n(x, 1)| \le C(1 + x + n^{-2})^{-(2\beta+1)/4} \prod_{i=1}^{N} (|x - t_i| + n^{-1})^{-\gamma_i/2}; \tag{11}$$

if -1 is a mass point,

$$|L_n(x, -1)| \le C(1 - x + n^{-2})^{-(2\alpha+1)/4} \prod_{i=1}^{N} (|x - t_i| + n^{-1})^{-\gamma_i/2}. \tag{12}$$

It is not difficult to see that these inequalities, together with (7), (8), and (9), lead to (b) and (c). ■

Corollary 6. *With the notation of Theorem* 5, *the uniform boundedness*

$$\|uS_n(v^{-1}f)\|_{L^p(d\nu)} \le C\|f\|_{L^p(d\nu)}$$

holds for every $n \ge 0$ *and* $f \in L^p(d\nu)$ *if and only if the inequalities* (7), (8), *and* (9) *are verified.*

Proof. The "if" part follows directly from the theorem and the "only if" part can be proved again as in [GPRV1]. ■

Corollary 7. *Let v be a weight verifying* (6). *Then,*

$$v(x)S_n(v^{-1}f)(x) \to f(x), \qquad \nu\text{-a.e.},$$

for every $f \in L^p(d\nu)$ if and only if the inequality (7) *holds.*

Proof. The "only if" part is a consequence of [GPRV2, Theorem 3]. For the "if" part, we can take a weight u such that the pair (u, v) satisfies the conditions of Theorem 5 and it follows by standard arguments. Notice that the weight u does not play any role for the almost everywhere convergence. ■

3. The Commutator $[M_b, S_n]$ for a Generalized Jacobi Weight with Mass Points on $[-1, 1]$

In this section we will adopt the notation of Section 2, with the additional restriction $\gamma_i \geq 0$, $i = 1, \dots, N$. We will write $I = [-1, 1]$.

The space $\mathrm{BMO}(I)$ (in the sequel, BMO) consists of the functions (modulus the constants) of bounded mean oscillation, i.e., with

$$\|b\|_{\mathrm{BMO}} = \sup_J \frac{1}{|J|}\int_J |b - b_J| < \infty,$$

where the supremum is taken over all the intervals $J \subseteq I$, $|J|$ means the Lebesgue measure of J and

$$b_J = \frac{1}{|J|}\int_J b$$

(the integrals are taken with respect to the Lebesgue measure). Given $1 < p < \infty$, we also have (see [GR])

$$\|b\|_{\mathrm{BMO}} \sim \sup_J \left(\frac{1}{|J|}\int_J |b - b_J|^p\right)^{1/p}. \tag{13}$$

Given $1 < p < \infty$ and a weight φ in Muckenhoupt's class A_p, the commutator $[H, M_b]$ of the Hilbert transform on I is bounded in $L^p(\varphi)$ if and only if $b \in \mathrm{BMO}$ (see [BL]). The norm of the commutator depends on the A_p constant of φ. We refer the reader to [GR] for further references on A_p weights.

Our result is the following:

Theorem 8. *If $b \in \mathrm{BMO}$, $1 < p < \infty$, and the inequalities* (7), (8), *and* (9) *hold* (*with $\gamma_i \geq 0$, $i = 1, \dots, N$*), *then there exists some constant $C > 0$ such that*:

$$\|u[M_b, S_n](v^{-1}f)\|_{L^p(d\nu)} \leq C\|f\|_{L^p(d\nu)}$$

for each $n \geq 0$ and $f \in L^p(d\nu)$.

For the proof of Theorem 8 we first state the following result:

Lemma 9. *With the hypothesis of Theorem* 8, *we have*:

(a) $u(a_i)\|v^{-1}L_n(x,a_i)[b(x)-b(a_i)]\|_{L^{p'}(d\mu)} \le C, \quad n \ge 0, i = 1, \dots, k$; *and*
(b) $v(a_i)^{-1}\|uL_n(x,a_i)[b(x)-b(a_i)]\|_{L^p(d\mu)} \le C, \quad n \ge 0, i = 1, \dots, k$.

Proof. Take r, s such that $1/r + 1/s = 1/p'$. By Hölder's inequality,

$$\|v^{-1}L_n(x,a_i)[b(x)-b(a_i)]\|_{L^{p'}(d\mu)} \le \|v^{-1}L_n(x,a_i)\|_{L^r(d\mu)}\|[b(x)-b(a_i)]\|_{L^s(d\mu)}.$$

From the John–Nirenberg inequality (13) it follows

$$\|b(x)-b(a_i)\|_{L^s(d\mu)} \le C,$$

while (10), (11), and (12) lead to

$$\|v^{-1}L_n(x,a_i)\|_{L^r(d\mu)} \le C$$

provided r is near enough to p'. This proves (a). Part (b) follows in a similar way. ∎

Now, according to Lemma 3, we only need to prove

$$\|u[M_b, T_n](v^{-1}f)\|_{L^p(d\mu)} \le C\|f\|_{L^p(d\mu)}, \qquad f \in L^p(d\mu), \quad n \ge 0.$$

From the analogue of Lemma 4, it is enough to show

$$\|uw^A[M_b, \widetilde{S}_n^A]([vw^A]^{-1}f)\|_{L^p(d\mu^A)} \le C\|f\|_{L^p(d\mu^A)}$$

for each $A \subseteq \{a_1, \dots, a_k\}$. For the sake of simplicity, we will prove this inequality only for $A = \emptyset$. Then, $w^A = 1$, $d\mu^A = d\mu$. Let us denote $\widetilde{S}_n = \widetilde{S}_n^{\emptyset}$.

Lemma 10. *With the hypothesis of Theorem* 8, *we have*

$$\|u[M_b, \widetilde{S}_n](v^{-1}f)\|_{L^p(d\mu)} \le C\|f\|_{L^p(d\mu)}$$

for each $n \ge 0$, $f \in L^p(d\mu)$.

Proof. Let us denote by $K_n(x,y)$ the nth kernel relative to $d\mu = w(x)\,dx$, by $\widetilde{P}_n$ the orthonormal polynomials relative to $d\mu$, and by $\widetilde{Q}_n$ the orthonormal polynomials relative to $(1-x^2)\,d\mu$. Then,

$$\widetilde{S}_n g(x) = \int_I K_n(x,y)g(y)w(y)\,dy.$$

Also,

$$|\widetilde{P}_n(x)| \le C(1-x+n^{-2})^{-(\alpha/2+1/4)}(1+x+n^{-2})^{-(\beta/2+1/4)}\prod_{i=1}^{N}(|x-t_i|+n^{-1})^{-\gamma_i/2},$$

$$|\widetilde{Q}_n(x)| \le C(1-x+n^{-2})^{-(\alpha/2+3/4)}(1+x+n^{-2})^{-(\beta/2+3/4)}\prod_{i=1}^{N}(|x-t_i|+n^{-1})^{-\gamma_i/2}$$

(see [B]). Now, we have Pollard's decomposition of K_n (see [P] and [M1]):

$$\begin{aligned} K_n(x,y) &= r_n \widetilde{P}_{n+1}(x)\widetilde{P}_{n+1}(y) \\ &\quad + s_n \widetilde{P}_{n+1}(x)\frac{(1-y^2)\widetilde{Q}_n(y)}{x-y} \\ &\quad + s_n(1-x^2)\widetilde{Q}_n(x)\frac{\widetilde{P}_{n+1}(y)}{y-x}, \end{aligned}$$

for some bounded sequences $\{r_n\}$, $\{s_n\}$ of real numbers. Actually, from $\mu' > 0$ a.e., it follows $\lim r_n = -1/2$, $\lim s_n = 1/2$ (it can be deduced from [P] and either [R] or [MNT]). Therefore, we can write

$$[M_b, \widetilde{S}_n] = r_n\Psi_{1,n} - r_n\Psi_{2,n} + s_n\Psi_{3,n} - s_n\Psi_{4,n},$$

where

$$\begin{aligned} \Psi_{1,n}g(x) &= [b(x)-b_I]\widetilde{P}_{n+1}(x)\int_I \widetilde{P}_{n+1}(y)g(y)w(y)\,dy, \\ \Psi_{2,n}g(x) &= \widetilde{P}_{n+1}(x)\int_I [b(y)-b_I]\widetilde{P}_{n+1}(y)g(y)w(y)\,dy, \\ \Psi_{3,n}g(x) &= \widetilde{P}_{n+1}(x)\,[M_b, H]((1-y^2)\widetilde{Q}_n g w)(x), \\ \Psi_{4,n}g(x) &= (1-x^2)\widetilde{Q}_n(x)\,[M_b, H](\widetilde{P}_{n+1} g w)(x). \end{aligned}$$

Lemma 11 below shows that the operators $\Psi_{3,n}$ are uniformly bounded; the proof for $\Psi_{4,n}$ is entirely similar. Lemma 12 shows that the operators $\Psi_{1,n}$ are uniformly bounded and the proof for $\Psi_{2,n}$ is again similar. ∎

Lemma 11. *With the hypothesis of Theorem* 8, *there exists a constant* $C > 0$ *such that*:

$$\|u\Psi_{3,n}(v^{-1}f)\|_{L^p(d\mu)} \le C\|f\|_{L^p(d\mu)}$$

for each $n \ge 0$ *and* $f \in L^p(d\mu)$.

Proof. According to the definition of $\Psi_{3,n}$, we must show

$$\|[M_b, H]g\|_{L^p(u^p|\widetilde{P}_{n+1}|^p w)} \le C\|g\|_{L^p(v^p|\widetilde{Q}_n|^{-p}(1-x^2)^{-p}w^{1-p})}.$$

By the result of S. Bloom [BL], it is enough to find weights $\{\varphi_n\}$ and positive constants $K_1, K_2 > 0$ such that:

(a) $K_1 u(x)^p|\widetilde{P}_{n+1}(x)|^p w(x) \le \varphi_n(x) \le K_2 v(x)^p|\widetilde{Q}_n(x)|^{-p}(1-x^2)^{-p}w(x)^{1-p}$; and
(b) $\varphi_n \in A_p(-1,1)$, with an A_p-constant independent of n.

We take φ_n of the form

$$\begin{aligned} \varphi_n(x) &= (1-x)^{r_0}(1-x+n^{-2})^{s_0} \\ &\quad \times \prod_{i=1}^{N}|x-t_i|^{r_i}(|x-t_i|+n^{-1})^{s_i} \\ &\quad \times (1+x)^{r_{N+1}}(1+x+n^{-2})^{s_{N+1}} \end{aligned}$$

Then, condition (b) is equivalent to

$$-1 < r_i < p-1, \qquad -1 < r_i + s_i < p-1 \qquad (i = 0, 1, \dots, N+1) \tag{14}$$

(see [GPV2]). Now, it is not difficult to see from (7), (8), (9), and $\gamma_i \geq 0$ that we can take r_i such that

$$-1 < r_i < p-1 \qquad (i = 0, 1, \dots, N+1),$$

$$\begin{aligned} Ap - p + \alpha(1-p) &\leq r_0 \leq ap + \alpha, \\ Bp - p + \beta(1-p) &\leq r_{N+1} \leq bp + \beta, \end{aligned}$$

$$G_i p + \gamma_i(1-p) \leq r_i \leq g_i p + \gamma_i \qquad (i = 1, \dots, N),$$

and then s_i such that

$$-1 < r_i + s_i < p-1 \qquad (i = 0, 1, \dots, N+1),$$

$$\begin{aligned} Ap - p + \alpha(1-p) + p\left(\frac{\alpha}{2} + \frac{3}{4}\right) &\leq r_0 + s_0 \leq ap + \alpha - p\left(\frac{\alpha}{2} + \frac{1}{4}\right), \\ Bp - p + \beta(1-p) + p\left(\frac{\beta}{2} + \frac{3}{4}\right) &\leq r_{N+1} + s_{N+1} \leq bp + \beta - p\left(\frac{\beta}{2} + \frac{1}{4}\right), \\ G_i p + \gamma_i(1-p) + p\frac{\gamma_i}{2} \leq r_i + s_i &\leq g_i p + \gamma_i - p\frac{\gamma_i}{2} \qquad (i = 1, \dots, N). \end{aligned}$$

Using the estimates for $\widetilde{P}_n$ and $\widetilde{Q}_n$ and (14), we can see that these conditions imply (a) and (b). ∎

Lemma 12. *With the hypothesis of Theorem* 8, *there exists a constant* $C > 0$ *such that*

$$\|u\Psi_{1,n}(v^{-1}f)\|_{L^p(d\mu)} \leq C\|f\|_{L^p(d\mu)}$$

for each $n \geq 0$ *and* $f \in L^p(d\mu)$.

Proof. Taking any $r > p$ and applying Hölder's inequality twice and then (13), it follows:

$$\|u\Psi_{1,n}(v^{-1}f)\|_{L^p(d\mu)} \leq C\|b\|_{\mathrm{BMO}}\|u\widetilde{P}_{n+1}w^{1/p}\|_{L^r(dx)}\|v^{-1}\widetilde{P}_{n+1}\|_{L^{p'}(w)}\|f\|_{L^p(w)}.$$

Therefore, it is enough to have, for some $r > p$,

$$\|u\widetilde{P}_{n+1}w^{1/p}\|_{L^r(dx)}\;\|v^{-1}\widetilde{P}_{n+1}\|_{L^{p'}(w)} < C.$$

This can be verified using the estimates for $\widetilde{P}_n$. ∎

4. Weak Behavior for Jacobi Weights with Mass Points on the Interval $[-1, 1]$

Let us now consider, as a particular case, a measure of the form

$$d\nu = (1-x)^{\alpha}(1+x)^{\beta}\,dx + \sum_{i=1}^{k} M_i \delta_{a_i}$$

and $u = v = 1$. If either $\alpha > -1/2$ or $\beta > -1/2$, then Corollary 6 determines an open interval of mean convergence (p_0, p_1), where $1 < p_0 < p_1 < \infty$, and the S_n are not uniformly bounded in $L^{p_0}(d\nu)$ or $L^{p_1}(d\nu)$. By symmetry, we can suppose $\alpha > -1/2$, $\alpha \geq \beta > -1$, so that

$$p_0 = \frac{4(\alpha+1)}{2\alpha+3}, \qquad p_1 = \frac{4(\alpha+1)}{2\alpha+1}.$$

In the absolutely continuous case and $\alpha = \beta = 0$ (i.e., for Legendre polynomials), when the mean convergence interval is $(4/3, 4)$, Chanillo proved (see [C]) that the partial sums S_n are not of weak type for $p = 4$, that is, there exists no constant $C > 0$ such that for every $n \geq 0$ and $f \in L^4(dx)$

$$\|S_n f\|_{L^{4,\infty}(dx)} \leq C\|f\|_{L^4(dx)}.$$

It was also shown that these operators are of restricted weak type for $p = 4$ (and $p = 4/3$, by duality), that is, the previous inequality is verified if we replace the L^4 norm by the $L^{p,1}$ norm. Actually, this is equivalent to the inequality

$$\|S_n \chi_E\|_{L^{4,\infty}(dx)} \leq C\|\chi_E\|_{L^4(dx)}$$

for every measurable set E (see [SW, Theorem 3.13]). The authors obtained (see [GPV1], [GPV2]) similar results for Jacobi weights. The weak boundedness at the endpoints has also been considered for other operators of Fourier Analysis. An important previous paper on the subject is due to Kenig and Tomas [KT], who studied the disk multiplier for radial functions.

We can now prove that these results also hold with mass points:

Theorem 13. *Let $\alpha > -1/2$, $\alpha \geq \beta > -1$. If*

$$p = \frac{4(\alpha+1)}{2\alpha+1} \quad \text{or} \quad p = \frac{4(\alpha+1)}{2\alpha+3},$$

then there exists no constant C such that for every $n \geq 0$ and $f \in L^p(d\nu)$

$$\|S_n f\|_{L^{p,\infty}(d\nu)} \leq C\|f\|_{L^p(d\nu)}.$$

Theorem 14. *Under the hypothesis of Theorem 13, there exists a constant $C > 0$ such that for every measurable set $E \subseteq [-1, 1]$ and every $n \geq 0$*

$$\|S_n \chi_E\|_{L^{p,\infty}(d\nu)} \leq C\|\chi_E\|_{L^p(d\nu)}.$$

Let us take now

$$u(x) = (1-x)^a (1+x)^b, \qquad x \in (-1,1);$$

$$0 < u(\pm 1) < \infty.$$

We can extend Theorems 13 and 14 to the weighted case, when both α and β are greater than or equal to $-1/2$.

Theorem 15. *Let $\alpha, \beta \geq -1/2$, $1 < p < \infty$. If there exists a constant $C > 0$ such that*

$$\|uS_n(u^{-1}f)\|_{L^{p,\infty}(d\nu)} \leq C\|f\|_{L^p(d\nu)}$$

for every $n \geq 0$ and $f \in L^p(d\nu)$, then the inequalities

$$\left|a + (\alpha+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| < \frac{1}{4}, \qquad \left|b + (\beta+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| < \frac{1}{4},$$

are verified.

Theorem 16. *Let $\alpha, \beta \geq -1/2$, $1 < p < \infty$. If the inequalities*

$$-\frac{1}{4} \leq a + (\alpha+1)\left(\frac{1}{p} - \frac{1}{2}\right) < \frac{1}{4}, \qquad -\frac{1}{4} \leq b + (\beta+1)\left(\frac{1}{p} - \frac{1}{2}\right) < \frac{1}{4},$$

hold, then there exists a constant $C > 0$ such that

$$\|uS_n(u^{-1}\chi_E)\|_{L^{p,\infty}(d\nu)} \leq C\|\chi_E\|_{L^p(d\nu)}$$

for every $n \geq 0$ and every measurable set $E \subseteq [-1,1]$.

Remark. By standard arguments of duality (see [GPV2]), Theorem 16 also holds when

$$-\frac{1}{4} < a + (\alpha+1)\left(\frac{1}{p} - \frac{1}{2}\right) \leq \frac{1}{4}, \qquad -\frac{1}{4} < b + (\beta+1)\left(\frac{1}{p} - \frac{1}{2}\right) \leq \frac{1}{4}.$$

Proof of Theorems 14 and 16. We only need to show that conditions (a), (b), and (c) in Lemma 1 are verified, with $p = q$, $r = \infty$, $s = 1$. The estimates (10), (11), (12) are now

$$|L_n(x, a_i)| \leq C(1 - x + n^{-2})^{-(2\alpha+1)/4}(1 + x + n^{-2})^{-(2\beta+1)/4}$$

if $a_i \neq \pm 1$;

$$|L_n(x, 1)| \leq C(1 + x + n^{-2})^{-(2\beta+1)/4}$$

if 1 is a mass point; and

$$|L_n(x, -1)| \leq C(1 - x + n^{-2})^{-(2\alpha+1)/4}$$

if -1 is a mass point, with C independent of $n \geq 0$ and $x \in [-1,1]$. Since either

$$(1 - x + n^{-2})^{-(2\alpha+1)/4} \leq C$$

or

$$(1-x+n^{-2})^{-(2\alpha+1)/4} \leq (1-x)^{-(2\alpha+1)/4},$$

conditions (b) and (c) can be checked out taking into account that

$$(1-x)^r \in L^{p,\infty}((1-x)^s dx) \iff pr+s+1 \geq 0, \qquad (r,s) \neq (0,-1)$$

(see [GPV2], for example).

In order to prove condition (a), we can use Pollard's decomposition for the kernels L_n (see [P] and [M1]) and write

$$T_n f = r_n W_{1,n} f + s_n W_{2,n} f - s_n W_{3,n} f,$$

with

$$\lim r_n = -1/2, \qquad \lim s_n = 1/2,$$

$$\begin{aligned} W_{1,n} f(x) &= P_{n+1}(x) \int_{-1}^{1} P_{n+1}(y) f(y) w(y)\, dy, \\ W_{2,n} f(x) &= P_{n+1}(x) H((1-y)^2 Q_n(y) f(y) w(y), x), \\ W_{3,n} f(x) &= (1-x^2) Q_n(x) H(P_{n+1}(y) f(y) w(y), x), \end{aligned}$$

where $\{P_n\}_{n\geq 0}$ is the sequence of orthonormal polynomials relative to $d\nu$, $\{Q_n\}_{n\geq 0}$ is the sequence associated to $(1-x^2)\,d\nu$, $w(x)=(1-x)^\alpha(1+x)^\beta$, and H is the Hilbert transform on the interval $[-1,1]$.

The polynomials $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ have the estimates (see [GPRV3])

$$|P_n(x)| \leq C(1-x+n^{-2})^{-(2\alpha+1)/4}(1+x+n^{-2})^{-(2\beta+1)/4},$$

and

$$|Q_n(x)| \leq C(1-x+n^{-2})^{-(2\alpha+3)/4}(1+x+n^{-2})^{-(2\beta+3)/4},$$

with C independent of $n \geq 0$ and $x \in [-1,1]$, as in the absolutely continuous case. We can now proceed exactly as in this case and show that $W_{1,n}$, $W_{2,n}$, and $W_{3,n}$ are of restricted weak type (see [GPV2]). ∎

Proof of Theorems 13 and 15. If

$$\|uS_n(u^{-1}f)\|_{L^{p,\infty}(d\nu)} \leq C\|f\|_{L^p(d\nu)}$$

for every $n \geq 0$ and $f \in L^p(d\nu)$, then

$$\begin{aligned} u &\in L^{p,\infty}(d\nu), \\ u^{-1} &\in L^{p'}(d\nu), \end{aligned}$$

$$uw^{-1/2}(1-x^2)^{-1/4} \in L^{p,\infty}(w),$$

and

$$u^{-1}w^{-1/2}(1-x^2)^{-1/4} \in L^{p'}(w),$$

with $w(x) = (1-x)^\alpha(1+x)^\beta$ (see [GPV1]). This proves Theorem 13 for $p = 4(\alpha+1)/(2\alpha+3)$ and implies

$$-\frac{1}{4} \le a + (\alpha+1)\left(\frac{1}{p} - \frac{1}{2}\right) < \frac{1}{4}, \qquad -\frac{1}{4} \le b + (\beta+1)\left(\frac{1}{p} - \frac{1}{2}\right) < \frac{1}{4}$$

in Theorem 15. For

$$-\frac{1}{4} = a + (\alpha+1)\left(\frac{1}{p} - \frac{1}{2}\right)$$

or

$$-\frac{1}{4} = b + (\beta+1)\left(\frac{1}{p} - \frac{1}{2}\right)$$

in Theorem 15 and $p = 4(\alpha+1)/(2\alpha+1)$ in Theorem 13, it can be proved that $W_{1,n}$ and $W_{3,n}$ are of weak type, while $W_{2,n}$ is not (like in [GPV1], [GPV2]). ■

5. Laguerre Weights with a Positive Mass on 0

Lemma 1 is also useful to study the mean boundedness of the Fourier series in the polynomials orthonormal with respect to the measure

$$d\nu = e^{-x}x^\alpha\,dx + M\delta_0$$

on $[0,\infty)$ (i.e., a Laguerre weight with a mass $M > 0$ at 0). In this case, parts (b) and (c) can be handled having in mind that the kernels $L_n(x,0)$ admit the formula

$$L_n(x,0) = r_n Q_n(x), \tag{15}$$

where Q_n is the nth orthonormal Laguerre polynomial relative to the measure $e^{-x}x^{\alpha+1}\,dx$. This formula follows from the fact that

$$\int_0^\infty L_n(x,0)xR_{n-1}(x)[e^{-x}x^\alpha\,dx + M\delta_0] = 0$$

and

$$\int_0^\infty Q_n(x)xR_{n-1}(x)[e^{-x}x^\alpha\,dx + M\delta_0] = 0$$

for any polynomial R_{n-1} of degree at most $n-1$. The constants $r_n = L_n(0,0)/Q_n(0)$ can be asymptotically estimated. As we mentioned in the proof of Lemma 1, $\{L_n(0,0)\}$ is an increasing, bounded sequence, since 0 is a mass point (see [N, p. 4]). On the other hand, if we denote by $\{L_n^{\alpha+1}\}$ the classical, not normalized Laguerre polynomials relative to $e^{-x}x^{\alpha+1}\,dx$, then it is well known that

$$L_n^{\alpha+1}(0) = \frac{\Gamma(n+\alpha+2)}{n!\,\Gamma(\alpha+2)}$$

(see [S] and [M3]) and

$$\|L_n^{\alpha+1}\|_{L^2(e^{-x}x^{\alpha+1}dx)} = \frac{\Gamma(n+\alpha+2)}{n!},$$

what, with our notation, implies

$$Q_n(0) = \frac{\Gamma(n+\alpha+2)^{1/2}}{\Gamma(\alpha+2)(n!)^{1/2}} \sim n^{(\alpha+1)/2}.$$

Therefore,

$$r_n \sim n^{-(\alpha+1)/2}. \tag{16}$$

According to (15) and (16), in order to find bounds for the kernels $L_n(x,0)$ we only need bounds for the normalized classical Laguerre polynomials. These bounds, as well as boundedness results for Laguerre series, can be found in Muckenhoupt's paper [M3]. Thus, we can use Lemma 1 as in the generalized Jacobi case to find that Muckenhoupt's result [M3, Theorem 7] remains valid in the case of a Laguerre weight with a positive mass on 0. The same can be done for Hermite series (see [M3, Theorem 1]).

Acknowledgment. This research was supported by DGICYT under grant PB93-0228-C02-02 and by IER.

References

[AW] R. Askey, S. Wainger (1965): *Mean convergence of expansions in Laguerre and Hermite series*. Amer. J. Math., **87**:695–708.

[AE] F. V. Atkinson, W. N. Everitt (1981): *Orthogonal polynomials which satisfy second order differential equations*. In: E. B. Christoffel, The Influence of His Work on Mathematics and the Physical Sciences (P. L. Butzer, F. Fehér, eds.). Basel: Birkhäuser, pp. 173–181.

[B] V. M. Badkov (1974): *Convergence in the mean and almost everywhere of Fourier series in polynomials orthogonal on an interval*. Math. USSR Sb., **24**:223–256.

[BL] S. Bloom (1985): *A commutator theorem and weighted* BMO. Trans. Amer. Math. Soc., **292**:103–122.

[C] S. Chanillo (1981): *On the weak behaviour of partial sums of Legendre series*. Trans. Amer. Math. Soc., **268**:367–376.

[Ch] T. S. Chihara (1985): *Orthogonal polynomials and measures with end point masses*. Rocky Mountain J. Math., **15**:705–719.

[CM] R. R. Coifman, M. A. M. Murray (1989): *Uniform analyticity of orthogonal projections*. Trans. Amer. Math. Soc., **312**:779–817.

[CRW] R. R. Coifman, R. Rochberg, G. Weiss (1976): *Factorization theorems for Hardy spaces in several variables*. Ann. of Math., **103**:611–635.

[Co] L. Colzani (1993): *Expansions in Legendre polynomials and Laguerre interpolation*. Acta Math. Hung., **61**:289–302.

[GHST] J. García-Cuerva, E. Harboure, C. Segovia, J. L. Torrea (1991): *Weighted norm inequalities for commutators of strongly singular integrals*. Indiana Univ. Math. J., **40**:1397–1420.

[GR] J. García-Cuerva, J. L. Rubio de Francia (1985): Weighted Norm Inequalities and Related Topics. Amsterdam, The Netherlands: North-Holland.

[G] J. E. Gilbert (1969): *Maximal theorems for some orthogonal series*. I. Trans. Amer. Math. Soc., **145**:495–515.

[GPRV1] J. J. Guadalupe, M. Pérez, F. J. Ruiz, J. L. Varona (1991): *Weighted L^p-boundedness of Fourier series with respect to generalized Jacobi weights*. Publicacions Matemàtiques, **35**:449–459.

[GPRV2] J. J. Guadalupe, M. Pérez, F. J. Ruiz, J. L. Varona (1992): *Two notes on convergence and divergence a.e. of Fourier series with respect to some orthogonal systems*. Proc. Amer. Math. Soc., **116**:457–464.

[GPRV3] J. J. Guadalupe, M. Pérez, F. J. Ruiz, J. L. Varona (1993): *Asymptotic behaviour of orthogonal polynomials relative to measures with mass points*. Mathematika, **40**:331–344.

[GPV1] J. J. GUADALUPE, M. PÉREZ, J. L. VARONA (1990): *Weak behaviour of Fourier–Jacobi series*. J. Approx. Theory, **61**:222–238.

[GPV2] J. J. GUADALUPE, M. PÉREZ, J. L. VARONA (1992): *Weighted weak behaviour of Fourier–Jacobi series*. Math. Nachr., **158**:161–174.

[KT] C. E. KENIG, P. A. TOMAS (1980): *The weak behavior of spherical means*. Proc. Amer. Math. Soc., **78**:48–50.

[K] T. KOORNWINDER (1984): *Orthogonal polynomials with weight function* $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$. Canad. Math. Bull., **27**:205–214.

[Kr] A. M. KRALL (1981): *Orthogonal polynomials satisfying fourth order differential equations*. Proc. Roy. Soc. Edinburgh, **87**:271–288.

[L] E. LAENG (1991): *Analytic dependence of orthogonal polynomials*. Rev. Mat. Iberoamericana, **7**:287–312.

[Li] L. L. LITTLEJOHN (1984): *On the classification of differential equations having orthogonal polynomial solutions*. Ann. Mat. Pura Appl., **138**:35–53.

[MNT] A. MÁTÉ, P. NEVAI, V. TOTIK (1985): *Asymptotics for the ratio of the leading coefficients of orthogonal polynomials on the unit circle*. Constr. Approx., **1**:63–69.

[M1] B. MUCKENHOUPT (1969): *Mean convergence of Jacobi series*. Proc. Amer. Math. Soc., **23**:306–310.

[M2] B. MUCKENHOUPT (1970): *Mean convergence of Hermite and Laguerre series*. I. Trans. Amer. Math. Soc., **147**:419–431.

[M3] B. MUCKENHOUPT (1970): *Mean convergence of Hermite and Laguerre series*. II. Trans. Amer. Math. Soc., **147**:433–460.

[N] P. NEVAI (1979): *Orthogonal polynomials*. Mem. Amer. Math. Soc., **213**.

[P] H. POLLARD (1948): *The mean convergence of orthogonal series*. II. Trans. Amer. Math. Soc., **63**:355–367.

[R] E. A. RAHMANOV (1983): *On the asymptotics of the ratio of orthogonal polynomials*. II. Math. USSR. Sb., **46**:105–117.

[ST1] C. SEGOVIA, J. L. TORREA (1989): *Vector-valued commutators and applications*. Indiana Univ. Math. J., **38**:959–971.

[ST2] C. SEGOVIA, J. L. TORREA (1991): *Weighted inequalities for commutators of fractional and singular integrals*. Publicacions Matemàtiques, **35**:209–235.

[ST3] C. SEGOVIA, J. L. TORREA (1993): *Higher order commutators for vector-valued Calderón-Zygmund operators*. Trans. Amer. Math. Soc., **336**:537–556.

[SW] E. M. STEIN, G. WEISS (1975): Introduction to Fourier Analysis on Euclidean Spaces. Princeton, NJ: Princeton University Press.

[S] G. SZEGŐ (1975): Orthogonal Polynomials, 4th ed. Providence, RI: American Mathematical Society.

J. J. Guadalupe
Departamento de Matemáticas y Computación
Universidad de La Rioja
26004 Logroño
Spain

M. Pérez
Departamento de Matemáticas
Universidad de Zaragoza
50009 Zaragoza
Spain
mperez@cc.unizar.es

F. J. Ruiz
Departamento de Matemáticas
Universidad de Zaragoza
50009 Zaragoza
Spain

J. L. Varona
Departamento de Matemáticas y Computación
Universidad de La Rioja
26004 Logroño
Spain
jvarona@siur.unirioja.es