ON COMPLETENESS OF ORTHOGONAL SYSTEMS AND DIRAC DELTAS

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Given a positive measure μ supported on a set $\Omega \subseteq \mathbb{C}$, an orthonormal system $\{\varphi_n\}_{n\geq 0}$ and a point $a \in \Omega$, we study the relationship among $\mu(\{a\})$, the kernels $K_n(a,a) = \sum_{k=0}^{n} \varphi_k(a)\overline{\varphi_k(a)}$ and the denseness of $\operatorname{span}\{\varphi_n\}_{n\geq 0}$ *in $L^2(\mu)$ and in $L^2(\nu)$, where $\nu = \mu + M\delta_a$.*

0. INTRODUCTION

Let μ be a positive measure supported on a subset $\Omega \subseteq \mathbb{C}$ and $\{\varphi_n : \Omega \to \mathbb{C}\}_{n\geq 0}$ an orthonormal system in $L^2(\mu) = L^2(\Omega, \mu)$. Then

$$\int_\Omega \varphi_n \overline{\varphi}_m \, d\mu = \begin{cases} 0, & \text{if } n \neq m; \\ 1, & \text{if } n = m. \end{cases}$$

The system $\{\varphi_n\}_{n\geq 0}$ is said to be complete in $L^2(\mu)$ if the set $\operatorname{span}\{\varphi_n\}_{n\geq 0}$ of finite linear combinations is dense in $L^2(\mu)$ or, in other terms, if for each $\Phi \in L^2(\mu)$

$$\int_\Omega \Phi \overline{\varphi}_n \, d\mu = 0 \quad \forall n \geq 0 \iff \Phi = 0 \quad \mu\text{-a.e.}$$

(the orthogonality is not required here).

For each $n \geq 0$, set $\Pi_n = \Big\{ \sum_{k=0}^{n} \lambda_k \varphi_k ; \lambda_0, \dots, \lambda_n \in \mathbb{C} \Big\}$. The best $L^2(\mu)$ approximant in Π_n of any $f \in L^2(\mu)$ is given by the n-th partial sum of its Fourier series with respect to the set $\{\varphi_n\}_{n\geq 0}$. Thus the best approximant is

$$S_n(f,x) = \sum_{k=0}^{n} c_k(f) \varphi_k(x) = \int_\Omega f(y) K_n(x,y) \, d\mu(y),$$

where

$$c_k(f) = \int_\Omega f \overline{\varphi}_k \, d\mu, \quad K_n(x,y) = \sum_{k=0}^{n} \varphi_k(x) \overline{\varphi_k(y)}.$$

Furthermore,

$$\sum_{k=0}^{\infty} |c_k(f)|^2 \leq \int_\Omega |f|^2 \, d\mu \quad \forall f \in L^2(\mu) \tag{1}$$

(Bessel's inequality). If the system $\{\varphi_n\}_{n\geq 0}$ is complete in $L^2(\mu)$, then (1) becomes an equality (Parseval's equality) and the approximants $\{S_n(f)\}_{n\geq 0}$ converge to f in $L^2(\mu)$. This leads to an elementary proof of the following result (see [1, p. 63, 114], [6, p. 45] for more elaborate proofs and only in the case of systems of polynomials):

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Proposition 1. *Let $\{\varphi_n\}_{n\geq 0}$ be an orthonormal system in $L^2(\mu)$ and let $a \in \Omega$. Then*

$$\frac{1}{K_n(a,a)} \geq \mu(\{a\}) \quad \forall n \geq 0. \tag{2}$$

If $\mu(\{a\}) > 0$ and $\{\varphi_n\}_{n\geq 0}$ is complete, then

$$\lim_n \frac{1}{K_n(a,a)} = \mu(\{a\}). \tag{3}$$

Proof. We can assume $0 < \mu(\{a\}) < \infty$, otherwise the statement holds trivially. Let f be the characteristic function at the point a; then $c_k(f) = \overline{\varphi_k(a)}\mu(\{a\})$, so that

$$\sum_{k=0}^{\infty} |c_k(f)|^2 = \mu(\{a\})^2 \sum_{k=0}^{\infty} |\varphi_k(a)|^2$$

and

$$\int_\Omega |f|^2 \, d\mu = \mu(\{a\}).$$

Now, (2) and (3) follow from Bessel's and Parseval's formula, respectively. ∎

Concerning the measure μ and the kernels $K_n(a,a)$, we have the following well-known, elementary result (see [3, p. 38, theorem 7.3] or [4, p. 4], for example). In fact, inequality (2) can also be obtained as a corollary of lemma 1.

Lemma 1. *Let $\{\varphi_n\}_{n\geq 0}$ be an orthonormal system in $L^2(\mu)$ and let $a \in \Omega$. Then*

$$\frac{1}{K_n(a,a)} = \min \int_\Omega |R_n|^2 \, d\mu,$$

where the minimum is taken over all $R_n \in \Pi_n$ such that $R_n(a) = 1$. Furthermore, this minimum is attained for $R_n(x) = K_n(x,a)/K_n(a,a)$.

Our aim is to use proposition 1 and lemma 1 to obtain, using elementary techniques, some relations between $\lim_n K_n(a,a)^{-1}$ and certain properties of completeness of the system $\{\varphi_n\}_{n\geq 0}$.

1. Addition of a mass point

Obviously, one can not expect (3) to hold if the system $\{\varphi_n\}_{n\geq 0}$ is not complete. If $\{\varphi_n\}_{n\geq 0}$ is complete but $\mu(\{a\}) = 0$, it can also fail, for the values $\varphi_n(a)$ are μ-meaningless; in this case, we will prove that (3) holds if and only if the system $\{\varphi_n\}_{n\geq 0}$ is complete in $L^2(\nu)$. Here $\nu = \mu + M\delta_a$, where δ_a is a Dirac delta on a, $M > 0$ and as a consequence

$$\int_\Omega f \, d\nu = \int_\Omega f \, d\mu + M f(a).$$

The system $\{\varphi_n\}_{n\geq 0}$ may not be orthogonal in $L^2(\nu)$, but it can be orthonormalized so as to get an orthonormal system $\{\psi_n\}_{n\geq 0}$ in $L^2(\nu)$, such that $\psi_n = \sum_{k=0}^{n} \lambda_{n,k}\varphi_k$, with $\lambda_{n,n} \neq 0$. Clearly, if $\{\psi_n\}_{n\geq 0}$ (or, equivalently, $\{\varphi_n\}_{n\geq 0}$) is complete in $L^2(\nu)$, then $\{\varphi_n\}_{n\geq 0}$ is also complete in $L^2(\mu)$, but the converse is not true, in general.

In view of (3), we will mainly deal with the case $\mu(\{a\}) = 0$. However, note that if $\mu(\{a\}) > 0$ then the measures ν and μ are equivalent and so $\{\varphi_n\}_{n\geq 0}$ is complete in $L^2(\mu)$ if and only if $\{\psi_n\}_{n\geq 0}$ is complete in $L^2(\nu)$ (for the same reason, M could be taken equal to 1).

Let us state our first result (another proof of part "*a*) $\Longrightarrow$ *b*)" can be found in [2, lemma 2]):

THEOREM 1. *If $\{\varphi_n\}_{n\geq 0}$ is a complete orthonormal system in $L^2(\mu)$, $\mu(\{a\}) = 0$ and $\nu = \mu + M\delta_a$, then the following properties are equivalent:*

a) $\lim_n \dfrac{1}{K_n(a,a)} = 0$;

b) $\{\psi_n\}_{n\geq 0}$ *is a complete orthonormal system in* $L^2(\nu)$.

Proof. *a*) $\Longrightarrow$ *b*): Suppose $\{\psi_n\}_{n\geq 0}$ is not complete in $L^2(\nu)$; then, there exists $\Phi \in L^2(\nu)$, $\Phi \neq 0$, such that

$$\int_\Omega \Phi\overline{\psi}_n \, d\nu = 0 \quad \forall n \geq 0.$$

We can also assume

$$\int_\Omega |\Phi|^2 \, d\nu = 1,$$

so that $\{\Phi\} \cup \{\psi_n\}_{n\geq 0}$ is an orthonormal system in $L^2(\nu)$. Furthermore, $\Phi(a) \neq 0$, otherwise it would be orthogonal to $\{\varphi_n\}_{n\geq 0}$ in $L^2(\mu)$ and therefore $\Phi = 0$ μ and ν a.e.

Put $D_n(a,a) = \sum_{k=0}^{n} |\psi_k(a)|^2$. Then, by (2) applied to $\{\Phi\}\cup\{\psi_n\}_{n\geq 0}$, the fact $\Phi(a) \neq 0$, and lemma 1 respectively, we have the chain of inequalities

$$M = \nu(\{a\}) \leq \lim_n \frac{1}{|\Phi(a)|^2 + D_n(a,a)} < \lim_n \frac{1}{D_n(a,a)} = \lim_n \frac{1}{K_n(a,a)} + M$$

and so

$$\lim_n \frac{1}{K_n(a,a)} > 0,$$

which is a contradiction.

b) $\Longrightarrow$ *a*): By (3) applied to $\{\Phi\} \cup \{\psi_n\}_{n\geq 0}$ and lemma 1,

$$M = \lim_n \frac{1}{D_n(a,a)} = \lim_n \frac{1}{K_n(a,a)} + M$$

which gives a). ∎

Under the conditions of theorem 1, the orthonormal system $\{\psi_n\}_{n\geq 0}$ may not be complete in $L^2(\nu)$, but in this case it becomes complete by adding just one new function:

Proposition 2. *Let $\{\varphi_n\}_{n\geq 0}$ be a complete system in $L^2(\mu)$, $\mu(\{a\})=0$ and $\nu=\mu+M\delta_a$ and suppose $\{\psi_n\}_{n\geq 0}$ is not complete in $L^2(\nu)$. Then, the system $\{\Phi\}\cup\{\psi_n\}_{n\geq 0}$ is orthogonal (Φ is not normalized) and complete in $L^2(\nu)$, where*

$$\Phi(x)=\begin{cases}\displaystyle\sum_{k=0}^{\infty}\varphi_k(x)\overline{\varphi_k(a)}, & \text{if } x\neq a;\\ -\frac{1}{M}, & \text{if } x=a.\end{cases}$$

Proof. By theorem 1,

$$\sum_{k=0}^{\infty}|\varphi_k(a)|^2<\infty.$$

Then, as

$$\int_\Omega\Big|\sum_{k=n}^{m}\varphi_k(x)\overline{\varphi_k(a)}\Big|^2\,d\nu(x)$$

$$=M\sum_{k=n}^{m}|\varphi_k(a)|^2+\int_\Omega\Big|\sum_{k=n}^{m}\varphi_k(x)\overline{\varphi_k(a)}\Big|^2\,d\mu(x)=(M+1)\sum_{k=n}^{m}|\varphi_k(a)|^2,$$

the series $\sum_{k=0}^{\infty}\varphi_k(x)\overline{\varphi_k(a)}$ converges in $L^2(\nu)$ because its partial sums constitute a Cauchy sequence in $L^2(\nu)$. So Φ is a well defined function in $L^2(\nu)$. Now,

$$\int_\Omega\varphi_n\overline{\Phi}\,d\nu=M\varphi_n(a)\overline{\Phi(a)}+\sum_{k=0}^{\infty}\varphi_k(a)\int_\Omega\varphi_n\overline{\varphi}_k\,d\mu=0$$

for every $n\geq 0$; therefore, we also have

$$\int_\Omega\psi_n\overline{\Phi}\,d\nu=0\quad\forall n\geq 0$$

and $\{\Phi\}\cup\{\psi_n\}_{n\geq 0}$ is an orthogonal system in $L^2(\nu)$. In order to prove that it is complete in $L^2(\nu)$, it is enough to check that

$$f\in L^2(\nu),\quad\int_\Omega f\overline{\psi}_n\,d\nu=0\quad\forall n\geq 0\quad\Longrightarrow\quad f=C\Phi\quad\nu\text{-a.e.}$$

If

$$\int_\Omega f\overline{\psi}_n\,d\nu=0\quad\forall n\geq 0$$

then

$$\int_\Omega f\overline{\varphi}_n\,d\nu=0\quad\forall n\geq 0.$$

Thus,

$$\int_\Omega (f + Mf(a)\Phi)\overline{\varphi}_n \, d\mu = \int_\Omega (f + Mf(a)\Phi)\overline{\varphi}_n \, d\nu = 0 \quad \forall n \geq 0,$$

i.e.,

$$f + Mf(a)\Phi = 0 \quad \mu\text{-a.e.}$$

and so

$$f + Mf(a)\Phi = 0 \quad \nu\text{-a.e.} \ \blacksquare$$

Example. Let $\mathbb{T}$ be the unit circle, $\Omega = \mathbb{T} \cup \{0\}$ and μ the measure

$$\int_\Omega f \, d\mu = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \, d\theta.$$

Let $\{\varphi_n\}_{n\in\mathbb{Z}}$ be given by $\varphi_n(z) = z^n$ for $n \geq 0$ and $\varphi_n(z) = (\overline{z})^{-n}$ for $n < 0$ (the system is indexed in $\mathbb{Z}$, but this makes no difference). This system is orthonormal and complete in $L^2(\mu)$. Take $\nu = \mu + \delta_0$. Then, the system $\{\varphi_n\}_{n\in\mathbb{Z}}$ is not complete in $L^2(\nu)$ and $\Phi(0) = -1$, $\Phi(z) = 1$ for $z \in \mathbb{T}$.

Given any finite positive measure ν supported on the unit circle, the orthonormal system obtained from $\{z^n\}_{n\in\mathbb{Z}}$ is complete in $L^2(\nu)$ (see, e.g. [1, p. 180, theorem 5.1.2]). Then, if we consider a finite positive measure μ supported on the unit circle and the orthonormal system $\{\varphi_n\}_{n\in\mathbb{Z}}$ obtained from $\{z^n\}_{n\in\mathbb{Z}}$, part b) in theorem 1 holds, so that part a) also holds.

In contrast, the real case is more interesting, as we see in the next section.

2. Orthogonal polynomials on the real line

In the following we will consider a system $\{p_n\}_{n\geq 0}$ of polynomials (p_n of degree n) orthonormal with respect to some positive measure μ on $\mathbb{R}$.

The measure μ is said to be determinate if there does not exist any other positive measure η on $\mathbb{R}$ such that

$$\int_\mathbb{R} x^n \, d\mu(x) = \int_\mathbb{R} x^n \, d\eta(x) \qquad \forall n \geq 0;$$

otherwise, μ is said to be indeterminate.

The system $\{p_n\}_{n\geq 0}$ is complete in $L^2(d\mu)$ if and only if μ is *N-extremal* (see [5]). Every determinate measure is N-extremal, and every indeterminate N-extremal measure is a countable sum of Dirac deltas (see [2]).

If μ is determinate, then

$$\lim_n \frac{1}{K_n(a,a)} = \mu(\{a\}) \qquad \forall a \in \mathbb{R} \tag{4}$$

(see, e.g., [6, p. 45, corollary 2.6]), while if μ is indeterminate then

$$\lim_n \frac{1}{K_n(a,a)} > 0 \qquad \forall a \in \mathbb{C} \tag{5}$$

(see, e.g., [1, p. 50] and [6, p. 50, corollary 2.7]).

Now, the previous results provide a simple proof of the following:

Theorem 2. *Let $\{p_n\}_{n\geq 0}$ be a system of polynomials orthonormal with respect to a positive measure μ on $\mathbb{R}$. Let $a \in \mathbb{R}$, $M > 0$, $\nu = \mu + M\delta_a$. Then:*

a) μ indeterminate N-extremal, $\mu(\{a\}) = 0 \Longrightarrow \nu$ indeterminate not N-extremal.
b) μ indeterminate N-extremal, $\mu(\{a\}) > 0 \Longrightarrow \nu$ indeterminate N-extremal.
c) μ indeterminate not N-extremal $\Longrightarrow \nu$ indeterminate not N-extremal.
d) μ determinate, $\mu(\{a\}) = 0 \Longrightarrow \nu$ determinate or indeterminate N-extremal.
e) μ determinate, $\mu(\{a\}) > 0 \Longrightarrow \nu$ determinate.

Proof. *a*), *b*) and *c*): since μ is indeterminate, $\nu = \mu + M\delta_a$ is also indeterminate. Now, if μ is not N-extremal (i.e. the polynomials are not dense in $L^2(\mu)$) then clearly ν is not N-extremal. If μ is N-extremal and $\mu(\{a\}) > 0$, then ν is also N-extremal, for both measures are equivalent. Finally, if $\mu(\{a\}) = 0$, from (5) and theorem 1 it follows that the polynomials are not dense in $L^2(\nu)$.

d) and *e*): if μ is determinate, from (4) and theorem 1 it follows that the polynomials are dense in $L^2(\nu)$, so that ν is either determinate or indeterminate N-extremal. This proves d). Now, assume $\mu(\{a\}) > 0$. Then, μ and ν are equivalent measures. Take $b \in \mathbb{R}$ such that $\mu(\{b\}) = 0$. Applying d), the measure $\mu + \delta_b$ is N-extremal. Since $\mu + \delta_b$ is equivalent to $\nu + \delta_b$, this measure is also N-extremal. By part a), ν can not be indeterminate N-extremal, so that it is determinate. ■

Remark. Both cases in part d) can actually occur. Indeed, if

$$\nu = \sum_{k=0}^{\infty} M_k \delta_{a_k}$$

is either indeterminate N-extremal (every indeterminate N-extremal measure is of this form) or determinate (take, for example, $\{a_k\}_{k\geq 0}$ bounded), it can be shown that the measure

$$\mu = \sum_{k=1}^{\infty} M_k \delta_{a_k}$$

is determinate. A proof can be seen in [2]; it is also a consequence of inequality (5) and theorem 1.

In this context, let us mention that, in case d), if the measure μ is not discrete, then ν is not discrete; therefore, it must be also determinate.

References

[1] N. I. Akhiezer, *The Classical Moment Problem* (Oliver and Boyd, Edinburgh, 1965).
[2] C. Berg and J. P. R. Christensen, Density questions in the classical theory of moments, *Ann. Inst. Fourier* **31** (1981), 99–114.
[3] T. S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).
[4] P. Nevai, Orthogonal Polynomials, *Mem. Amer. Math. Soc.* **213** (1979).
[5] M. Riesz, Sur le problème des moments et le théorème de Parseval correspondant, *Acta Litt. Ac. Sci. Szeged* **1** (1923), 209–225.

[6] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (Math. Surveys No. 1, Amer. Math. Soc., Providence, RI, 1970).

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