FOURIER SERIES OF FUNCTIONS WHOSE HANKEL TRANSFORM IS SUPPORTED ON $[0,1]$

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Abstract. Let J_μ denote the Bessel function of order μ. For $\alpha > -1$, the system $x^{-\alpha/2-1/2}J_{\alpha+2n+1}(x^{1/2})$, $n = 0, 1, 2, \dots$ is orthogonal on $L^2((0,\infty), x^\alpha\,dx)$. In this paper we study the mean convergence of Fourier series with respect to this system for functions whose Hankel transform is supported on $[0,1]$.

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§1. Introduction.

Let $J_\mu(x)$ stand for the Bessel function of order μ and $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$ for the Jacobi polynomials (see [15] and Ch. VII and X in [5]). It is well known that the Jacobi polynomials are orthogonal on (-1,1) with respect to the weight $(1-x)^\alpha(1+x)^\beta$, $\alpha,\beta > -1$, and the Bessel functions satisfy the orthogonality relation

$$\int_0^\infty J_{\alpha+2n+1}(x)J_{\alpha+2m+1}(x)\,\frac{dx}{x} = \frac{\delta_{nm}}{2(2n+\alpha+1)}, \quad n,m = 0,1,2,\dots \qquad (\alpha > -1).$$

If we denote

$$j_n^\alpha(x) = \sqrt{\alpha+2n+1}\,J_{\alpha+2n+1}(\sqrt{x})x^{-\alpha/2-1/2}, \quad n = 0,1,2,\dots \tag{1}$$

then the system $\{j_n^\alpha\}_{n=0}^\infty$ is orthonormal on $L^2(x^\alpha) = L^2((0,\infty), x^\alpha\,dx)$.

The Bessel functions and the Jacobi polynomials are related by the formula (see [6])

$$\int_0^\infty J_{\alpha+2n+1}(t)J_\alpha(xt)\,dt = x^\alpha P_n^{(\alpha,0)}(1-2x^2)\chi_{[0,1]}(x).$$

Following [4] we define the Hankel transform $\mathcal{H}_\alpha$ of order $\alpha > -1$ to be the integral operator

$$\mathcal{H}_\alpha(f,x) = \frac{x^{-\alpha/2}}{2}\int_0^\infty f(t)J_\alpha(\sqrt{xt})t^{\alpha/2}\,dt, \quad x > 0, \tag{2}$$

for suitable functions f.

This means that

$$\mathcal{H}_\alpha(j_n^\alpha, x) = \sqrt{\alpha+2n+1}\,P_n^{(\alpha,0)}(1-2x)\chi_{[0,1]}(x), \tag{3}$$

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and therefore $\operatorname{supp}(\mathcal{H}_\alpha(j_n^\alpha)) \subseteq [0,1]$.

We consider the partial sums of the Fourier series with respect to the system $\{j_n^\alpha\}_{n=0}^\infty$:

$$S_n(f,x) = \sum_{k=0}^{n} c_k(f) j_k^\alpha(x), \qquad c_k(f) = \int_0^\infty f(t) j_k^\alpha(t) t^\alpha \, dt.$$

They can be written as

$$S_n(f,x) = \int_0^\infty f(t) K_n(x,t) t^\alpha \, dt, \quad \text{where} \quad K_n(x,t) = \sum_{k=0}^{n} j_k^\alpha(x) j_k^\alpha(t). \tag{4}$$

Series of this kind are a particular case of series $\sum_{n\geq 0} a_n J_{\alpha+n}$, which are usually called Neumann series. A study of their pointwise convergence can be found in [16] and [17].

The main aim in this paper is to study the convergence of $S_n f$ in the $L^p(x^\alpha)$-norm. This involves two problems:

a) To obtain uniform boundedness of the operators $S_n f$ in $L^p(x^\alpha)$.

b) To find the subspace of $L^p(x^\alpha)$ consisting of the functions f which can be approximated in the $L^p(x^\alpha)$-norm by its Fourier series, that is, to describe the space

$$B_{p,\alpha} = \overline{\operatorname{span}}\{j_n^\alpha(x)\}_{n=0}^\infty \quad (\text{closure in } L^p(x^\alpha)).$$

In order to solve a) the kernel K_n is decomposed in a suitable way which reduces the problem to show the boundedness of the Hilbert transform with weights, and hence some estimates for the Bessel functions and some results on A_p theory are needed. Some of these ideas have been used in the literature (see [1], [8], [9], [10], [12], [13]).

Regarding to b), looking at (3) we only need to deal with functions with Hankel transform supported on $[0,1]$. This leads us to consider, in a natural way, the analogous of the disc multiplier for the Hankel transform, i. e., the operator M_α defined by

$$\mathcal{H}_\alpha(M_\alpha f, x) = \mathcal{H}_\alpha(f,x)\chi_{[0,1]}(x).$$

Our problem of expanding a function whose Hankel transform is supported on $[0,1]$ with respect to an orthogonal system is in some sense similar to expanding a function whose Fourier transform is supported on $[-1,1]$, which has been treated in [1] using as an orthogonal system the spherical Bessel functions $\sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)$. The method we use to establish our results is easier than the one in [1]. Our development in §2 can be adapted to simplify some of the proofs in [1] and [2].

The paper is organized as follows: In section §2, we solve problem a). In section §3 we prove that the operator M_α, defined for suitable functions f, can be extended from $L^p(x^\alpha)$ into itself and it turns out to be the projection operator. This allows us to study problem b) obtaining, in section §4, a characterization of $B_{p,\alpha}$ in terms of M_α and solving the convergence of $S_n f$ to f in the $L^p(x^\alpha)$-norm.

§2. Uniform boundedness of the partial sums.

In what follows, $\alpha \geq -\frac{1}{2}$ and $p_0 = \frac{4(\alpha+1)}{2\alpha+3}$, $p_1 = \frac{4(\alpha+1)}{2\alpha+1}$. From the well known estimates (see [5] or [15])

$$J_\mu(x) = \frac{x^\mu}{2^\mu \Gamma(\mu+1)} + O(x^{\mu+2}), \quad x \to 0+$$

and

$$J_\mu(x) = \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{\mu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1}) \right], \quad x \to \infty \tag{5}$$

it follows

$$|J_\alpha(x)| \leq C_\alpha x^\alpha, \quad x \in (0,\infty) \tag{6}$$

and

$$|J_\alpha(x)| \leq C_\alpha x^{-1/2}, \quad x \in (0,\infty). \tag{7}$$

Given $p \in (1,\infty)$ and a fixed interval (a,b), a weight w is said to belong to the $A_p(a,b)$ class if

$$\left(\int_I w(x)\,dx\right)\left(\int_I w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leq C|I|^p$$

for every interval $I \subseteq (a,b)$, with C independent of I. An important application of A_p theory lies in its relation with the boundedness of the Hilbert transform

$$H(f,x) = \int_a^b \frac{f(t)}{x-t}\,dt.$$

Indeed, in [11] (see also [7] for further information) it is proved that

$$H : L^p((a,b),w) \longrightarrow L^p((a,b),w) \text{ bounded} \quad \Longleftrightarrow \quad w \in A_p(a,b).$$

Besides, the norm of the Hilbert transform operator and the constant in the A_p definition depend only one on each other. This allows us to use the uniform A_p theory in a similar way to [10]. Let us suppose that a family of weights $\{w_n\}_{n=0}^\infty$ defined in the same interval (a,b) satisfies the A_p condition with the same constant C (in this case we will say that $w_n \in A_p(a,b)$ uniformly). Then, the Hilbert transform H is uniformly bounded from $L^p((a,b),w_n)$ into itself, that is, with constant independent of n.

It is well known

$$x^\beta \in A_p(0,1) \Longleftrightarrow x^\beta \in A_p(1,\infty) \Longleftrightarrow x^\beta \in A_p(0,\infty) \Longleftrightarrow -1 < \beta < p-1. \tag{8}$$

Moreover, by making the change of variable $x = r_n z$ we easily obtain that, if $r_n \searrow 0$, then

$$(|x| + r_n)^\beta \in A_p(-1,1) \text{ uniformly} \iff -1 < \beta < p-1. \tag{9}$$

By using (8), it is easy to prove that, if $\alpha \geq -1/2$ and $p_0 < p < p_1$, then

$$x^{\alpha-\alpha p/2+p/4} \in A_p(0,\infty) \quad \text{and} \quad x^{\alpha-\alpha p/2-p/4} \in A_p(0,\infty).$$

The following result will be used later.

Lemma 1. *Let $\alpha \geq -1/2$, $\max\{\frac{4}{3}, p_0\} < p < \min\{4, p_1\}$ and $0 < s_n \nearrow \infty$. Then*

$$x^{\alpha-\alpha p/2+p/8}\left(|x^{1/2}-s_n| + s_n^{1/3}\right)^{p/4} \in A_p(0,\infty) \text{ uniformly} \tag{10}$$

and

$$x^{\alpha-\alpha p/2-p/8}\left(|x^{1/2}-s_n| + s_n^{1/3}\right)^{-p/4} \in A_p(0,\infty) \text{ uniformly}. \tag{11}$$

Proof.
Let us prove (11); the proof of (10) is similar. Making the change of variable $x = s_n^2 z$ in the A_p definition, it is easy to show that (11) is equivalent to proving

$$z^{\alpha-\alpha p/2-p/8}\left(|z^{1/2}-1| + s_n^{-2/3}\right)^{-p/4} \in A_p(0,\infty) \text{ uniformly}.$$

Taking into account the behaviour of this expression on the intervals $(0,1/2)$, $(1/2,3/2)$ and $(3/2,\infty)$, it is not difficult to check that we only need to see

$$z^{\alpha-\alpha p/2-p/8} \in A_p(0,\tfrac{1}{2}), \quad \left(|z^{1/2}-1| + s_n^{-2/3}\right)^{-p/4} \in A_p(\tfrac{1}{2},\tfrac{3}{2}), \quad z^{\alpha-\alpha p/2-p/4} \in A_p(\tfrac{3}{2},\infty).$$

The first and the third ones are true by (8). Finally, the second one follows from (9) by using $|z^{1/2}-1| \sim |z-1|$ and making a change of variable.

Theorem 1. *Let $1 < p < \infty$ and $\alpha \geq -1/2$. Then, there exists a constant C independent of n and f, such that*

$$\|S_n f\|_{L^p(x^\alpha)} \leq C\|f\|_{L^p(x^\alpha)} \qquad \forall f \in L^p(x^\alpha) \tag{12}$$

if and only if $\max\{\frac{4}{3}, p_0\} < p < \min\{4, p_1\}$.

Proof.
To find necessary conditions for (12) we apply the standard argument used for the first time in [12]. The uniform boundedness implies that of the operator $T_n = S_n - S_{n-1}$,

$$T_n(f,x) = c_n(f) j_n^\alpha(x) = j_n^\alpha(x) \int_0^\infty f(t) j_n^\alpha(t) t^\alpha \, dt.$$

Thus, by using duality we can easily obtain that (12) implies

$$\|j_n^\alpha\|_{L^p(x^\alpha)} \|j_n^\alpha\|_{L^q(x^\alpha)} \leq C. \tag{13}$$

Taking $n = 0$ and applying (5) and (1), we find that $p_0 < p < p_1$. Now, provided that $p_0 < p < p_1$, asymptotic estimates for $J_{\alpha+2n+1}(x)$ with n and x large enough allow us to show that

$$\|j_n^\alpha\|_{L^p(x^\alpha)} \sim \begin{cases} n^{-1-\alpha+2\alpha/p+2/p} & \text{if } p < 4 \\ n^{-1/2-\alpha/2}(\log n)^{1/4} & \text{if } p = 4 \\ n^{-5/6-\alpha+2\alpha/p+4/(3p)} & \text{if } p > 4 \end{cases}.$$

This, together with (13), implies $4/3 < p < 4$.

On the other hand, let us suppose that $\max\{\frac{4}{3}, p_0\} < p < \min\{4, p_1\}$ and prove the uniform boundedness of $S_n f$.

We need a suitable decomposition of the kernel $K_n(x,t)$. In [16] it is proved that

$$\phi_\mu(x,t) = \sum_{k=0}^{\infty} 2(\mu+2k+1) J_{\mu+2k+1}(x) J_{\mu+2k+1}(t)$$

satisfies

$$\phi_\mu(x,t) = \frac{xt}{x^2-t^2}\left\{x J_{\mu+1}(x) J_\mu(t) - t J_\mu(x) J_{\mu+1}(t)\right\}.$$

Consequently, by (4) and (1) it is clear that

$$K_n(x,t) = \frac{1}{2} x^{-\alpha/2-1/2} t^{-\alpha/2-1/2}\left\{\phi_\alpha(x^{1/2}, t^{1/2}) - \phi_{\alpha+2n+2}(x^{1/2}, t^{1/2})\right\}.$$

Now, by using $zJ_{\alpha+2n+3}(z) = (\alpha+2n+2)J_{\alpha+2n+2}(z) - zJ'_{\alpha+2n+2}(z)$ we obtain

$$\phi_{\alpha+2n+2}(x,t) = \frac{xt}{x^2-t^2}\left\{J_{\alpha+2n+2}(x) t J'_{\alpha+2n+2}(t) - J_{\alpha+2n+2}(t) x J'_{\alpha+2n+2}(x)\right\}.$$

Hence it follows

$$K_n(x,t) = \frac{x^{-\alpha/2} t^{-\alpha/2}}{2(x-t)}\left\{x^{1/2} J_{\alpha+1}(x^{1/2}) J_\alpha(t^{1/2}) - t^{1/2} J_\alpha(x^{1/2}) J_{\alpha+1}(t^{1/2})\right\}$$

$$+\frac{x^{-\alpha/2} t^{-\alpha/2}}{2(x-t)}\left\{x^{1/2} J'_{\alpha+2n+2}(x^{1/2}) J_{\alpha+2n+2}(t^{1/2}) - t^{1/2} J_{\alpha+2n+2}(x^{1/2}) J'_{\alpha+2n+2}(t^{1/2})\right\}.$$

This enables us to write

$$S_n(f,x) = W_1(f,x) - W_2(f,x) + W_{3,n}(f,x) - W_{4,n}(f,x),$$

where

$$W_1(f,x) = \frac{1}{2}\int_0^\infty x^{-\alpha/2+1/2} t^{-\alpha/2} \frac{J_{\alpha+1}(x^{1/2}) J_\alpha(t^{1/2})}{x-t} f(t) t^\alpha \, dt,$$

$$W_2(f,x) = \frac{1}{2}\int_0^\infty x^{-\alpha/2} t^{-\alpha/2+1/2} \frac{J_\alpha(x^{1/2}) J_{\alpha+1}(t^{1/2})}{x-t} f(t) t^\alpha \, dt,$$

$$W_{3,n}(f,x) = \frac{1}{2}\int_0^\infty x^{-\alpha/2+1/2}t^{-\alpha/2}\frac{J'_\nu(x^{1/2})J_\nu(t^{1/2})}{x-t}f(t)t^\alpha\,dt,$$

$$W_{4,n}(f,x) = \frac{1}{2}\int_0^\infty x^{-\alpha/2}t^{-\alpha/2+1/2}\frac{J_\nu(x^{1/2})J'_\nu(t^{1/2})}{x-t}f(t)t^\alpha\,dt$$

and $\nu = \alpha + 2n + 2$.

Therefore, to prove (12) it is sufficient to see

$$\|W_i f\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}, \qquad i = 1, 2$$

and

$$\|W_{i,n} f\|_{L^p(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}, \qquad i = 3, 4.$$

In order to apply A_p theory we will use the following estimates for Bessel functions and their derivatives:

(14) $$|J_\nu(x)| \le Cx^{-1/4}\left(|x-\nu| + \nu^{1/3}\right)^{-1/4},$$

(15) $$|J'_\nu(x)| \le Cx^{-3/4}\left(|x-\nu| + \nu^{1/3}\right)^{1/4},$$

where, again, $\nu = \alpha + 2n + 2$ and the constant C depends only on α. These inequalities can be easily deduced from those used in [1] (see also [15]).

Now, the boundedness of the operators W_1 and W_2 is equivalent to

$$\|x^{1/2-\alpha/2}J_{\alpha+1}(x^{1/2})H(t^{\alpha/2}J_\alpha(t^{1/2})f(t),x)\|_{L^p(x^\alpha)} \le C\|f(x)\|_{L^p(x^\alpha)}$$

and

$$\|x^{-\alpha/2}J_\alpha(x^{1/2})H(t^{1/2+\alpha/2}J_{\alpha+1}(t^{1/2})f(t),x)\|_{L^p(x^\alpha)} \le C\|f(x)\|_{L^p(x^\alpha)}.$$

By using (7) and $x^{\alpha-\alpha p/2-p/4} \in A_p(0,\infty)$ it follows

$$\|x^{-\alpha/2}J_\alpha(x^{1/2})H(t^{1/2+\alpha/2}J_{\alpha+1}(t^{1/2})f(t),x)\|^p_{L^p(x^\alpha)}$$

$$\le C_1\int_0^\infty \left|x^{-\alpha/2-1/4}H(t^{1/2+\alpha/2}J_{\alpha+1}(t^{1/2})f(t),x)\right|^p x^\alpha\,dx$$

$$\le C_2\int_0^\infty \left|x^{1/2+\alpha/2}J_{\alpha+1}(x^{1/2})f(x)\right|^p x^{\alpha-\alpha p/2-p/4}\,dx$$

$$\le C_3\int_0^\infty \left|x^{1/2+\alpha/2-1/4}f(x)\right|^p x^{\alpha-\alpha p/2-p/4}\,dx = C_3\|f(x)\|^p_{L^p(x^\alpha)}.$$

The verification of the other inequality is similar by using (7) and $x^{\alpha-\alpha p/2+p/4} \in A_p(0,\infty)$.

To finish the proof we will prove the uniform boundedness of $W_{4,n}$; that of $W_{3,n}$ can be obtained in a similar way. Taking $g(t) = J'_\nu(t^{1/2})t^{\alpha/2+1/2}f(t)$ and using consecutively (14), (11) and (15) we have

$$\|W_{4,n}f\|^p_{L^p(x^\alpha)} = 2^{-p}\int_0^\infty \left|\int_0^\infty x^{-\alpha/2}t^{\alpha/2+1/2}\frac{J_\nu(x^{1/2})J'_\nu(t^{1/2})}{x-t}f(t)\,dt\right|^p x^\alpha\,dx$$

$$= 2^{-p}\int_0^\infty |H(g,x)|^p \left|J_\nu(x^{1/2})\right|^p x^{-\alpha p/2+\alpha}\,dx$$

$$\le C_1\int_0^\infty |H(g,x)|^p x^{\alpha-\alpha p/2-p/8}\left(|x^{1/2}-\nu|+\nu^{1/3}\right)^{-p/4}\,dx$$

$$\le C_2\int_0^\infty |g(x)|^p x^{\alpha-\alpha p/2-p/8}\left(|x^{1/2}-\nu|+\nu^{1/3}\right)^{-p/4}\,dx \le C_3\|f\|^p_{L^p(x^\alpha)}$$

and the result follows.

§3. The projection operator M_α.

By using (6), the integral operator (2) exists for every $f \in L^1(x^\alpha)$ and $\mathcal{H}_\alpha f \in L^\infty(x^\alpha)$. Actually, we have

$$\|\mathcal{H}_\alpha f\|_{L^\infty(x^\alpha)} \le \tfrac{1}{2}C_\alpha\|f\|_{L^1(x^\alpha)}. \tag{16}$$

However (2) does not exist for $f \in L^p(x^\alpha)$ in general and so we will introduce the operator M_α in a similar way to that of the disc multiplier.

We consider the space

$$S^+ = \left\{f \in C^\infty(0,\infty) : \forall k,n \ge 0,\ |t^k f^{(n)}(t)| < C_{k,n}\right\}$$

with the topology generated by the seminorms $\|\cdot\|_{k,n}$, $k,n \in \mathbb{N}$, defined by $\|f\|_{k,n} = \sup_{t\in(0,\infty)} t^k|f^{(n)}(t)|$ (see [3] and [4]). It is easy to identify S^+ with the functions f such that $f(t) = \phi(t)$, $t \ge 0$, for some $\phi(t)$ in the Schwartz class S.

With this notation $\mathcal{H}_\alpha$ is an isomorphism of S^+ onto itself and $\mathcal{H}_\alpha^2$ is the identity map. Moreover, Fubini's Theorem implies the multiplication formula for the Hankel transform:

$$\int_0^\infty \mathcal{H}_\alpha(f,x)g(x)x^\alpha\,dx = \int_0^\infty \mathcal{H}_\alpha(g,x)f(x)x^\alpha\,dx, \quad f,g \in S^+. \tag{17}$$

Now, taking $g(x) = \mathcal{H}_\alpha(f,x)$ and using that $\mathcal{H}_\alpha^2 = \mathrm{Id}$, we have Parseval's formula $\|\mathcal{H}_\alpha f\|_{L^2(x^\alpha)} = \|f\|_{L^2(x^\alpha)}$.

Since S^+ is dense in $L^2(x^\alpha)$, the operator $\mathcal{H}_\alpha : S^+ \to S^+$ can be extended to $\mathcal{H}_\alpha : L^2(x^\alpha) \to L^2(x^\alpha)$ satisfying

$$\|\mathcal{H}_\alpha f\|_{L^2(x^\alpha)} = \|f\|_{L^2(x^\alpha)} \quad\text{and}\quad \mathcal{H}_\alpha^2 = \mathrm{Id}\,. \tag{18}$$

Let M_α be the operator defined by

$$M_\alpha(f,x) = \mathcal{H}_\alpha(\chi_{[0,1]}\mathcal{H}_\alpha f, x), \quad f \in S^+.$$

Note that if $f \in S^+$ then $\mathcal{H}_\alpha f \in S^+$; thus $\chi_{[0,1]}\mathcal{H}_\alpha f \in L^2(x^\alpha)$ and $M_\alpha f$ is well defined $\forall f \in S^+$.

Theorem 2. *Let $\alpha \geq -1/2$ and $p_0 < p < p_1$. Then there exists a constant $C_{p,\alpha}$ such that*

$$\|M_\alpha f\|_{L^p(x^\alpha)} \leq C_{p,\alpha}\|f\|_{L^p(x^\alpha)}, \quad \forall f \in S^+.$$

Therefore, M_α can be extended to an operator (also denoted M_α) bounded on $L^p(x^\alpha)$ such that

i) $\mathcal{H}_\alpha(M_\alpha f) = \mathcal{H}_\alpha(f)\chi_{[0,1]}$ *for all* $f \in L^2(x^\alpha) \cap L^p(x^\alpha)$.

ii) $M_\alpha^2 f = M_\alpha f$ *for all* $f \in L^p(x^\alpha)$.

iii) *Moreover, for $f \in L^p(x^\alpha)$ and $g \in L^q(x^\alpha)$, $1/p + 1/q = 1$, we have*

$$\int_0^\infty f(x)M_\alpha(g,x)x^\alpha\,dx = \int_0^\infty g(x)M_\alpha(f,x)x^\alpha\,dx.$$

Proof.
By using (2) and Fubini's Theorem we obtain

$$M_\alpha(f,x) = \frac{1}{4}\int_0^\infty \left(\int_0^1 J_\alpha(\sqrt{yt})J_\alpha(\sqrt{yx})\,dy\right) t^{\alpha/2}x^{-\alpha/2}f(t)\,dt.$$

Moreover, a change of variable in Lommel's formula

$$\begin{aligned}\int_0^1 J_\alpha(yt)J_\alpha(yx)y\,dy &= \frac{1}{t^2-x^2}\left(J_\alpha(t)xJ'_\alpha(x) - J_\alpha(x)tJ'_\alpha(t)\right)\\ &= \frac{1}{t^2-x^2}\left(tJ_{\alpha+1}(t)J_\alpha(x) - xJ_\alpha(t)J_{\alpha+1}(x)\right)\end{aligned}$$

(for the last equality, use $zJ'_\alpha(z) = \alpha J_\alpha(z) - zJ_{\alpha+1}(z)$) leads us to

$$\begin{aligned}M_\alpha(f,x) &= \frac{1}{2}\int_0^\infty \frac{t^{1/2}J_{\alpha+1}(t^{1/2})J_\alpha(x^{1/2}) - x^{1/2}J_\alpha(t^{1/2})J_{\alpha+1}(x^{1/2})}{t-x} t^{\alpha/2}x^{-\alpha/2}f(t)\,dt\\ &= W_1(f,x) - W_2(f,x),\end{aligned}$$

where W_1 and W_2 are bounded operators (see the proof of Theorem 1).

Now, taking into account that $\mathcal{H}_\alpha^2 = \mathrm{Id}$ and using standard density arguments, the statements i) and ii) follow easily. Let us now prove iii):

From (17), it easily follows that, for $f, g \in L^2(x^\alpha)$, we have

$$\int_0^\infty \mathcal{H}_\alpha(f,x)\mathcal{H}_\alpha(g,x)x^\alpha\,dx = \int_0^\infty f(x)g(x)x^\alpha\,dx. \tag{19}$$

Now, let U_1 and U_2 be the bilinear functionals on $L^p(x^\alpha) \times L^q(x^\alpha)$ defined by

$$U_1(f,g) = \int_0^\infty f(x)M_\alpha(g,x)x^\alpha\,dx$$

and

$$U_2(f,g) = \int_0^\infty g(x)M_\alpha(f,x)x^\alpha\,dx.$$

It suffices to show that U_1 and U_2 are bounded and coincide on the subset $\big(L^2(x^\alpha) \times L^2(x^\alpha)\big) \cap \big(L^p(x^\alpha) \times L^q(x^\alpha)\big)$, which is dense in $L^p(x^\alpha) \times L^q(x^\alpha)$. The boundedness of U_1 and U_2 is clear by Hölder's inequality and $p_0 < p, q < p_1$. Furthermore, by using (19), i) twice, and (19) once again, we have

$$\int_0^\infty f(x)M_\alpha(g,x)x^\alpha\,dx = \int_0^\infty \mathcal{H}_\alpha(f,x)\mathcal{H}_\alpha(M_\alpha g,x)x^\alpha\,dx = \int_0^1 \mathcal{H}_\alpha(f,x)\mathcal{H}_\alpha(g,x)x^\alpha\,dx$$

$$= \int_0^\infty \mathcal{H}_\alpha(M_\alpha f,x)\mathcal{H}_\alpha(g,x)x^\alpha\,dx = \int_0^\infty M_\alpha(f,x)g(x)x^\alpha\,dx$$

and so the proof is complete.

§4. Main consequences.

Definition. *Let $\alpha \geq -1/2$ and $p_0 < p < p_1$. We define*

$$E_{p,\alpha} = \{f \in L^p(x^\alpha) : M_\alpha f = f\}$$

endowed with the topology induced by $L^p(x^\alpha)$.

Proposition 1. *Let $\alpha \geq -1/2$, $p_0 < s < r < p_1$. Then $E_{s,\alpha} \subset E_{r,\alpha}$ and the inclusion is continuous and dense.*

Proof.
From (16) and (18), by using interpolation (see [14]), it follows

$$\|\mathcal{H}_\alpha f\|_{L^q(x^\alpha)} \leq C\|f\|_{L^p(x^\alpha)}, \quad 1 \leq p \leq 2 \leq q \leq \infty, \quad \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

On the other hand, if $p \leq 2$ and $p_0 < p < p_1$, for $f \in S^+$ we have

$$\|M_\alpha f\|_{L^\infty(x^\alpha)} = \|\mathcal{H}_\alpha(\chi_{[0,1]}\mathcal{H}_\alpha f)\|_{L^\infty(x^\alpha)}$$

$$\leq C_1\|\chi_{[0,1]}\mathcal{H}_\alpha f\|_{L^1(x^\alpha)} \leq C_2\|\mathcal{H}_\alpha f\|_{L^q(x^\alpha)} \leq C_3\|f\|_{L^p(x^\alpha)}.$$

By density,

$$\|M_\alpha f\|_{L^\infty(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}.$$

Since M_α is bounded from $L^p(x^\alpha)$ into itself, $p_0 < p < p_1$, by interpolation we obtain $\|M_\alpha f\|_{L^r(x^\alpha)} \le C\|f\|_{L^p(x^\alpha)}$, $r \ge p$. Interpolating one more time, we find

$$\|M_\alpha f\|_{L^r(x^\alpha)} \le C\|f\|_{L^s(x^\alpha)}, \quad p_0 < s < r < p_1. \tag{20}$$

If $f \in E_{s,\alpha}$ then $M_\alpha f = f$ and so (20) leads to $\|f\|_{L^r(x^\alpha)} \le C\|f\|_{L^s(x^\alpha)}$, which implies $E_{s,\alpha} \subset E_{r,\alpha}$.

To prove that the inclusion is dense, let $f \in E_{r,\alpha}$. Since $L^r(x^\alpha) \cap L^s(x^\alpha)$ is dense in $L^r(x^\alpha)$, for each $\varepsilon > 0$ there exists a function $g \in L^r(x^\alpha) \cap L^s(x^\alpha)$ such that $\|f - g\|_{L^r(x^\alpha)} < \varepsilon$. Taking $h = M_\alpha g$ it follows that $h \in E_{p,\alpha}$. Then

$$\|f - h\|_{L^r(x^\alpha)} = \|M_\alpha f - M_\alpha g\|_{L^r(x^\alpha)} \le \|M_\alpha\| \, \|f - g\|_{L^r(x^\alpha)} < C\varepsilon$$

and the proof is complete.

If $1/p + 1/q = 1$, the following result says that the dual space $(E_{p,\alpha})'$ is isomorphic to $E_{q,\alpha}$ in the standard sense

Proposition 2. *Let $\alpha \ge -1/2$ and T a bounded linear operator on $E_{p,\alpha}$, $p_0 < p < p_1$. There exists a unique function $g \in E_{q,\alpha}$, $1/p + 1/q = 1$, such that*

$$T(f) = \int_0^\infty f(x)g(x)x^\alpha \, dx \quad \forall f \in E_{p,\alpha}. \tag{21}$$

Furthermore, $C\|g\|_{L^q(x^\alpha)} \le \|T\| \le \|g\|_{L^q(x^\alpha)}$ for a constant $C > 0$ depending only on the norm of M_α in $L^q(x^\alpha)$.

Proof.

Let $T \in (E_{p,\alpha})'$. By Hahn-Banach's Theorem, T can be extended to $T \in (L^p(x^\alpha))'$ preserving its norm $\|T\|$. By duality between $L^p(x^\alpha)$ and $L^q(x^\alpha)$, there exists $h \in L^q(x^\alpha)$ such that

$$T(f) = \int_0^\infty f(x)h(x)x^\alpha \, dx \quad \forall f \in L^p(x^\alpha)$$

and $\|h\|_{L^q(x^\alpha)} = \|T\|$.

If $g = M_\alpha h$ from ii) in Theorem 2 it follows $g \in E_{q,\alpha}$. We are going to check that this function satisfies our purposes. If $f \in E_{p,\alpha}$, from iii) in Theorem 2 we have

$$T(f) = \int_0^\infty f(x)h(x)x^\alpha \, dx = \int_0^\infty M_\alpha(f,x)h(x)x^\alpha \, dx$$

$$= \int_0^\infty M_\alpha(h,x)f(x)x^\alpha \, dx = \int_0^\infty g(x)f(x)x^\alpha \, dx.$$

By using $p_0 < q < p_1$ and Hölder's inequality we have $\|T\| \le \|M_\alpha h\|_{L^q(x^\alpha)}$. Thus, the equivalence of norms $\|g\|_{L^q(x^\alpha)} \sim \|T\|$ follows immediately.

To prove the uniqueness, suppose $p \ge 2$ and there exist g and g' in $E_{q,\alpha}$ satisfying (21). Then

$$\int_0^\infty f(x)(g(x) - g'(x))x^\alpha \, dx = 0 \quad \forall f \in E_{p,\alpha}.$$

Taking $f = g - g' \in E_{q,\alpha} \subset E_{p,\alpha}$ it follows $g - g' = 0$ a. e. Now, the case $p < 2$ is a simple consequence of this considering that $L^q(x^\alpha)$ is reflexive and, since $E_{q,\alpha}$ is closed, then $E_{q,\alpha}$ is also reflexive.

As it was pointed out in Introduction, let $B_{p,\alpha}$ stands for the closure in $L^p(x^\alpha)$ of the space span $\{j_n^\alpha\}_{n=0}^\infty$. The following results show the main consequences in this section.

Corollary. *Let* $\alpha \ge -1/2$. *Then* $\lim_{n\to\infty} \|S_n f - f\|_{L^p(x^\alpha)} = 0$ *for every* $f \in B_{p,\alpha}$ *if and only if* $\max\left\{\frac{4}{3}, p_0\right\} < p < \min\{4, p_1\}$. *Moreover, if it is the case, then* $B_{p,\alpha} = E_{p,\alpha}$.

Proof.

i) By using (3), (18) and the completeness of the Jacobi system it follows that $\{j_n^\alpha\}_{n=0}^\infty$ is complete in $E_{2,\alpha}$ and so $B_{2,\alpha} = E_{2,\alpha}$. Hence the convergence for $p = 2$ is clear. When $p_0 < p < p_1$ then $j_n^\alpha \in E_{p,\alpha}$ and therefore $B_{p,\alpha} \subset E_{p,\alpha}$.

If $2 < p < p_1$ and $f \in E_{p,\alpha}$ it follows that for each $\varepsilon > 0$ there exists a function $g \in L^2(x^\alpha) \cap L^p(x^\alpha)$ such that $\|f - g\|_{L^p(x^\alpha)} < \varepsilon$. Taking $h = M_\alpha g$ we have $M_\alpha h = h$ and so $h \in E_{2,\alpha} \cap E_{p,\alpha} = B_{2,\alpha} \cap E_{p,\alpha}$. Since M_α is continuous then $\|f - h\|_{L^p(x^\alpha)} = \|M_\alpha f - M_\alpha g\|_{L^p(x^\alpha)} < C\varepsilon$. As $h \in B_{2,\alpha}$ it follows that there exists $h' \in \operatorname{span}\{j_n^\alpha\}_{n=0}^\infty$ such that $\|h - h'\|_{L^2(x^\alpha)} < \varepsilon$. By applying triangle inequality and Proposition 1 we obtain $\|f - h'\|_{L^p(x^\alpha)} < C_1\varepsilon$ and therefore $E_{p,\alpha} \subset B_{p,\alpha}$. Then the mean convergence for $2 < p < \min\{4, p_1\}$ is an immediate consequence of Theorem 1 and Banach-Steinhaus Theorem.

If $\max\left\{\frac{4}{3}, p_0\right\} < p < 2$, the result is easily obtained by duality.

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