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Endpoint weak boundedness of some polynomial expansions *

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Abstract

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Let $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ on $[-1, 1]$, $\alpha, \beta \geqslant -\frac{1}{2}$, and for each function f let $S_n f$ be the nth expansion in the corresponding orthonormal polynomials. We show that the operators $f \to u S_n(u^{-1} f)$ are not of weak (p, p)-type, where u is another Jacobi weight and p is an endpoint of the interval of mean convergence. The same result is shown for expansions associated to measures of the form $\mathrm{d}\nu = w(x)\,\mathrm{d}x + \sum_{i=1}^{k} M_i \delta_{a_i}$.

Keywords: Fourier–Jacobi series; weak boundedness; orthogonal polynomials

1. Introduction and main results

Let μ be a positive measure on $\mathbb{R}$ with infinitely many points of increase and such that all the moments

$$\int_{\mathbb{R}} x^n \,\mathrm{d}\mu, \quad n = 0, 1, \ldots,$$

exist. Let $\{P_n\}_{n \geqslant 0}$ stand for the corresponding orthonormal polynomials. For $f \in L^1(\mathrm{d}\mu)$, let $S_n f$ denote the nth partial sum of the orthonormal Fourier expansion of f in $\{P_n\}_{n \geqslant 0}$:

$$S_n(f, x) = \int_{\mathbb{R}} f(y) K_n(x, y)\,\mathrm{d}\mu(y), \quad K_n(x, y) = \sum_{k=1}^{n} P_k(x) P_k(y).$$

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The problem of the uniform boundedness of the partial sum operators S_n in weighted L^p spaces, that is,

$$\|uS_nf\|_{L^p(\mathrm{d}\mu)} \leqslant C\|uf\|_{L^p(\mathrm{d}\mu)}, \quad \forall n \geqslant 0,\ \forall f \in L^p(u^p\,\mathrm{d}\mu), \tag{1}$$

has been completely solved only in some specific cases (this boundedness implies, in rather general situations, the L^p convergence of S_nf to f). For example, Badkov [3] gave necessary and sufficient conditions for (1) when $\mathrm{d}\mu$ and u are generalized Jacobi weights (earlier results can be found in [15,21,22,24]). Orthogonal Hermite and Laguerre series were studied in [1,16,17].

Let us consider the case of a Jacobi weight on the interval $[-1, 1]$, that is, $\mathrm{d}\mu = w(x)\,\mathrm{d}x$,

$$w(x) = (1-x)^\alpha(1+x)^\beta,$$

and let $1 < p < \infty$. If $\alpha, \beta \geqslant -\frac{1}{2}$, then (see [15])

$$\|S_nf\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)}, \quad \forall n \geqslant 0,\ \forall f \in L^p(w), \tag{2}$$

if and only if p belongs to the open interval (p_0, p_1), where

$$p_0 = \frac{4(\alpha+1)}{2\alpha+3}, \qquad p_1 = \frac{4(\alpha+1)}{2\alpha+1},$$

when $\alpha \geqslant \beta$ (and the analogous formulas with α replaced by β if $\beta \geqslant \alpha$).

If both $\alpha, \beta > -\frac{1}{2}$, the authors proved (see [8]) that the nth partial sum operators are not of weak (p, p)-type when p is an endpoint of the interval of mean convegence. In Theorem 1 we extend this result to the weighted case $f \to uS_n(u^{-1}f)$, where u is also a Jacobi weight, $u(x) = (1-x)^a(1+x)^b$, $a, b \in \mathbb{R}$. Now, the weighted uniform boundedness (1) holds (see [15]) if and only if

$$\begin{aligned} \left|a + (\alpha+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| &< \min\{\tfrac{1}{4}, \tfrac{1}{2}(\alpha+1)\}, \\ \left|b + (\beta+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| &< \min\{\tfrac{1}{4}, \tfrac{1}{2}(\beta+1)\}. \end{aligned} \tag{3}$$

Let us state our first result.

Theorem 1. *Let $\alpha, \beta \geqslant -\frac{1}{2}$, $w(x) = (1-x)^\alpha(1+x)^\beta$, $u(x) = (1-x)^a(1+x)^b$, $1 < p < \infty$. Let S_n be the partial sum operators associated to w. If there exists a constant $C > 0$ such that for every $f \in L^p(u^pw)$ and for every $n \geqslant 0$,*

$$\|uS_nf\|_{L^p_*(w)} \leqslant C\|uf\|_{L^p(w)},$$

then the inequalities

$$\left|a + (\alpha+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| < \frac{1}{4}, \qquad \left|b + (\beta+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| < \frac{1}{4}$$

are verified.

On the other hand, we also study the weak boundedness of the operators S_n associated to a measure $d\nu = d\mu + \sum_{i=1}^{k} M_i \delta_{a_i}$, where $\mu\{a_i\} = 0$. In the particular case of a Jacobi weight and two mass points on 1 and -1, the corresponding orthonormal polynomials were studied in [10] from the point of view of differential equations (see also [2,4,11,12]). The authors have found (see [6]) some estimates for the orthonormal polynomials and kernels relative to this type of measures.

In this context, let us consider the polynomial expansion associated to a measure $d\nu = w(x)\,dx + \sum_{i=1}^{k} M_i \delta_{a_i}$, where $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $M_i > 0$, and take $u(x) = (1-x)^{a}(1+x)^{b}$ for $x \neq a_i$, $0 < u(a_i) < \infty$. With this notation, we can state the following result.

Theorem 2. *Let $\alpha, \beta \geqslant -\frac{1}{2}$, $1 < p < \infty$. Then, there exists a constant $C > 0$ such that*

$$\|uS_n f\|_{L^p_*(d\nu)} \leqslant C\|uf\|_{L^p(d\nu)}, \quad \forall f \in L^p(u^p\,d\nu),\ \forall n \geqslant 0,$$

if and only if the inequalities

$$\left|a + (\alpha+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| < \frac{1}{4}, \qquad \left|b + (\beta+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| < \frac{1}{4}$$

are verified.

2. Preliminary lemmas

A basic tool in the study of Fourier series on the interval $[-1, 1]$ is Pollard's decomposition of the kernels $K_n(x, t)$ (see [15,22]): if $\{P_n\}_{n\geqslant 0}$ is the sequence of polynomials orthonormal with respect to $w(x)\,dx$ and $\{Q_n\}_{n\geqslant 0}$ is the sequence of polynomials relating to $(1-x^2)w(x)\,dx$, then

$$K_n(x, t) = r_n T_{1,n}(x, t) + s_n T_{2,n}(x, t) + s_n T_{3,n}(x, t),$$

where

$$T_{1,n}(x, t) = P_{n+1}(x)P_{n+1}(t),$$

$$T_{2,n}(x, t) = (1-t^2)\frac{P_{n+1}(x)Q_n(t)}{x-t}, \qquad T_{3,n}(x, t) = (1-x^2)\frac{P_{n+1}(t)Q_n(x)}{t-x},$$

and $\{r_n\}$, $\{s_n\}$ are bounded sequences. In fact, for any measure μ on $[-1, 1]$ with $\mu' > 0$ a.e. (in particular, for $w(x)\,dx$),

$$\lim_{n\to\infty} r_n = -\tfrac{1}{2}, \qquad \lim_{n\to\infty} s_n = \tfrac{1}{2}$$

(this can be deduced from [13,22,23]). Therefore, we can write

$$S_n f = r_n W_{1,n} f + s_n W_{2,n} f - s_n W_{3,n} f,$$

where

$$W_{1,n} f(x) = P_{n+1}(x)\int_{-1}^{1} P_{n+1}(t) f(t) w(t)\,dt,$$

$$W_{2,n} f(x) = P_{n+1}(x) H\big((1-t^2)Q_n(t) f(t) w(t), x\big)$$

and

$$W_{3,n}f(x)=(1-x^2)Q_n(x)H(P_{n+1}(t)f(t)w(t),\ x),$$

H being the Hilbert transform on the interval $[-1, 1]$. Thus, the study of S_n can be reduced to that of $W_{i,n}$, $i=1, 2, 3$.

The boundedness of the Hilbert transform can be stated in terms of Muckenhoupt's A_p classes of weights (see [9,19]; throughout this paper, the Hilbert transform, as well as the A_p classes, are taken on the interval $[-1, 1]$): if u is a weight on $[-1, 1]$ and $1<p<\infty$, then $u\in A_p$ if and only if H is a bounded operator in $L^p(u)$, with a constant which depends only on the A_p constant of u.

Concerning mixed weak-norm inequalities for the Hilbert transform, we can state the following property, which can be proved in the same way as [18, Theorem 3]: assume that $u_1(x)$, $u_2(x)$, $v(x)\geqslant 0$, $1<p<\infty$ and there is a constant $C>0$ such that

$$\|u_2 Hg\|_{L^p_*(u_1)}\leqslant C\|g\|_{L^p(v)},\quad \forall g\in L^p(v);$$

then, there exists another constant $B>0$ which depends only on C, such that for every interval I,

$$\|u_2\chi_I\|_{L^p_*(u_1)}\left(\int_{-1}^{1}\frac{v(x)^{-1/(p-1)}}{(|I|+|x-x_I|)^q}\,\mathrm{d}x\right)^{1/q}\leqslant B, \tag{4}$$

x_I being the centre of I and $1/p+1/q=1$.

The polynomials P_n satisfy the estimate

$$|P_n(x)|\leqslant C(1-x)^{-(2\alpha+1)/4}(1+x)^{-(2\beta+1)/4},\quad \forall n,\ \forall x\in[-1, 1], \tag{5}$$

with a constant $C>0$ independent of x and n. A similar estimate is verified by Q_n, with $\alpha+1$ and $\beta+1$ instead of α and β:

$$|Q_n(x)|\leqslant C(1-x)^{-(2\alpha+3)/4}(1+x)^{-(2\beta+3)/4},\quad \forall n,\ \forall x\in[-1, 1]. \tag{6}$$

Thus, the following easy result will be useful.

Lemma 3. *Let* $r\in\mathbb{R}$. *Then,* $|x|^r\in A_p([-1, 1])\Leftrightarrow -1<r<p-1$.

The same property holds if we replace x by $x-a$, with $a\in[-1, 1]$. Even more, it is not difficult to show that in order to see whether a finite product of this type of expressions belongs to A_p, we only need to check the above inequalities for each factor separately.

We will eventually need to show that some of the operators are not of strong or weak type. In this sense, the following lemma (see [14]) will be used.

Lemma 4. *Let* supp $\mathrm{d}\alpha=[-1, 1]$, $\alpha'>0$ *a.e. in* $[-1, 1]$, *and* $0<p\leqslant\infty$. *There exists a constant* $C>0$ *such that if* g *is a Lebesgue-measurable function on* $[-1, 1]$, *then*

$$\left\|\alpha'(x)^{-1/2}(1-x^2)^{-1/4}\right\|_{L^p(|g|^p\mathrm{d}x)}\leqslant C\liminf_{n\to\infty}\|P_n\|_{L^p(|g|^p\mathrm{d}x)}.$$

There is a weak version of this property: it is a consequence of Kolmogorov's condition (see [5, Lemma V.2.8, p.485]) and the previous lemma.

Lemma 5. *Let* supp $\mathrm{d}\alpha=[-1,1]$, $\alpha'>0$ *a.e. in* $[-1,1]$, *and* $0<p<\infty$. *There exists a constant* $C>0$ *such that if* g, h *are Lebesgue-measurable functions on* $[-1,1]$, *then*

$$\left\|\alpha'(x)^{-1/2}(1-x^2)^{-1/4}g(x)\right\|_{L^p_*(|h|^p\,\mathrm{d}x)}\leqslant C\liminf_{n\to\infty}\|P_n g\|_{L^p_*(|h|^p\,\mathrm{d}x)}.$$

The following lemma will be useful to estimate some weighted L^p_* norms.

Lemma 6. *Let* $1\leqslant p<\infty$, r, $s\in\mathbb{R}$, $a>0$. *Then,*

$$\chi_{(0,a)}(x)x^r\in L^p_*(x^s\,\mathrm{d}x)\ \Leftrightarrow\ pr+s+1\geqslant 0,\quad (r,s)\neq(0,-1).$$

Moreover, in this case there is a constant K depending on r, s, p such that

$$\|\chi_{(0,a)}(x)x^r\|_{L^p_*(x^s\,\mathrm{d}x)}=Ka^{r+(s+1)/p}.$$

3. Proof of Theorem 1

The weak boundedness

$$\|uS_nf\|_{L^p_*(w)}\leqslant C\|uf\|_{L^p(w)}$$

implies the following conditions (see [8, Theorem 1], with the appropriate changes):

$$u\in L^p_*(w),\qquad u^{-1}\in L^q(w),$$
$$u(x)w(x)^{-1/2}(1-x^2)^{-1/4}\in L^p_*(w),$$
$$u(x)^{-1}w(x)^{-1/2}(1-x^2)^{-1/4}\in L^q(w),$$

where $1/p+1/q=1$. With the weight $u(x)=(1-x)^a(1+x)^b$ and having in mind that α, $\beta\geqslant-\frac{1}{2}$, this means

$$-\frac{1}{4}\leqslant a+(\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right)<\frac{1}{4},\qquad -\frac{1}{4}\leqslant b+(\beta+1)\left(\frac{1}{p}-\frac{1}{2}\right)<\frac{1}{4}.$$

Therefore, we only need to show that the equality cannot occur in the left-hand side of these equations. Assume, for example,

$$-\frac{1}{4}=a+(\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right).\tag{7}$$

Let us consider again Pollard's decomposition of the partial sums S_nf. We will prove that there exists a constant C such that

$$\|uW_{1,n}f\|_{L^p_*(w)}\leqslant C\|uf\|_{L^p(w)}\quad\text{and}\quad\|uW_{3,n}f\|_{L^p(w)}\leqslant C\|uf\|_{L^p(w)}.$$

This, together with the boundedness of S_n, implies the same property for $W_{2,n}$ and will lead to a contradiction.

(a) *Boundedness of* $W_{1,n}$. From its definition, we have

$$\|uW_{1,n}f\|_{L^p_*(w)} \leqslant \|uP_{n+1}\|_{L^p_*(w)} \|u^{-1}P_{n+1}\|_{L^q(w)} \|uf\|_{L^p(w)}.$$

So, we only need to prove

$$\|uP_n\|_{L^p_*(w)} \leqslant C, \ \forall n \in \mathbb{N}, \quad \text{and} \quad \|u^{-1}P_n\|_{L^q(w)} \leqslant C, \ \forall n \in \mathbb{N},$$

which follows from Lemma 6, (5) and the dominated convergence theorem.

(b) *Boundedness of* $W_{3,n}$. Using again (5) and (6), it is enough to obtain

$$\|Hg\|_{L^p(v)} \leqslant C\|g\|_{L^p(v)}, \quad \forall g \in L^p(v),$$

with

$$v(x) = (1-x)^{\alpha+ap+p(1-2\alpha)/4}(1+x)^{\beta+bp+p(1-2\beta)/4}.$$

Now, we only need to prove that $v \in A_p$. This can be deduced from Lemma 3.

(c) From (a), (b) and the hypothesis, we have a constant C such that for all $f \in L^p(u^p w)$ and every $n \in \mathbb{N}$,

$$\|uW_{2,n}f\|_{L^p_*(w)} \leqslant C\|uf\|_{L^p(w)},$$

that is,

$$\|uP_{n+1}Hg\|_{L^p_*(w)} \leqslant C\left\|u(x)(1-x^2)^{-1}Q_n(x)^{-1}w(x)^{-1}g\right\|_{L^p(w)}.$$

Applying (4), we have

$$\|uP_{n+1}\chi_I\|_{L^p_*(w)}\left(\int_{-1}^{1}\frac{u(x)^{-q}(1-x^2)^q|Q_n(x)|^q w(x)}{(|I|+|x-x_I|)^q}\,\mathrm{d}x\right)^{1/q} \leqslant C,$$

for every interval $I \subseteq [-1, 1]$, with a constant $C > 0$ independent of n and I; by Lemma 5 with $I = [1-\epsilon, 1]$, it follows

$$\left\|x^{a-\alpha/2-1/4}\chi_{[0,\epsilon]}\right\|_{L^p_*(x^\alpha)}\left(\int_0^1\frac{x^{-aq+q/4+\alpha(1-q/2)}}{(\epsilon+|x-\frac{1}{2}\epsilon|)^q}\,\mathrm{d}x\right)^{1/q} \leqslant C. \tag{8}$$

Now, by Lemma 6 and (7),

$$\left\|x^{a-\alpha/2-1/4}\chi_{[0,\epsilon]}\right\|_{L^p_*(x^\alpha)} = K \tag{9}$$

and

$$\int_0^1\frac{x^{-aq+q/4+\alpha(1-q/2)}}{(\epsilon+|x-\frac{1}{2}\epsilon|)^q}\,\mathrm{d}x = \int_0^1\frac{x^{1/(p-1)}}{(\epsilon+|x-\frac{1}{2}\epsilon|)^q}\,\mathrm{d}x \geqslant C\int_\epsilon^1 x^{1/(p-1)-q}\,\mathrm{d}x = C|\log\epsilon|,$$

which, together with (9), leads to a contradiction in (8). Therefore, (7) cannot be true and the theorem is proved.

4. Adding mass points

Let $d\mu$ be a positive measure on $\mathbb{R}$, $d\nu = d\mu + \sum_{i=1}^{k} M_i \delta_{a_i}$, where $M_i > 0$, $\mu\{a_i\} = 0$. Let also u be a weight such that $0 < u(a_i) < \infty$, $i = 1, \ldots, k$. We will denote by $\{K_n(x, y)\}$ the kernels relative to $d\mu$ and by $\{L_n(x, y)\}$ the kernels relative to $d\nu$. Then, the nth partial sum of the Fourier series with respect to $d\nu$ is given by

$$S_n f(x) = \int_{\mathbb{R}} L_n(x, y) f(y)\, d\nu(y).$$

Let us take $1 < p < \infty$, $1/p + 1/q = 1$ and

$$T_n f(x) = \int_{\mathbb{R}} L_n(x, y) f(y)\, d\mu(y).$$

Then the following theorem holds.

Theorem 7. *With the above notation, there exists a constant C such that*

$$\|uS_n f\|_{L^p_*(d\nu)} \leqslant C\|uf\|_{L^p(d\nu)}, \quad \forall n \geqslant 0,\ \forall f \in L^p(u^p\, d\nu), \tag{10}$$

if and only if there exists another constant C such that

(a) $\|uT_n f\|_{L^p_*(d\mu)} \leqslant C\|uf\|_{L^p(d\mu)}$, $\forall f \in L^P(u^p\, d\mu)$,
(b) $u(a_i)\|u^{-1}L_n(x, a_i)\|_{L^q(d\mu)} \leqslant C$, $\forall n \geqslant 0$, $i = 1, \ldots, k$,
(c) $\|uL_n(x, a_i)\|_{L^p_*(d\mu)} \leqslant Cu(a_i)$, $\forall n \geqslant 0$, $i = 1, \ldots, k$.
The same holds when replacing $L^p_*(d\nu)$ by $L^p(d\nu)$ *and* $L^p_*(d\mu)$ *by* $L^p(d\mu)$.

Proof. From the definition, it follows

$$S_n f(x) = T_n f(x) + \sum_{i=1}^{k} M_i L_n(x, a_i) f(a_i). \tag{11}$$

Now, suppose (10) holds. If $f \in L^p(u^p\, d\nu)$, let us define $g(x) = f(x)$ for $x \neq a_i$, $i = 1, \ldots, k$, and $g(a_i) = 0$, $i = 1, \ldots, k$. Since $\mu(\{a_i\}) = 0$, we have $S_n g = T_n f$ and

$$\|ug\|_{L^p(d\nu)} = \|uf\|_{L^p(d\mu)}.$$

Therefore, (10) implies

$$\|uT_n f\|_{L^p_*(d\nu)} \leqslant C\|uf\|_{L^p(d\mu)}, \quad \forall n \geqslant 0, \quad \forall f \in L^p(u^p\, d\nu). \tag{12}$$

Taking now $f = \chi_{\{a_i\}}$, we obtain $S_n f(x) = M_i L_n(x, a_i)$ and $\|uf\|_{L^p(d\nu)} = M_i^{1/p} u(a_i)$. Thus, (10) also implies

$$\|uL_n(x, a_i)\|_{L^p_*(d\nu)} \leqslant Cu(a_i), \quad \forall n \geqslant 0,\ i = 1, \ldots, k. \tag{13}$$

Actually, since $\|uf\|_{L^p(d\mu)}$, $u(a_i)|f(a_i)| \leqslant \|uf\|_{L^p(d\nu)}$ it is immediate from (11) that (12) and (13) imply (10). So, we only need to show that (12) is equivalent to (a) and (b) and that (13) is the same as (c).

It is easy to see that

$$\|uL_n(x, a_i)\|^p_{L^p_*(d\nu)} \leqslant \|uL_n(x, a_i)\|^p_{L^p_*(d\mu)} + \sum_{j=1}^{k} M_j u(a_j)^p |L_n(a_j, a_i)|^p.$$

Now, by Schwarz inequality we have

$$|L_n(a_j, a_i)| \leqslant L_n(a_j, a_j)^{1/2} L_n(a_i, a_i)^{1/2}$$

and $\{L_n(a_i, a_i)\}_{n \geqslant 0}$ is a bounded sequence, since $\mu(\{a_i\}) > 0$. Therefore,

$$\|uL_n(x, a_i)\|^p_{L^p_*(\mathrm{d}\mu)} \leqslant \|uL_n(x, a_i)\|^p_{L^p_*(\mathrm{d}\nu)} \leqslant \|uL_n(x, a_i)\|^p_{L^p_*(\mathrm{d}\mu)} + C,$$

and (13) is actually equivalent to (c).

Let us examine now condition (12). It is easy to see that

$$\|uT_n f\|^p_{L^p_*(\mathrm{d}\nu)} \leqslant \|uT_n f\|^p_{L^p_*(\mathrm{d}\mu)} + \sum_{i=1}^{k} M_i u(a_i)^p |T_n f(a_i)|^p,$$

$$\|uT_n f\|^p_{L^p_*(\mathrm{d}\mu)} \leqslant \|uT_n f\|^p_{L^p_*(\mathrm{d}\nu)} \quad \text{and} \quad M_i u(a_i)^p |T_n f(a_i)|^p \leqslant \|uT_n f\|^p_{L^p_*(\mathrm{d}\nu)}.$$

Thus, (12) holds if and only if condition (a) holds together with

$$u(a_i)|T_n f(a_i)| \leqslant C\|uf\|_{L^p(\mathrm{d}\mu)}, \quad \forall n \geqslant 0, \ \forall f \in L^p(u^p \, \mathrm{d}\mu), \ i = 1, \ldots, k.$$

Taking into acount that

$$T_n f(a_i) = \int_{\mathbb{R}} L_n(a_i, x) f(x) \, \mathrm{d}\mu(x),$$

this last inequality is simply (b). □

The proof can be rewritten with L^p norms instead of L^p_* norms.

The operators T_n can be handled in a similar way to expansions with respect to $\mathrm{d}\mu$. Regarding parts (b) and (c), let us introduce the following notation:

- $\mathrm{d}\mu^c(x) = (x - c)^2 \, \mathrm{d}\mu(x)$;
- $\{P_n^c\}$ is the sequence of orthonormal polynomials relative to $\mathrm{d}\mu^c$;
- $P_n^c(x) = k_n^c x^n + \cdots$, $k_n^c > 0$;
- $\{K_n^c(x, y)\}$ is the sequence of kernels relative to $\mathrm{d}\mu^c$.

Then the following proposition holds (see [6]).

Proposition 8. *Let* $\mathrm{d}\mu$ *be a positive measure on* $\mathbb{R}$, $c \in \mathbb{R}$, $M > 0$. *Let* $\{\tilde{P}_n\}_{n \geqslant 0}$ *be the polynomials orthonormal with respect to* $\mathrm{d}\mu + M\delta_c$. *Then, for each* $n \in \mathbb{N}$ *there exist two constants* A_n, $B_n \in (0, 1)$ *such that*

$$\tilde{P}_n(x) = A_n P_n(x) + B_n(x - c) P_{n-1}^c(x).$$

Furthermore, if $\operatorname{supp} \mathrm{d}\mu = [-1, 1]$, $\mu' > 0$ *a.e. and* $c \in [-1, 1]$, *then*

$$\lim_{n \to \infty} A_n K_{n-1}(c, c) = \frac{1}{\lambda(c) + M} \quad \text{and} \quad \lim_{n \to \infty} B_n = \frac{M}{\lambda(c) + M},$$

where

$$\lambda(c) = \lim_{n \to \infty} \frac{1}{K_n(c, c)}.$$

We can also find some relations which involve the kernels.

Proposition 9. *let* $d\mu$ *be a positive measure on* $\mathbb{R}$, $c \in \mathbb{R}$ *and* $M > 0$. *Let* $\{\tilde{K}_n\}_{n \geqslant 0}$ *be the kernels relative to* $d\mu + M\delta_c$. *Then* $\forall n \in \mathbb{N}$,

$$\tilde{K}_n(x, y) = \frac{1}{1 + MK_n(c, c)} K_n(x, y) + \frac{MK_n(c, c)}{1 + MK_n(c, c)} (x - c)(y - c) K_{n-1}^c(x, y).$$

Propositions 8 and 9 lead to bounds for $\tilde{P}_n$ and $\tilde{K}_n$, provided bounds for P_n, P_n^c, K_n, K_n^c are known. These bounds, together with Theorem 7, can be used to prove Theorem 2 (see [7]).

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