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Bergman and Bloch spaces of vector-valued functions

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We investigate Bergman and Bloch spaces of analytic vector-valued functions in the unit disc. We show how the Bergman projection from the Bochner-Lebesgue space $L_p(\mathbb{D}, X)$ onto the Bergman space $B_p(X)$ extends boundedly to the space of vector-valued measures of bounded p-variation $V_p(X)$, using this fact to prove that the dual of $B_p(X)$ is $B_p(X^*)$ for any complex Banach space X and $1 < p < \infty$. As for $p = 1$ the dual is the Bloch space $\mathcal{B}(X^*)$. Furthermore we relate these spaces (via the Bergman kernel) with the classes of p-summing and positive p-summing operators, and we show in the same framework that $B_p(X)$ is always complemented in $\ell_p(X)$.

1 Introduction

Throughout the paper X will be a complex Banach space, $1 \leq p < \infty$, $\mathcal{H}(\mathbb{D}, X)$ (resp. $\mathcal{P}(X)$) denotes the space of analytic functions (resp. polynomials) on the unit disc $\mathbb{D}$ taking values in X and $L_p(m, X)$ stands for the Bochner-Lebesgue p-integrable functions on $\mathbb{D}$ where m is the normalized Borel-Lebesgue measure on $\mathbb{D}$. We write $H_p(X)$ and $B_p(X)$ for the Hardy and Bergman spaces of vector-valued analytic functions respectively, which, using the notation $M_p(f, r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(re^{it})\|^p \, dt\right)^{1/p}$, consist of those functions in $\mathcal{H}(\mathbb{D}, X)$ where $\sup_{0<r<1} M_p(f, r) = \|f\|_{H_p(X)} < \infty$ and $\left(\int_0^1 M_p^p(f, r) r \, dr\right)^{1/p} = \|f\|_{B_p(X)} < \infty$.

A limiting case in the scale of Bergman spaces, which is useful for many purposes, is the Bloch space $\mathcal{B}(X)$, its elements being all functions in $\mathcal{H}(\mathbb{D}, X)$ such that $\sup_{|z|<1} \left(1 - |z|^2\right) \|f'(z)\| < \infty$.

For $X = \mathbb{C}$ the reader is referred to [3], [15] and [22] for the scalar-valued theory on these spaces, to [8], [9] or [10] for several properties of Bloch functions and their connection with multipliers between H_1 and $BMOA$ in the vector-valued setting, and finally to the paper [4] for properties on Taylor coefficient of functions in $B_p(X)$ and different results on multipliers between vector-valued Bergman spaces.

In this paper we shall study questions such as the boundedness of Bergman projection, the duality or the atomic decomposition in the vector-valued setting. The relationship between vector-valued analytic functions, vector measures and operators is also considered.

The paper is divided into four sections. In the first one we prove some elementary facts on the spaces $B_p(X)$ and $\mathcal{B}(X)$, showing that the norm of a function in $B_p(X)$ can be described in terms of its derivatives, in particular that $f \in B_p(X)$ if and only if $(1 - |z|) \|f'(z)\| \in L_p(m)$, which makes natural to introduce $\mathcal{B}(X)$ in the scale as a limiting case.

The second section is devoted to analyze the Bergman projection in the vector-valued setting. We see that the Bergman projection is bounded not only on $L_p(m, X)$ but even on the space of vector measures of bounded p-variation $V_p(m, X)$. This allows us, as in the scalar-valued case, to get the duality $(B_p(X))^* = B_{p'}(X^*)$ without conditions on the space X. It is also shown that the Bergman projection is bounded from $V_\infty(m, X)$ onto $\mathcal{B}(X)$, and a projection from the space of vector-valued measures of bounded variation $M(X)$ onto $B_1(X)$

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is also presented. Then we prove that $B_1(X)$ coincides with the projective tensor product $B_1\hat{\otimes}X$ and that $\mathcal{B}(X)$ can be identified with $\mathcal{L}(B_1, X)$. As a consequence the duality $(B_1(X))^* = \mathcal{B}(X^*)$ is obtained.

Next section is devoted to relate vector-valued analytic functions and operators. For any $\mathcal{L}(X,Y)$-valued analytic function $F(z) = \sum_{n=0}^{\infty} T_n z^n$ we can associate two linear operators, $T_F(x) = F_x$ where $F_x(z) = \sum_{n=0}^{\infty} T_n(x) z^n$ which maps elements in X into Y-valued analytic functions and $S_F(g) = \sum_{n=0}^{\infty} \frac{T_n(x_n)}{n+1}$ for any $g \in \mathcal{P}(X)$ such that $g(z) = \sum_{n\geq 0} x_n z^n$, which maps X-valued polynomials into vectors in Y. Under these identifications it is shown that $\mathcal{B}(\mathcal{L}(X,Y))$ can be regarded either as $\mathcal{L}(X, \mathcal{B}(Y))$ or as $\mathcal{L}(B_1(X), Y)$. Of course if $F \in B_p(\mathcal{L}(X,Y))$ then $T_F \in \mathcal{L}(X, B_p(Y))$ and $S_F \in \mathcal{L}(B_{p'}(X), Y)$ but the converse does not hold true in general. Some connections with the theory of p-summing and positive p-summing operators are provided. It is observed that $B_p(\Pi_p(X,Y))$ is continuosly embedded into $\Pi_p(X, B_p(Y))$ but again the converse is false. As a final result of our considerations we see that if $T \in \mathcal{L}(B_{p'}, X)$ and $f_T(z) = T(K_z)$, where K_z stands for the Bergman kernel, then f_T belongs to $B_p(X)$ if and only if the composition with the Bergman projection TP gives a positive p-summing operator from $L_{p'}(m)$ into X.

Finally, in the last section we show that $B_p(X)$ is always isomorphic to a complemented subspace of $\ell_p(X)$.

We write $\mathcal{L}(X,Y)$ (resp. $\mathcal{K}(X,Y)$) for the space of bounded (resp. compact) linear operators between the spaces X and Y, we denote x^*x the duality pairing in (X^*, X), $u_n(z) = z^n$ for $n \geq 0$ and any $f \in \mathcal{P}(X)$ is written $f = \sum_{n=0}^{N} u_n \otimes x_n$ for some $N \in \mathbb{N}$ where $(\phi \otimes x)(z) = \phi(z)x$ for $\phi \in \mathcal{H}(\mathbb{D}, \mathbb{C})$ and $x \in X$. As usual we use p' for the conjugate exponent, i.e. $1/p + 1/p' = 1$, and C denotes a constant that may vary from line to line.

2 Preliminaries

Definition 2.1 Let $1 \leq p < \infty$ and let X be a complex Banach space. $B_p(X)$ is defined as the space of X-valued analytic functions on the unit disc $\mathbb{D}$ such that

$$\|f\|_{B_p(X)} = \left(\int_{\mathbb{D}} \|f(z)\|^p \, dm(z) \right)^{1/p} < \infty .$$

As in the scalar-valued case one gets the following facts, whose proofs are left to the reader.

Proposition 2.2 *Let $1 \leq p < \infty$ and let X be a complex Banach space.*
(i) *$B_p(X)$ is a Banach space.*
(ii) *If $f \in B_p(X)$ then $\lim_{r\to 1} \|f - f_r\|_{B_p(X)} = 0$, where $f_r(z) = f(rz)$.*
(iii) *The space of X-valued analytic polynomials $\mathcal{P}(X)$ is dense in $B_p(X)$.*

Remark 2.3 If H is a complex Hilbert space, then $B_2(H)$ is also a Hilbert space under the scalar product given by

$$\langle\langle f, g \rangle\rangle = \int_{\mathbb{D}} \langle f(z), g(\bar{z}) \rangle \, dm(z) \quad (f, g \in B_2(H))$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on H.

For any $f \in B_2(H)$ such that $f(z) = \sum_{n=0}^{\infty} x_n z^n$ we have

$$\|f\|_{B_2(H)} = \left(\sum_{n=0}^{\infty} \frac{\|x_n\|^2}{n+1} \right)^{1/2} . \tag{2.1}$$

This shows that $B_2(H)$ is isometrically isomorphic to $\ell_2(H)$. Actually, if H is separable with an orthonormal basis $(e_n)_{n\geq 0}$ then $\left(\sqrt{n+1}\, u_n \otimes e_k\right)_{n,k\geq 0}$ is an orthonormal basis of $B_2(H)$.

Let us mention that (2.1) is no longer true for Banach spaces, as follows from the next easy example.

Example 2.4 Let $2 < p < \infty$ and let (e_n) be the canonical basis of ℓ_p. If $f(z) = \sum_{n=0}^{\infty} e_n z^n = (z^n)_{n=0}^{\infty}$ then $f \in B_2(\ell_p)$ but $\sum_{n=0}^{\infty} \frac{\|e_n\|^2}{n+1} = \infty$.

The reader is referred to [4] for further results on Taylor coefficients of functions in vector valued Bergman spaces and for connections with geometry of Banach spaces.

Let us point out that, as in the scalar-valued case, we have that for $f \in \mathcal{H}(\mathbb{D}, X)$, $0 < r < 1$ and $1 \le q \le \infty$, the following inequalities hold true:

$$r^2 M_q\big(f', r^2\big) \;\le\; \frac{M_q(f,r)}{1-r}\,. \tag{2.2}$$

$$M_q(f,r) \;\le\; \|f(0)\| + \int_0^r M_q(f',s)\,ds\,. \tag{2.3}$$

These facts can be used to get an equivalent norm in $B_p(X)$ by looking at the derivatives of the function rather than the function itself.

Theorem 2.5 (See [22].) *Let $f \in \mathcal{H}(\mathbb{D}, X)$, $n \in \mathbb{N}$, $1 \le p < \infty$. Then $f \in B_p(X)$ if and only if the function $z \mapsto \big(1-|z|^2\big)^n f^{(n)}(z) \in L_p(m, X)$.*

P r o o f. Let us show that for any $g \in \mathcal{H}(\mathbb{D}, X)$ and $k \ge 0$, the function $\big(1-|z|^2\big)^k g(z)$ belongs to $L_p(m,X)$ if and only if $\big(1-|z|^2\big)^{k+1} g'(z)$ also does. Then a recurrence argument gives the statement.

Note that $\big(1-|z|^2\big)^{k+1} g'(z) \in L_p(m,X)$ if and only if

$$\int_{\mathbb{D}} \big(1-|z|^2\big)^{pk+p} \|zg'(z)\|^p \, dm(z) \;<\; \infty\,.$$

Let us denote $h(z) = zg'(z) = \sum_{n=0}^{\infty} n x_n z^n$, and observe that for each $r < 1$ one has that $h_{r^2} = g_r * \lambda_r$, where $\lambda_r\big(e^{i\theta}\big) = re^{i\theta}\big(1-re^{i\theta}\big)^{-2}$.

Since $M_1(\lambda, r) = \frac{r}{1-r^2}$ and $M_p\big(h, r^2\big) \le M_1(\lambda, r) M_p(g,r)$, one gets that

$$\begin{aligned}
\int_{\mathbb{D}} \big(1-|z|^2\big)^{pk+p} \|zg'(z)\|^p \, dm(z) \;&=\; \int_0^1 4r^3\big(1-r^4\big)^{pk+p} M_p^p\big(h,r^2\big)\,dr \\
&\le\; \int_0^1 8r\big(1-r^2\big)^{pk} M_p^p(g,r)\,dr \\
&=\; C\int_{\mathbb{D}} \big(1-|z|^2\big)^{pk} \|g(z)\|^p\,dm(z)\,.
\end{aligned}$$

Conversely, let us take g such that $\big(1-r^2\big)^{k+1} M_p(g',r) \in L_p((0,1),dr)$. We may assume that

$$\int_0^1 (1-r)^{(k+1)p} M_p^p(g',r)\,dr \;=\; 1$$

and also that $g(0) = 0$.

Thanks to (2.3) we have

$$\begin{aligned}
\int_{\mathbb{D}} \big(1-|z|^2\big)^{kp} \|g(z)\|^p\,dm(z) \;&=\; \int_0^1 2r\big(1-r^2\big)^{kp} M_p^p(g,r)\,dr \\
&\le\; \int_0^1 2r\big(1-r^2\big)^{kp}\left(\int_0^r M_p(g',s)\,ds\right)^p dr \\
&\le\; C\int_0^1 (1-r)^{kp}\left(\int_0^r M_p(g',s)\,ds\right)^p dr\,.
\end{aligned}$$

For $p = 1$ we get

$$\begin{aligned}
\int_{\mathbb{D}} \big(1-|z|^2\big)^{k} \|g(z)\|\,dm(z) \;&\le\; C\int_0^1 (1-r)^k\left(\int_0^r M_1(g',s)\,ds\right) dr \\
&=\; C\int_0^1 (1-s)^{k+1} M_1(g',s)\,ds \;=\; C\,.
\end{aligned}$$

For $p > 1$, we write for each $t \in (0,1)$

$$I_t \;=\; \int_0^t (1-r)^{kp} \left(\int_0^r M_p(g', s)\, ds \right)^p dr\,.$$

Let

$$u(r) \;=\; -\frac{1}{pk+1}(1-r)^{pk+1} \quad \text{and} \quad v(r) \;=\; \left(\int_0^r M_p(g', s)\, ds \right)^p .$$

Since $u(t)v(t) < 0$ and $v(0) = 0$, we have

$$I_t \;=\; \int_0^t u'(r)v(r)\, dr \;\leq\; -\int_0^t u(r)v'(r)\, dr\,.$$

That is

$$\begin{aligned} I_t \;&\leq\; \frac{p}{pk+1} \int_0^t (1-r)^{pk+1} M_p(g', r) \left(\int_0^r M_p(g', s)\, ds \right)^{p-1} dr \\ &=\; \frac{p}{pk+1} \int_0^t (1-r)^{k+1} M_p(g', r)(1-r)^{(p-1)k} \left(\int_0^r M_p(g', s)\, ds \right)^{p-1} dr\,. \end{aligned}$$

Then the assumption and Hölder's inequality show that $I_t \leq C I_t^{1/p'}$. Hence $I_t \leq C$ for all t and the proof is finished. □

Taking the formulation in terms of the first derivative, it makes sense to look at the extreme case $p = \infty$ of Bergman spaces as functions in $\mathcal{H}(\mathbb{D}, X)$ such that the function $(1-|z|)^2 f'(z)$ belongs to $L_\infty(m, X)$.

Definition 2.6 The *Bloch space* $\mathcal{B}(X)$ is defined as the set of all functions in $\mathcal{H}(\mathbb{D}, X)$ for which $\sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \|f'(z)\| < \infty$. Under the norm

$$\|f\|_{\mathcal{B}(X)} \;=\; \|f(0)\| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \|f'(z)\|$$

it becomes a Banach space.

The *little Bloch space* $\mathcal{B}_0(X)$ is the subspace of $\mathcal{B}(X)$ given by those functions for which

$$\lim_{r\to 1} \left(1-r^2\right) M_\infty(f', r) \;=\; 0\,.$$

Remark 2.7 $f \in \mathcal{B}(X)$ if and only if $x^* f \in \mathcal{B}$ for all $x^* \in X^*$.

And, interchanging the suprema, we have that

$$\|f\|_{\mathcal{B}(X)} \;\approx\; \sup_{\|x^*\|=1} \|x^* f\|_{\mathcal{B}} \tag{2.4}$$

where $x^* f(z) = \langle f(z), x^* \rangle$.

Proposition 2.8 *If* $f \in \mathcal{B}(X)$ *then* $\|f\|_{\mathcal{B}(X)} = \lim_{r\to 1} \|f_r\|_{\mathcal{B}(X)}$.

P r o o f. Note that

$$\left(1-|z|^2\right) \|f_r'(z)\| \;=\; r\left(1-|z|^2\right) \|f'(rz)\| \;\leq\; \left(1-|rz|^2\right) \|f'(rz)\|$$

what implies that $\|f_r\|_{\mathcal{B}(X)} \leq \|f\|_{\mathcal{B}(X)}$ for all $0 < r < 1$.

Now, given $\varepsilon > 0$ take $z_0 \in \mathbb{D}$ such that $\left(1-|z_0|^2\right) \|f'(z_0)\| > \|f\|_{\mathcal{B}(X)} - \varepsilon/2$ and take r_0 verifying that $r\left(1-|z_0|^2\right) \|f'(rz_0)\| > \left(1-|z_0|^2\right) \|f'(z_0)\| - \varepsilon/2$ for any $r > r_0$. Hence

$$\|f_r\|_{\mathcal{B}(X)} \;>\; \|f\|_{\mathcal{B}(X)} - \varepsilon\,.$$

□

Theorem 2.9 *Let $f \in \mathcal{B}(X)$. The following are equivalent.*
(i) $f \in \mathcal{B}_0(X)$.
(ii) $\lim_{r\to 1} \|f - f_r\|_{\mathcal{B}(X)} = 0$.
(iii) *f belongs to the closure of $\mathcal{P}(X)$.*

Proof. (i) $\Rightarrow$ (ii). Assume that $\lim_{s\to 1} \big(1 - s^2\big) M_\infty(f', s) = 0$. Note that for all $0 < s < 1$ we have

$$\sup_{|z|<1} \big(1 - |z|^2\big) \|f'(z) - rf'(rz)\| \ \le\ 2 \sup_{|z|>s} \big(1 - |z|^2\big) M_\infty(f', |z|) + \sup_{|z|\le s} \|f'(z) - f_r'(z)\| \,.$$

Hence, given $\varepsilon > 0$ choose $s_0 < 1$ such that $\sup_{|z|>s_0} \big(1 - |z|^2\big) M_\infty(f', |z|) < \frac{\varepsilon}{4}$ and then use that f_r' converges uniformly on compact sets to get $r_0 < 1$ such that $\sup_{|z|\le s_0} \|f'(z) - f_r'(z)\| < \frac{\varepsilon}{2}$ for $r > r_0$. Then

$$\|f - f_r\|_{\mathcal{B}(X)} \ <\ \varepsilon \quad \text{for} \quad r \ >\ r_0 \,.$$

(ii) $\Rightarrow$ (iii). Assume now that, for each $\varepsilon > 0$, there exists $r_0 < 1$ such that $\|f - f_{r_0}\|_{\mathcal{B}(X)} < \varepsilon/2$. Now we can take a Taylor polynomial of f_{r_0} $P_N = P_N(f_{r_0})$ such that $\|f_{r_0} - P_N\|_{H_\infty(X)} < \varepsilon/2$. Therefore

$$\|f - P_N(f_{r_0})\|_{\mathcal{B}(X)} \ \le\ \|f - f_{r_0}\|_{\mathcal{B}(X)} + \|f_{r_0} - P_N\|_{H_\infty(X)} \ <\ \varepsilon \,.$$

(iii) $\Rightarrow$ (i). Note that $\mathcal{P}(X) \subset \mathcal{B}_0(X)$, because if $P \in \mathcal{P}(X)$ then

$$\big(1 - r^2\big) M_\infty(P', r) \ \le\ 2(1 - r) \max_{|z|\le 1} \|P'(z)\| \,.$$

Since $\mathcal{B}_0(X)$ is closed the result is proved. □

3 Bergman kernels and projections

Let us write $K(z, w) = \frac{1}{(1-zw)^2}$ and $K_z(w) = K(z, w)$ for $z, w \in \mathbb{D}$. That is

$$K_z \ =\ \sum_{n=0}^{\infty} (n + 1) u_n z^n \,.$$

Since $\|u_n\|_{H_p} = 1$, $\|u_n\|_{B_p} \sim n^{-1/p}$ and $\|u_n\|_{\mathcal{B}} \sim e^{-1}$ we have that, for each $|z| < 1$, the series $\sum_{n=0}^{\infty} (n + 1) u_n z^n$ is absolutely convergent considered as a $\mathcal{B}$, H_p or B_p-valued function. This allows us to consider $K : \mathbb{D} \to X$ given by $K(z) = K_z$ as an X-valued analytic function where X is either $\mathcal{B}$, H_p or B_p for $1 \le p \le \infty$.

We will call $K(z, w)$ the *Bergman kernel*, and the map $K : \mathbb{D} \to X$ the *Bergman function*. Of course $(n + 1)u_n$ are its Taylor coefficients, and its derivative is given by $K'(z) = \sum_{n=1}^{\infty} (n + 1) n u_n z^{n-1}$, with $K'(z)(w) = \frac{2w}{(1-zw)^3}$.

In order to estimate the norms of K in different spaces we simply need the following lemmas.

Lemma 3.1 (See [15], page 65.) *Let $J_\alpha(r) = \int_0^1 \frac{dt}{|1-re^{it}|^\alpha}$ for $\alpha > 0$. Then*
(i) *$J_\alpha(r)$ is bounded in $(0, 1)$ for $\alpha < 1$,*
(ii) *$J_\alpha(r) \sim \log \frac{1}{1-r}$ as $r \to 1$ for $\alpha = 1$, and*
(iii) *$J_\alpha(r) \sim \frac{1}{(1-r)^{\alpha-1}}$ as $r \to 1$ for $\alpha > 1$.*

Lemma 3.2 (See [22], 4.2.2.) *Let $I_{\alpha,\beta}(r) = \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-rw|^\beta} \, dm(w)$ for $\beta > 0$ and $\alpha > -1$. Then*
(i) *$I_{\alpha,\beta}(r)$ is bounded in $(0, 1)$ for $\beta - \alpha < 2$,*
(ii) *$I_{\alpha,\beta}(r) \sim \log \frac{1}{1-r}$ as $r \to 1$ for $\beta - \alpha = 2$, and*
(iii) *$I_{\alpha,\beta}(r) \sim \frac{1}{(1-r)^{\beta-\alpha-2}}$ as $r \to 1$ for $\beta - \alpha > 2$.*

From these lemmas we get the next estimates as $|z| \to 1$:

$$\|K(z)\|_{B_1} \sim \log\frac{1}{1-|z|} \quad \text{and} \quad \|K(z)\|_{B_p} \sim \frac{1}{(1-|z|)^{2/p'}} \text{ for } p > 1\,, \tag{3.1}$$

$$\|K(z)\|_{H_p} \sim \frac{1}{(1-|z|)^{2-1/p}} \quad \text{for} \quad p \geq 1\,, \tag{3.2}$$

$$\|K'(z)\|_{B_p} \sim \frac{1}{(1-|z|)^{3-2/p}} \quad \text{and} \quad \|K'(z)\|_{H_p} \sim \frac{1}{(1-|z|)^{3-1/p}}\,, \tag{3.3}$$

$$\|K(z)\|_{\mathcal{B}} \sim \frac{1}{(1-|z|)^2} \quad \text{and} \quad \|K'(z)\|_{\mathcal{B}} \sim \frac{1}{(1-|z|)^3}\,. \tag{3.4}$$

Proposition 3.3 *Let $1 \leq p, q < \infty$. Let $X \in \{B_q, H_q, \mathcal{B}, 1 \leq q \leq \infty\}$.*
(i) *The Bergman function $K \in B_p(X)$ if and only if $X = B_q$ and $2p < q'$.*
(ii) *The Bergman function $K \in \mathcal{B}(X)$ if and only if $X = B_1$.*
(iii) *The Bergman function $K \notin H_p(X)$.*

Definition 3.4 (See [14] or [12].) For any Banach space X, we denote by $M(X)$ the Banach space of vector (X-valued) measures of bounded variation defined on the Borel subsets of $\mathbb{D}$, with norm given by $\|G\|_1 = |G|(\mathbb{D})$.
For $1 < p < \infty$: A measure G is said to have *bounded p-variation,* $G \in V_p(m, X)$, if

$$\|G\|_p = \sup_\pi \left(\sum_{A\in\pi} \frac{\|G(A)\|^p}{m(A)^{p-1}} \right)^{\frac{1}{p}} < \infty\,,$$

where the supremum is taken over all finite partitions π of $\mathbb{D}$ into Borel sets of positive measure.

For $p = \infty$ we have that $G \in V_\infty(m, X)$ if there exists a constant $C > 0$ such that $\|G(A)\| \leq Cm(A)$ for any Borel set A, and its norm is given by

$$\|G\|_\infty = \sup\left\{ \frac{\|G(A)\|}{m(A)} : m(A) > 0 \right\}.$$

Remark 3.5 Given an X-valued simple measurable function $f = \sum_{k=1}^n x_k \chi_{A_k}$ and a X^*-valued measure G we denote by

$$\langle f, G\rangle = \sum_{k=1}^n x_k^* x_k$$

where $x_k^* = G(\bar{A}_k)$ and $\bar{A}_k = \{z \in \mathbb{D} : \bar{z} \in A_k\}$.

It is not difficult to see that if $G \in V_p(m, X^*)$ this extends to a linear functional in $L_{p'}(m, X)$ and actually we have the duality $(L_{p'}(m, X))^* = V_p(m, X^*)$ under this pairing (see [14]).

If G is an X-valued measure of bounded variation, and $\phi = \sum_{k=1}^n \alpha_k \chi_{E_k}$ is a simple function then we define

$$\int_{\mathbb{D}} \phi\, dG = \sum_{k=1}^n \alpha_k G(E_k)\,.$$

Since $\left\| \int_{\mathbb{D}} \phi\, dG \right\| \leq \|G\|_1 \|\phi\|_\infty$, using the density of simple functions we extend the definition of $\int_{\mathbb{D}} \phi\, dG$ for any bounded function ϕ.

Definition 3.6 Let $G \in M(X)$. We define the *Bergman projection* of the measure G as the analytic function in the disc given by

$$PG(z) = \int_{\mathbb{D}} K_z(\overline{w})\, dG(w) \in X\,.$$

Since $\sup_{w\in\mathbb{D}} \|K_z(w)\| \leq \frac{1}{(1-|z|)^2}$ then $PG(z)$ is well defined. Actually since the series

$$K_z \;=\; \sum_{n=0}^{\infty}(n+1)u_n z^n$$

is absolutely convergent in $L_\infty(m)$ for each $|z| < 1$ then for $z \in \mathbb{D}$ we have

$$PG(z) \;=\; \sum_{n=0}^{\infty} x_n z^n\,,$$

where $x_n = (n+1)\int_{\mathbb{D}} \overline{w}^n\, dG(w)$.

Remark 3.7 If $f \in L_1(m,X)$ then $Pf(z) = \int_{\mathbb{D}} f(w)K_z(\overline{w})\, dm(w)\,.$

In particular we have that $Pf = f$ for $f \in B_1(X)$.

Indeed, if $f(z) = \sum_{n=0}^{\infty} x_n z^n$ then

$$\begin{aligned}(n+1)\int_{\mathbb{D}} f(w)\overline{w}^n\, dm(w) \;&=\; (n+1)\int_0^1 2r^{n+1}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(re^{i\theta}\big)e^{in\theta}\, d\theta\Big)dr\\ &=\; (n+1)\Big(\int_0^1 2r^{2n+1}\, dr\Big)\, x_n = x_n.\end{aligned}$$

This shows that the Taylor coefficients of f and Pf coincide.

Theorem 3.8 *Let X be a complex Banach space and $1 < p < \infty$. Then the Bergman projection P is bounded from $V_p(m,X)$ onto $B_p(X)$.*

P r o o f. Since $G \in V_p(m,X)$ there exists a nonnegative ϕ in $L_p(m)$ such that $d|G| = \phi\, dm$ and $\|\phi\|_p = \|G\|_p$ (see [14], page 243).

Now, for each $z \in \mathbb{D}$ we have

$$\|PG(z)\| \;=\; \Big\|\int_{\mathbb{D}} K_z(\overline{w})\, dG(w)\Big\| \;\leq\; \int_{\mathbb{D}} \big|K_z(\overline{w})\big|\, d|G|(w) \;=\; \int_{\mathbb{D}} \big|K_z(\overline{w})\big|\phi(w)\, dm(w)\,.$$

Now to finish the proof, let us recall that if $P^*(f)(z) = \int_{\mathbb{D}} |K(z,\overline{w})|\, f(w)\, dm(w)$ then $P^* : L_p(m) \to L_p(m)$ defines a bounded operator for any $1 < p < \infty$ (see for instance [22] for a proof).

Therefore $\|PG\|_{B_p(X)} \leq \|P^*(\phi)\|_{L_p} \leq C\, \|\phi\|_{L_p} = C\, \|G\|_p$. □

This allows, as in the scalar valued case, to get the duality result for vector-valued Bergman spaces.

Theorem 3.9 *Let X be a complex Banach space and $1 < p < \infty$. Then $(B_p(X))^*$ is isometrically isomorphic to $B_{p'}(X^*)$.*

P r o o f. Let us define the linear operator $J : B_{p'}(X^*) \to (B_p(X))^*$ given by

$$(Jg)(f) \;=\; \int_{\mathbb{D}} g(z)f(\bar{z})\, dm(z)\,.$$

It follows from Hölder's inequality that J is bounded. Let us see that it is injective. If g verifies that $Jg = 0$, then for each $n \in \mathbb{N}$ and $x \in X$ we have

$$(Jg)(f_n) \;=\; \Big(\int_{\mathbb{D}} g(z)\bar{z}^n\, dm(z)\Big)x \;=\; 0\,,$$

where $f_n = u_n \otimes x$. This shows that $\displaystyle\int_{\mathbb{D}} g(z)\bar{z}^n\, dm(z) = \frac{1}{(n+1)!}\, g^{(n)}(0) = 0$ for all $n \in \mathbb{N}$ and hence $g = 0$.

Let us now show that J is surjective.

Given $\xi \in (B_p(X))^*$, the Hahn-Banach theorem gives an extension $\tilde{\xi} \in (L_p(m,X))^*$ with the same norm. Using duality (see Remark 3.5) there exists a vector valued measure $G \in V_{p'}(m, X^*)$, with p'-variation equal to $\|\xi\|$, for which $\tilde{\xi}\varphi = \int_{\mathbb{D}} \varphi \, dG$ for every $\varphi \in L_p(m,X)$.

Let G_c be the measure defined by $G_c(E) = G\big(\overline{E}\big)$ for each measurable set $E \subset \mathbb{D}$. Clearly G_c has the same p'-variation as G, and

$$\int_{\mathbb{D}} \psi(z)\, dG_c(z) \;=\; \int_{\mathbb{D}} \psi(\bar{z})\, dG(z)$$

for any simple function ψ. Define $g = PG_c$. From Theorem 3.8 we get $g \in B_{p'}(X^*)$. Let us see that $Jg = \xi$:

For any $f \in \mathcal{P}(X)$ we can write

$$\begin{aligned}
(Jg)(f) &= \int_{\mathbb{D}} PG_c(z) f(\bar{z})\, dm(z) &&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} K_z(\overline{w})\, dG_c(w) \right) f(\bar{z})\, dm(z) \\
&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} K_z(w)\, dG(w) \right) f(\bar{z})\, dm(z) &&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} K_z(w) f(\bar{z})\, dm(z) \right) dG(w) \\
&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} K_w(\bar{z}) f(z)\, dm(z) \right) dG(w) &&= \int_{\mathbb{D}} Pf(w)\, dG(w) \\
&= \int_{\mathbb{D}} f(w)\, dG(w) &&= \xi(f)\,.
\end{aligned}$$

□

Proposition 3.10 *P is not bounded neither on $M(X)$ nor on $V_\infty(m,X)$.*

Proof. Assume that P is bounded on $V_\infty(m,X)$. Using measures $dG = (\phi \otimes x)\, dm$ for $\phi \in L_\infty(m)$ and $x \in X$ we also have that the corresponding Bergman projection is bounded on $L_\infty(m)$. In such case

$$\sup_{|z|<1} \left| \int_{\mathbb{D}} K_z(w) \phi(w)\, dm(w) \right| \;\le\; C\, \|\phi\|_\infty$$

for all $\phi \in L_\infty(m)$. Hence $\sup_{|z|<1} \|K_z\|_{L_1(m)} \le C$, but we have previously noticed that $\|K_z\|_{L_1(m)} = \|K(z)\|_{B_1} \sim \log \frac{1}{1-|z|}$ as $|z| \to 1$.

The case $p = 1$ follows now looking at the adjoint operator. □

Theorem 3.11 *The Bergman projection P defines a bounded operator from $V_\infty(m,X)$ onto $\mathcal{B}(X)$.*

Proof. Let G belong to $V_\infty(m,X)$. Therefore there exists $C > 0$ such that

$$|G|(A) \;\le\; C m(A)$$

for all measurable sets A. Now from the Radon-Nikodym theorem there exists $\phi \in L_\infty(m)$ such that $d|G| = \phi\, dm$ and $\|\phi\|_{L_\infty} = \|G\|_\infty$.

On the other hand $PG(z) = \sum_{n=0}^\infty x_n z^n$, where $x_n = (n+1) \int_{\mathbb{D}} \bar{z}^n dG(z)$.

Since $(PG)'(z) = \int_{\mathbb{D}} \frac{2\overline{w}}{(1-\overline{w}z)^3}\, dG(w)$ we have

$$\|(PG)'(z)\| \;\le\; \int_{\mathbb{D}} \frac{2}{|1 - z\overline{w}|^3}\, \phi(w)\, dm(w) \;\le\; C\, \frac{\|\phi\|_{L_\infty}}{1-|z|}\,.$$

Let us prove the surjectivity. Let $f \in \mathcal{B}(X)$ with $f(0) = f'(0) = 0$.

If $f(z) = \sum_{n=2}^\infty x_n z^n$, let g be given by

$$g(z) \;=\; \frac{\big(1-|z|^2\big) f'(z)}{\bar{z}}\,.$$

We have that $g \in L_\infty(m, X)$ since $f \in \mathcal{B}(X)$ and $f'(0) = 0$. Now write $Pg(z) = \sum_{n=0}^{\infty} y_n z^n$ and take $n \geq 1$

$$\begin{aligned} y_n &= (n+1)\int_{\mathbb{D}} g(z)\bar{z}^n\, dm(z) \\ &= (n+1)\int_{\mathbb{D}} \big(1-|z|^2\big) f'(z)\bar{z}^{n-1}\, dm(z) \\ &= (n+1)\int_{\mathbb{D}} f'(z)\bar{z}^{n-1}\, dm(z) - (n+1)\int_{\mathbb{D}} z f'(z)\bar{z}^n\, dm(z) \;=\; x_n\,. \end{aligned}$$

Also $y_0 = 0$. That is $PG = f$ for $dG = g\, dm$.

The general case follows by writing $f = f(0) + u_1 \otimes f'(0) + f_1$ where f_1 is as above. So if $Pg_1 = f_1$ then $P(f(0) + u_1 \otimes f'(0) + g_1) = f$. □

Let us recall that the Riesz projection $R : L_p(\mathbb{T}) \to H_p(\mathbb{T})$ defined by $R(f) = \sum_{n\geq 0} \hat{f}(n) u_n$ gives, as happens for the Bergman projection on $L_p(m)$, a bounded operator only for $1 < p < \infty$. Nevertheless $H_1(\mathbb{T})$ is not isomorphic to any complemented subspace of $L_1(\mathbb{T})$, so we cannot define any bounded projection from $L_1(\mathbb{T})$ to $H_1(\mathbb{T})$, while we can define several bounded projections from $L_1(m)$ to B_1 (see [22]). Let us extend this also to the vector valued setting.

Definition 3.12 For any $G \in M(X)$, we can also define

$$\widetilde{P}G(z) \;=\; \int_{\mathbb{D}} \widetilde{K}_z(\overline{w})\, dG(w)\,,$$

where the kernel $\widetilde{K}_z(w) = \frac{2\left(1-|w|^2\right)}{(1-wz)^3} = \sum_{n=0}^{\infty}(n+1)(n+2)v_n(w)z^n$ and $v_n(w) = \big(1-|w|^2\big)w^n$.

Hence $\widetilde{P}(G)(z) = \sum_{n=0}^{\infty} x_n z^n$ where $x_n = (n+1)(n+2)\int_{\mathbb{D}} \big(1-|w|^2\big)\overline{w}^n\, dG(w)$.

Theorem 3.13 *$\widetilde{P}$ defines a bounded projection from $M(X)$ onto $B_1(X)$.*

P r o o f. Let us first see that $B_1(X)$ is left invariant under $\widetilde{P}$.

Let $dG(w) = f(w)\, dm(w)$ for some $f(z) = \sum_{n=0}^{\infty} x_n z^n$ in $B_1(X)$. Let us show that the Taylor coefficients of f and $\widetilde{P}(f)$ coincide.

$$\begin{aligned} \int_{\mathbb{D}} \big(1-|w|^2\big)\overline{w}^n dG(w) &= \int_{\mathbb{D}} \big(1-|w|^2\big)\overline{w}^n f(w)\, dm(w) \\ &= \int_0^1 2r^{n+1}\big(1-r^2\big)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(re^{i\theta}\big)e^{-in\theta} d\theta\right) dr \\ &= \left(\int_0^1 2r^{2n+1}\big(1-r^2\big)\, dr\right) x_n \\ &= \frac{x_n}{(n+1)(n+2)}\,. \end{aligned}$$

Given now $G \in M(X)$ we can write

$$\begin{aligned} \int_{\mathbb{D}} \big\|\widetilde{P}G(z)\big\|\, dm(z) &= \int_{\mathbb{D}} \left\|\int_{\mathbb{D}} \widetilde{K}_z(\overline{w})\, dG(w)\right\| dm(z) \\ &\leq \int_{\mathbb{D}}\left(\int_{\mathbb{D}} \big|\widetilde{K}_z(\overline{w})\big|\, d|G|(w)\right) dm(z) \\ &= \int_{\mathbb{D}}\left(\int_{\mathbb{D}} \big|\widetilde{K}_z(\overline{w})\big|\, dm(z)\right) d|G|(w)\,. \end{aligned}$$

Using Lemma 3.2 for $\alpha = 0$ and $\beta = 3$ we have

$$\int_{\mathbb{D}} \big|\widetilde{K}_z(\overline{w})\big|\, dm(z) \;=\; \int_{\mathbb{D}} \frac{2\big(1-|w|^2\big)}{|1-wz|^3}\, dm(z) \;\leq\; C$$

and then $\left\|\widetilde{P}G\right\|_{B_1(X)} \leq C\,|G|(\mathbb{D})$. □

Theorem 3.14 *$B_1(X)$ is isometrically isomorphic to $B_1\hat{\otimes}X$.*

Proof. Let $\widetilde{P}$ and $\widetilde{P}_X$ be the respective projections from $L_1(m)$ and $L_1(m,X)$ onto B_1 and $B_1(X)$ given above. By the properties of the projective tensor product, $\widetilde{P}\otimes \mathrm{id}_X$ is a projection from $L_1(m)\hat{\otimes}X$ onto $B_1\hat{\otimes}X$. Then the usual isometry J between $L_1(m)\hat{\otimes}X$ and $L_1(m,X)$ restricts to an operator $\tilde{J}$ from $B_1\hat{\otimes}X$ such that $\tilde{J}\big(\widetilde{P}\otimes \mathrm{id}_X\big) = \widetilde{P}_X J$, and $\tilde{J}$ is an isometry between $B_1\hat{\otimes}X$ and $B_1(X)$. □

Remark 3.15 The Bloch space was first shown to be a dual space in [1]. In fact one has that $(B_1)^* = \mathcal{B}$ and $(\mathcal{B}_0)^* = B_1$ (see [22]) under the pairing

$$\langle f, g\rangle \;=\; \int_{\mathbb{D}} f(z)\overline{g(z)}\,dm(z)\,,$$

which is well defined for polynomials and then extends by density for functions in B_1.

That it is well defined and bounded is seen as the first part in the following proposition:

Proposition 3.16 (i) *If $T\in\mathcal{L}(B_1,X)$ then $f_T(z) = T(K_z)\in\mathcal{B}(X)$.*
(ii) *If $f\in\mathcal{B}(X)$, the linear operator defined by $T_f(\phi) = \int_{\mathbb{D}} f(z)\phi(\bar{z})\,dm(z)$ for each polynomial ϕ extends to a bounded operator in $\mathcal{L}(B_1,X)$.*
(iii) *$\mathcal{B}(X)$ is isomorphic to $\mathcal{L}(B_1,X)$.*

Proof. (i) Let $g(z) = \frac{2w}{(1-zw)^3}$. One easily sees that $f_T'(z) = T(g_z)$ and $\|g_z\|_1 \sim 1/(1-|z|)$. This shows that $f_T\in\mathcal{B}(X)$.

(ii) From Remark 2.7 $x^*f\in\mathcal{B}$ and then $x^*T_f\in(B_1)^*$ for all $x^*\in X^*$. We have that T_f is bounded since $\|T_f(\phi)\| = \sup_{\|x^*\|=1}|x^*T_f(\phi)|$. Moreover

$$\left\|T_f\right\| \;=\; \sup_{\|x^*\|=1}\left\|x^*T_f\right\|_{(B_1)^*} \;\approx\; \sup_{\|x^*\|=1}\|x^*f\|_{\mathcal{B}} \;\approx\; \|f\|_{\mathcal{B}(X)}\,.$$

(iii) follows easily from (i) and (ii). □

We will explore further this interplay between functions and operators in Section 4.

Corollary 3.17 (See [7].) *$\mathcal{B}(X^*)$ is isomorphic to $(B_1(X))^*$.*

Proof. Since $(X\hat{\otimes}Y)^* = \mathcal{L}(X,Y^*)$, Proposition 3.16 and Theorem 3.14 give the result. □

4 Vector-valued functions and operators

Given two complex Banach spaces X and Y there are two natural ways of looking at a map $F:\mathbb{D}\times X\to Y$: it can be regarded as a map from X into $Y^{\mathbb{D}}$ or, alternatively, from $\mathbb{D}$ into Y^X (and vice-versa). More precisely, given $F:\mathbb{D}\times X\to Y$, we can define $F_x:\mathbb{D}\to Y$ and $F_z: X\to Y$ by

$$F_x(z) \;=\; F_z(x) \;=\; F(z,x)$$

for any $x\in X$ and $z\in\mathbb{D}$.

Proposition 4.1 *Let $F:\mathbb{D}\times X\to Y$ be a continuous map such that F_z is linear for all $z\in\mathbb{D}$. Then $F_z\in\mathcal{L}(X,Y)$ for all z, and the norm $\|F_z\|$ is locally bounded.*

Proof. First statement is immediate. To see the second one, let us assume there exists a compact set $K\subset\mathbb{D}$ where $\{\|F_z\|;\ z\in K\}$ is not bounded. By the Banach-Steinhaus theorem the set $A = \big\{x\in X;\ \sup_{z\in K}\|F(z,x)\|_Y = \infty\big\}$ will be dense in X. Then we can take two sequences $(x_j)\subset X$ and $(z_j)\subset K$ such that $x_j\to 0$ and $\left\|F\big(z_j,x_j\big)\right\|\geq j$. By the compactness of K, passing to a subsequence we see that we can assume that (z_j) converges to certain $z_0\in K$. But (z_j,x_j) tends to $(z_0,0)$ and $F(z_0,0)=0$, so F cannot be continuous. □

Theorem 4.2 *Let $F : \mathbb{D} \times X \to Y$ be continuous, such that F_z is linear for all $z \in \mathbb{D}$ and F_x is analytic for all $x \in X$. Then*

(i) *The map $z \mapsto F_z$ is an $\mathcal{L}(X, Y)$-valued analytic function.*

(ii) *The operator $x \mapsto F_x$ is linear, and continuous with respect to the topology of the uniform convergence on compact sets on the space $\mathcal{H}(\mathbb{D}, Y)$.*

Proof. For each $n \geq 0$ we define

$$T_n x \;=\; \frac{1}{2\pi i} \int_{|z|=r} \frac{F_z(x)}{z^{n+1}}\, dz\,.$$

Of course $F_x(z) = \sum_{n=0}^{\infty} (T_n x) z^n$.

It is clear that T_n is linear for each $n \in \mathbb{N}$. By the previous proposition, there exists C_r such that $\|F_z\| \leq C_r$ for all $z \in \overline{D(0,r)}$, and then $\|T_n x\| \leq C_r \|x\|/r^n$.

Now for each $z \in \mathbb{D}$ the series $\sum_{n\geq 0} T_n z^n$ is absolutely convergent in $\mathcal{L}(X, Y)$. Hence $z \mapsto \sum_{n\geq 0} T_n z^n$ defines an analytic function from $\mathbb{D}$ into $\mathcal{L}(X, Y)$.

To see (ii), note that the linearity is immediate, so it suffices to see that if $x_j \to 0$ then $F_{x_j} \to 0$ uniformly on compact sets. If $K \subset \overline{D(0,r)}$ is compact and we take $s \in (r, 1)$ and C such that $\|T_n\| \leq C/s^n$, then for any $z \in K$,

$$\big\|F_{x_j}(z)\big\|_Y \;\leq\; \sum_{n\geq 0} \|T_n\| \|x_j\| |z|^n \;\leq\; C\,\|x_j\| \sum_{n\geq 0} \Big(\frac{r}{s}\Big)^n \;=\; \frac{Cs}{s-r} \|x_j\|\,.$$

□

Definition 4.3 Let X, Y be two complex Banach spaces and let $F(z) = \sum_{n=0}^{\infty} T_n z^n$ be a function in $\mathcal{H}(\mathbb{D}, \mathcal{L}(X, Y))$. We denote by $T_F : X \to \mathcal{H}(\mathbb{D}, Y)$ the linear operator given by

$$(T_F x)(z) \;=\; (F(z))(x) \;=\; \sum_{n=0}^{\infty} (T_n x) z^n\,.$$

Theorem 4.4 *$\mathcal{B}(\mathcal{L}(X, Y))$ is isomorphic to $\mathcal{L}(X, \mathcal{B}(Y))$ (via the map $F \mapsto T_F$).*

Proof. Let $F \in \mathcal{B}(\mathcal{L}(X, Y))$. We have that

$$\sup_{\|x\|=1} \|F_x(0)\|_Y \;=\; \|F(0)\|_{\mathcal{L}(X,Y)}$$

and also

$$\begin{aligned} \sup_{\|x\|=1} \sup_{z\in\mathbb{D}} \big(1-|z|^2\big) \|F'(z)x\|_Y \;&=\; \sup_{z\in\mathbb{D}} \sup_{\|x\|=1} \big(1-|z|^2\big) \|F'(z)x\|_Y \\ &=\; \sup_{z\in\mathbb{D}} \big(1-|z|^2\big)\, \|F'(z)\|_{\mathcal{L}(X,Y)}\,. \end{aligned}$$

This shows that $\|T_F\|_{\mathcal{L}(X,\mathcal{B}(Y))} \approx \|F\|_{\mathcal{B}(\mathcal{L}(X,Y))}$.

Now given $T \in \mathcal{L}(X, \mathcal{B}(Y))$ we can define $F : \mathbb{D} \times X \to Y$ by $F(z, x) = Tx(z)$. Now by Theorem 4.2 we have that $z \mapsto F_z$ belongs to $\mathcal{H}(\mathbb{D}, \mathcal{L}(X, Y))$. Since $T_F = T$ the previous identities complete the proof. □

Proposition 4.5 *Let $1 \leq p < \infty$. $B_p(\mathcal{L}(X, Y)) \subset \mathcal{L}(X, B_p(Y))$ (via the mapping $F \mapsto T_F$). In general $B_p(\mathcal{L}(X, Y)) \neq \mathcal{L}(X, B_p(Y))$.*

Proof. Given $F \in B_p(\mathcal{L}(X, Y))$ we clearly have

$$\begin{aligned} \|T_F x\|_{B_p(Y)} \;&=\; \bigg(\int_{\mathbb{D}} \|F(z)x\|_Y^p\, dm(z)\bigg)^{1/p} \\ &\leq\; \|x\| \bigg(\int_{\mathbb{D}} \|F(z)\|_{\mathcal{L}(X,Y)}^p dm(z)\bigg)^{1/p} \;=\; \|F\|_{B_p(X)} \|x\|\,. \end{aligned}$$

To see that this inclusion is not surjective, let us take $X = \ell_1$ and $Y = \mathbb{C}$. The inclusion becomes $B_p(\ell_\infty) = B_p(\mathcal{L}(\ell_1,\mathbb{C})) \hookrightarrow \mathcal{L}(\ell_1, B_p) = \ell_\infty(B_p)$.

Let us consider the function $f(z) = \frac{1}{(1-z)^{1/p}}$. Clearly,

$$\|f\|_{B_p}^p \;\sim\; \int_0^1 \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\left|1-re^{i\theta}\right|}\,d\theta\right) dr \;\sim\; \int_0^1 \log\frac{1}{1-r}\,dr \;<\; \infty\,.$$

Now let (ζ_n) be a dense sequence in the unit circle, and define $f_n(z) = f(\zeta_n z)$ for each n and $F(z) = (f_n(z))_{n\in\mathbb{N}}$.

By the density of (ζ_n) one gets

$$\|F(z)\|_{\ell_\infty} \;=\; \sup_n |f_n(z)| \;=\; \sup_n |f(\zeta_n z)| \;=\; M_\infty(f,|z|) \;=\; \frac{1}{(1-|z|)^{1/p}}$$

for every $z \in \mathbb{D}$. Then, despite (f_n) obviously belongs to $\ell_\infty(B_p)$, we get that the vector valued function F does not belong to $B_p(\ell_\infty)$ since

$$\int_0^1 M_p^p(F,r)\,dr \;=\; \int_0^1 \frac{1}{1-r}\,dr \;=\; \infty\,.$$

□

Let us now introduce an interesting ideal of operators that play an important role in understanding the interpretation of vector-valued Bergman functions as operators.

Definition 4.6 Let X and Y be Banach spaces and $1 \le p < \infty$. A linear operator $T \in \mathcal{L}(X,Y)$ is said to be *p-summing* (denoted $T \in \Pi_p(X,Y)$) if there is a constant $C>0$ such that for every $k\in\mathbb{N}$ and $x_1, x_2, \ldots, x_k \in X$ we have

$$\left(\sum_{i=1}^k \|T(x_i)\|^p\right)^{1/p} \;\le\; C \sup_{\|x^*\|_{X^*}\le 1} \left(\sum_{i=1}^k |\langle x_i, x^*\rangle|^p\right)^{1/p}.$$

Its norm is given by the infimum of the constants C satisfying the previous inequality and is denoted by $\pi_p(T)$.

The reader is referred to [13], [21], [19] or [18] for results and references on these classes of operators. We simply include the following remark to be used in the sequel.

Remark 4.7 (See for instance [21].) Let (Ω,Σ,μ) be a measure space, let $f : \Omega \to X$ be a measurable function such that $x^* f \in L_p(\mu)$ for all $x^* \in X^*$ and $T \in \Pi_p(X,Y)$. Then $Tf : \Omega \to Y$ given by $Tf(\omega) = T(f(\omega))$ belongs to $L_p(\mu, Y)$.

Proposition 4.8 *Let $1 \le p < \infty$. Then $B_p(\Pi_p(X,Y)) \subset \Pi_p(X, B_p(Y))$ (via the mapping $F \mapsto T_F$). There exist infinite dimensional Banach spaces X and Y such that*

$$B_p(\Pi_p(X,Y)) \;\neq\; \Pi_p(X, B_p(Y))\,.$$

P r o o f. Let $x_1, x_2, ..., x_n$ be elements in X. Then

$$\begin{aligned}
\sum_{k=1}^n \|T_F x_k\|_{B_p(Y)}^p &= \sum_{k=1}^n \int_{\mathbb{D}} \|F(z)x\|_Y^p\,dm(z)\\
&= \int_{\mathbb{D}} \sum_{k=1}^n \|F(z)(x_k)\|^p\,dm(z)\\
&\le \int_{\mathbb{D}} \pi_p^p(F(z)) \sup_{\|x^*\|=1} \sum_{k=1}^n |\langle x^*, x_k\rangle|^p dm(z)\\
&= \|F\|_{B_p(\Pi_p(X,Y))}^p \sup_{\|x^*\|=1} \sum_{k=1}^n |\langle x^*, x_k\rangle|^p\,.
\end{aligned}$$

To see that the embedding is not surjective even for infinite dimensional Banach space we can take $p = 2$, $X = \ell_1$ and $Y = \ell_2$. It is well known (see [18]) that $\Pi_2(\ell_1, H) = \mathcal{L}(\ell_1, H)$ for any Hilbert space (actually $\Pi_1 \subseteq \Pi_2$, and Grothendieck theorem (see [13] or [18]) even says that $\Pi_1(\ell_1, H) = \mathcal{L}(\ell_1, H)$). Hence, in our situation $\Pi_2(X, Y) = \mathcal{L}(X, Y)$ and $\Pi_2(X, B_2(Y)) = \mathcal{L}(X, B_2(Y))$. Therefore we simply need to show that $B_2(\ell_\infty(\ell_2))$ is strictly contained in $\ell_\infty(B_2(\ell_2))$.

Let us define now $f_n(z) = \frac{1}{\log(n+1)} \sum_{k=1}^{\infty} (1 - 1/n)^k e_k z^k$ where e_k is the canonical basis of ℓ_2.

Using (2.1) one has that

$$\|f_n\|_{B_2(\ell_2)} = \frac{1}{\log(n+1)} \left(\sum_{k=1}^{\infty} \frac{(1-1/n)^{2k}}{k+1} \right)^{1/2} .$$

Hence $\sup_n \|f_n\|_{B_2(\ell_2)} < \infty$.

On the other hand, for all $n \in \mathbb{N}$

$$\begin{aligned} \|f_n(z)\|_{\ell_2} &= \frac{1}{\log(n+1)} \left(\sum_{k=1}^{\infty} (1-1/n)^{2k} |z|^{2k} \right)^{1/2} \\ &\geq C \frac{1}{\log(n+1)} \left(\sum_{k=n}^{\infty} |z|^{2k} \right)^{1/2} \\ &\geq C \frac{1}{\log(n+1)} \frac{|z|^n}{(1-|z|^2)^{1/2}} . \end{aligned}$$

This shows that $F \notin B_2(\ell_\infty(\ell_2)) = B_2(\Pi_2(X, Y))$ while $(f_n) \in \ell_\infty(B_2(\ell_2))$ and therefore

$$T_F \in \Pi_2(X, B_2(Y)) .$$

□

Definition 4.9 Let X, Y be two complex Banach spaces and let $F(z) = \sum_{n=0}^{\infty} T_n z^n$ be a function in $\mathcal{H}(\mathbb{D}, \mathcal{L}(X, Y))$. We denote by $S_F : \mathcal{P}(X) \to Y$ the linear operator given by

$$S_F(g) = \int_{\mathbb{D}} F(z)(g(\bar{z}))\, dm(z) = \sum_{n \geq 0} \frac{T_n(x_n)}{n+1}$$

for $g = \sum_{n \geq 0} u_n \otimes x_n$.

Theorem 4.10 (See [7].) *$\mathcal{B}(\mathcal{L}(X, Y)) = \mathcal{L}(B_1(X), Y)$ (via the map $F \mapsto S_F$) with equivalent norms.*

P r o o f. Theorem 3.14, Proposition 3.16 and $\mathcal{L}(X \hat{\otimes} Y, Z) = \mathcal{L}(X, \mathcal{L}(Y, Z))$ imply that

$$\mathcal{L}(B_1 \hat{\otimes} X, Y) = \mathcal{L}(B_1, \mathcal{L}(X, Y)) = \mathcal{B}(\mathcal{L}(X, Y)) .$$

It is rather clear that the mapping which gives the isomorphism is actually $F \mapsto S_F$. □

Proposition 4.11 *Let $1 < p < \infty$ and let X and Y be complex Banach spaces. Then $B_p(\mathcal{L}(X, Y))$ (resp. $\mathcal{B}_0(\mathcal{L}(X, Y))$) is isomorphically embedded in $\mathcal{K}(B_{p'}(X), Y)$ (resp. $\mathcal{K}(B_1(X), Y)$), via the map $F \mapsto S_F$.*

P r o o f. For the case $F \in \mathcal{B}_0(\mathcal{L}(X, Y))$, Theorem 4.10 gives $\|F\|_{\mathcal{B}(L(X,Y))} \approx \|S_F\|$.

In the case $F \in B_p(\mathcal{L}(X, Y))$. Clearly,

$$\|S_F(g)\| \leq \int_{\mathbb{D}} \|F(z)\| \, \|g(\bar{z})\| \, dm(z) \leq \|F\|_{B_p(\mathcal{L}(X,Y))} \|g\|_{B_{p'}(X)} .$$

So $\|S_F\| \leq C \, \|F\|_{B_p(\mathcal{L}(X,Y))}$. The compactness of S_F in both cases follows from the fact that $\mathcal{P}(X)$ is dense in the corresponding spaces and for polynomials F then S_F is a finite rank operator. □

Remark 4.12 Let X, Y be two complex Banach spaces. If $T : \mathcal{P}(X) \to Y$ is a linear operator such that the linear operators $T_n : X \to Y$ given by $T_n(x) = T(u_n \otimes x)$ are bounded and $\limsup_{n\to\infty} \|T_n\|^{1/n} \le 1$ then we can define the $\mathcal{L}(X, Y)$-valued analytic function

$$F_T(z) \;=\; \sum_{n\ge 0} (n+1) T_n z^n \,.$$

It is worth mentioning that this is the inverse map of $F \mapsto S_F$, so that $F_{S_F} = F$ and $S_{F_T} = T$.

Definition 4.13 Let $1 < p < \infty$, and let X be a complex Banach space and $T \in \mathcal{L}(B_{p'}, X)$. We define $f_T \in \mathcal{H}(\mathbb{D}, X)$ given by

$$f_T(z) \;=\; T(K_z)\,.$$

Remark 4.14 $T \in \mathcal{L}(B_1, X)$ if and only if $f_T \in \mathcal{B}(X)$ (see Proposition 3.16).
If $p > 2$ and $T \in \mathcal{L}(B_{p'}, X)$ then $f_T \in B_q(X)$ for $1 \le q < \frac{p}{2}$ (use Proposition 3.3).

We would like to find some properties of T to get that $f_T \in B_p(X)$.

For that purpose we need to use the following class of operators.

Definition 4.15 (See [6] and [5].) Let E be a Banach lattice and Y a Banach space. A linear operator $T \in \mathcal{L}(E, Y)$ is said to be *positive p-summing* (denoted $T \in \Lambda_p(E, Y)$) if there is a constant $C > 0$ such that for every $k \in \mathbb{N}$ and positive elements $e_1, e_2, \ldots, e_k \in E$ we have

$$\left(\sum_{i=1}^{k} \|T(e_i)\|^p\right)^{1/p} \;\le\; C \sup_{\|e^*\|_{E^*}\le 1} \left(\sum_{i=1}^{k} |\langle e_i, e^*\rangle|^p\right)^{1/p} .$$

Its norm is given by the infimum of the constants C satisfying the previous inequality and denoted by $\lambda_p(T)$.

Remark 4.16 In the case $p = 1$ these operators are also known as cone absolutely summing (c.a.s) operators (see [20]).

In this case, $T \in \Lambda_1(E, Y)$ if and only if there is a constant $C > 0$ such that for every $k \in \mathbb{N}$ and positive elements $e_1, e_2, \ldots, e_k \in E$ we have

$$\sum_{i=1}^{k} \|T(e_i)\| \;\le\; C \left\|\sum_{i=1}^{k} e_i\right\| . \tag{4.1}$$

It is easy to see that $\Lambda_{p_1}(E, Y) \subset \Lambda_{p_2}(E, Y)$ if $p_1 < p_2$, and it was shown in [5] that, for $E = L_p(\mu)$, we have $\Lambda_r(E, Y) = \Lambda_1(E, Y)$ for all $1 \le r \le p'$.

Theorem 4.17 *Let $1 < p < \infty$ and X a Banach space.*
(i) *If $T \in \mathcal{L}(B_{p'}, X)$ and $f_T \in B_p(X)$ then T is compact.*
(ii) *If $T \in \Pi_p(B_{p'}, X)$ then $f_T \in B_p(X)$.*
(iii) *If $T \in \Lambda_p(L_{p'}(m), X)$ and T_1 denotes its restriction to $B_{p'}$ then $f_{T_1} \in B_p(X)$.*

Proof. To see (i) we show that $T = S_{f_T}$ and then (ii) in Proposition 4.11 gives the compactness.
Indeed, since $f_T(z) = T(K_z) = \sum_{n=0}^{\infty} (n+1) T u_n z^n$, we have for any $m \in \mathbb{N}$

$$S_{f_T}(u_m) \;=\; \int_{\mathbb{D}} \sum_{n=0}^{\infty} (n+1) T u_n \bar{z}^n z^m \, dm(z) \;=\; T(u_m)\,.$$

To prove (ii) let us first observe that if $\phi \in B_p$

$$\langle K(z), \phi\rangle \;=\; \int_D K_z(w) \phi(\overline{w})\, dm(w) \;=\; \phi(z)\,.$$

Hence it follows that the function $K : \mathbb{D} \to B_{p'}$ verifies, for all $\phi \in (B_{p'})^*$, that $z \mapsto \langle K(z), \phi\rangle$ belongs to $L_p(m)$. Now Remark 4.7 gives that $f_T(z) = T(K_z) \in L_p(m, X)$.

To see (iii) let us observe first that the measure $G(E) = T(\chi_E)$ belongs to $V_p(m, X)$.
Indeed, for any partition π we have

$$\begin{aligned} \sum_{A\in\pi} \frac{\|G(A)\|^p}{m(A)^{p-1}} &= \sum_{A\in\pi} \left\| T\left(\frac{\chi_A}{m(A)^{1/p'}}\right)\right\|^p \\ &\le \lambda_p^p(T) \sup\left\{ \sum_{A\in\pi} \left| \int_A g(z)\,dm(z)\, \frac{1}{m(A)^{1/p'}}\right|^p : \|g\|_p = 1\right\} \\ &= \lambda_p^p(T) \sup\left\{ \left\| \sum_{A\in\pi} \frac{\int_A g(z)\,dm(z)}{m(A)}\,\chi_A \right\|_p^p : \|g\|_p = 1\right\} \\ &\le \lambda_p^p(T)\,. \end{aligned}$$

Given $z \in \mathbb{D}$ we get, taking $G_c(E) = G(\overline{E})$,

$$f_{T_1}(z) = T(K_z) = \int_{\mathbb{D}} K_z(w)\,dG(w) = \int_{\mathbb{D}} K_z(\overline{w})\,dG_c(w) = PG_c(z)\,,$$

and then $F_T = PG_c \in B_p(X)$ according to Theorem 3.8. □

Theorem 4.18 *Let $1 < p < \infty$, and let X be a complex Banach space and $F \in B_1(X)$. Then $F \in B_p(X)$ if and only if the linear operator $\Phi_F(\phi) = \int_{\mathbb{D}} F(z)\phi(\bar z)\,dm(z)$ defined on the subspace of simple functions extends to an operator in $\Lambda_p(L_{p'}(m), X)$.*
Moreover $\|F\|_{B_p(X)} \sim \lambda_p(\Phi_F)$.

Proof. Let us assume that $F \in B_p(X)$, which ensures that $\Phi_F \in \mathcal{L}(L_{p'}(m), X)$. Now take positive functions $\phi_1, \phi_2, \dots, \phi_n \in L_{p'}(m)$. We have that

$$\begin{aligned} \sum_{k=1}^n \|\Phi_F \phi_k\|^p &= \sum_{k=1}^n \left| \int_{\mathbb{D}} F(\bar z)\phi_k(z)\,dm(z)\right|^p \\ &\le \|F\|_{B_p(X)}^p \sum_{k=1}^n \left(\int_{\mathbb{D}} \frac{\|F(z)\|}{\|F\|_{B_p(X)}} \phi_k(z)\,dm(z)\right)^p \\ &\le \|F\|_{B_p(X)}^p \sup_{\|\psi\|_p = 1} \sum_{k=1}^n |\langle \psi, \phi_k\rangle|^p\,. \end{aligned}$$

This shows that $\lambda_p(\Phi_F) \le C\,\|F\|_{B_p(X)}$.
To see the converse let us observe that $f_S = F$, where S denotes the restriction of Φ_F to $B_{p'}$. Indeed, for all $z \in \mathbb{D}$

$$S(K_z) = \Phi_F(K_z) = \int_{\mathbb{D}} F(w) K_z(\overline{w})\,dm(w) = F(z)\,.$$

Now (iii) in Theorem 4.17 gives that $F \in B_p(X)$ and $\|F\|_{B_p(X)} \le C\lambda_p(\Phi_F)$. □

From Theorem 3.16 we have that $\mathcal{B}(X) = \mathcal{L}(B_1, X)$. The next result covers the cases $1 < p < \infty$.

Corollary 4.19 *Let $1 < p < \infty$. Then*

$$B_p(X) = \{T : B_{p'} \to X : TP \in \Lambda_p(L_{p'}(m), X))\}\,.$$

Moreover $\lambda_p(TP) \approx \|f_T\|_{B_p(X)}$.

Proof. If TP is positive p-summing then (iii) in Theorem 4.17 gives that $F \in B_p(X)$ for $F(z) = T(K_z)$.

To see the converse, assume that $F \in B_p(X)$. Let Φ_F as in Theorem 4.18 and let T be its restriction to $B_{p'}$. Now take the vector measure defined by $G(E) = T(P(\chi_E))$ and denote $G_c(E) = G\big(\overline{E}\big)$ for all measurable set E. We have that

$$\begin{aligned} G_c(E) \;&=\; T\Big(\int_{\overline{E}} K(.,\overline{w})\, dm(w)\Big) \;=\; T\Big(\int_E K(.,w)\, dm(w)\Big) \\ &=\; \int_E T(K(.,w))\, dm(w) \;\;=\; \int_E \tilde{\Phi}_F(K_w)\, dm(w) \;\;=\; \int_E F(w)\, dm(w)\,. \end{aligned}$$

Therefore $dG_c = F\, dm$.

This obviously implies that $TP(\phi) = \int_{\mathbb{D}} F(z)\phi(\bar{z})\, dm(z)$ for all $\phi \in L_{p'}(m)$, and, in particular

$$\|TP(\phi)\| \;\le\; \int_{\mathbb{D}} \|F(z)\|\, \phi(\bar{z})\, dm(z)$$

for all positive $\phi \in L_{p'}(m)$.

A simple computation using (4.1) now shows that TP is cone abolutely summing and hence also positive p-summing. □

5 $B_p(X)$ is complemented in $\ell_p(X)$

A classical result in the theory of Bergman spaces is the isomorphism between B_p and ℓ_p for each $p \ge 1$ (see [21]). It is enough to see that B_p is isomorphic to a complemented subspace of ℓ_p, since then it is automatically isomorphic to ℓ_p. In the vector case, Theorem 3.14 gives the isomorphism for $p = 1$:

Theorem 5.1 *For any complex Banach space X, $B_1(X)$ is isomorphic to $\ell_1(X)$.*

Proof. $B_1(X)$ is isomorphic to $B_1\hat{\otimes}X$, and then to $\ell_1\hat{\otimes}X = \ell_1(X)$. □

As for $p > 1$, we will show next that $B_p(X)$ is isomorphic to a complemented subspace of $\ell_p(X)$. The proof follows similar ideas to the ones used to get a so-called atomic decomposition of B_p (see [22], Theorem 4.4.6).

For each $z \in \mathbb{D}$, let φ_z the involutive Möbius transformation fixing the unit disc and verifying $\varphi_z(0) = z$ and $\varphi_z(z) = 0$, that is

$$\varphi_z(w) \;=\; \frac{z - w}{1 - \bar{z}w}\,.$$

The Bergman metric between z and w is defined by

$$\beta(z,w) \;=\; \frac{1}{2}\log\frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}\,.$$

Note that $|\varphi_z(w)|$ is the hyperbolic tangent of $\beta(z,w)$.

This distance β is not bounded on $\mathbb{D}$, and for any $z \in \mathbb{D}$ and $r > 0$ the β-ball

$$E(z,r) \;=\; \{w \in \mathbb{D};\ \beta(w,z) < r\}$$

is the euclidean disc with center $\frac{1-s^2}{1-s^2|z|^2}z$ and radius $\frac{1-|z|^2}{1-s^2|z|^2}s$, where $s = \tanh r$.

One relevant connection between Bergman metric and Bloch spaces is the following result:

Theorem 5.2 (See [22], 5.1.6.) $\beta(z,w) \sim \sup\{|f(z) - f(w)|;\ \|f\|_{\mathcal{B}} \le 1\}$ (*with constants independent from z and w in $\mathbb{D}$*).

In particular this allows us to get the following remark.

Corollary 5.3 *If $F \in \mathcal{B}(X)$ then $F : \mathbb{D} \to X$ is a Lipschitz map with respect to the Bergman metric.*

Proof. A look at Proposition 3.16 gives that $F(z) = T(K_z)$ for some $T \in \mathcal{L}(B_1, X)$. Hence

$$\begin{aligned} \|F(z) - F(w)\| &\leq C\,\|T\|\,\|K_z - K_w\|_{B_1} \\ &\leq C\,\|T\|\,\sup\{|\xi(K_z - K_w)|;\ \xi \in B_1^*\} \\ &\sim \sup\{|f(z) - f(w)|;\ \|f\|_{\mathcal{B}} = 1\} \\ &\sim \beta(z, w)\,. \end{aligned}$$

□

The key point in order to relate $B_p(X)$ to $\ell_p(X)$ is the use of sequences in $\mathbb{D}$ with good separation properties with respect to Bergman metric. The next lemma resumes some well known results (see for instance [22]):

Lemma 5.4 *There exists a number $N \in \mathbb{N}$ such that, for any $r \leq 1$, we can take a sequence (λ_n) in $\mathbb{D}$ and a decomposition of $\mathbb{D}$ into a disjoint union of measurable sets E_n such that*

(i) $E(\lambda_n, r/4) \subseteq E_n \subseteq E(\lambda_n, r)$ *for every* n,

(ii) *every point in $\mathbb{D}$ belongs to no more than N discs from* $\{E(\lambda_n, 2r)\}$,

(iii) $|E_n| \sim |E(\lambda_n, r)| \sim |E(\lambda_n, 2r)| \sim \big(1 - |\lambda_n|^2\big)^2 \sim |E(w, r)|$ *for any* $w \in E(\lambda_n, 2r)$ *and*

(iv) $1 - |\lambda_n|^2 \leq C\big(1 - |z|^2\big)$ *for each* $z \in E(\lambda_n, 2r)$.

The well known fact that, for all $0 < p < \infty$, $|f|^p$ is a subharmonic function with respect to β-balls for any analytic function f, also holds true in the vector valued setting.

Lemma 5.5 *Let X be any complex Banach space, let $f \in \mathcal{H}(\mathbb{D}, X)$ and $p > 0$. There exists a constant $C > 0$ such that we have*

$$\|f(z)\|^p \;\leq\; \frac{C}{|E(z,r)|}\int_{E(z,r)} \|f(w)\|^p\,dm(w)$$

for any $r \leq 1$ and $z \in \mathbb{D}$.

Proof. From the scalar valued case we get $C > 0$ such that

$$|x^* f(z)|^p \;\leq\; \frac{C}{|E(z,r)|}\int_{E(z,r)} |x^* f(w)|^p\,dm(w) \;\leq\; \frac{C}{|E(z,r)|}\int_{E(z,r)} \|f(w)\|^p\,dm(w)$$

for all $r \leq 1$ and $z \in \mathbb{D}$.

Now take the supremum over the unit ball of X^* to finish the proof. □

Corollary 5.6 *Let $r < 1$ and $p > 1$, and let X be a Banach space. Let $Q_r = Q_{r,p,X} : B_p(X) \to L_p(m, X)$ be defined by*

$$Q_r(f) \;=\; \sum_{n=1}^{\infty} f(\lambda_n)\chi_{E_n}\,.$$

Then Q_r is a bounded operator.

Proof. By Lemmas 5.5 and 5.4

$$\begin{aligned} \sum_{n=1}^{\infty} |E_n|\,\|f(\lambda_n)\|^p &\leq C\sum_{n=1}^{\infty}\int_{E(\lambda_n,r)} \|f(z)\|^p\,dm(z) \\ &= \int_{\mathbb{D}} \|f(z)\|^p \sum_{n=1}^{\infty} \chi_{E(\lambda_n,r)}(z)\,dm(z) \;\leq\; CN\int_{\mathbb{D}} \|f(z)\|^p\,dm(z)\,. \end{aligned}$$

This shows the boundedness of $Q_{r,p,X}$. □

Lemma 5.7 *Let $r \leq 1$. The linear operator*

$$f \longmapsto \sum_{n=1}^{\infty} (f - f(\lambda_n))\chi_{E_n}$$

is bounded from $B_p(X)$ to $L_p(m, X)$, and its norm is less or equal than $C \tanh r$.

Proof. Let $z \in E_n$, and observe that

$$\|f(z) - f(\lambda_n)\| = \left\| \int_{[\lambda_n, z]} f'(w)\, dw \right\| \leq \left(\sup_{w \in [\lambda_n, z]} \|f'(w)\| \right) |z - \lambda_n| \,.$$

Since $E(w, r) \subset E(\lambda_n, 2r)$ for any $z \in E_n$ and $w \in [\lambda_n, z]$, by Lemma 5.5 and the properties of (λ_n) we have that

$$\|f'(w)\|^p \leq \frac{C}{|E(w,r)|} \int_{E(w,r)} \|f'\|^p \, dm \leq \frac{C}{|E_n|} \int_{E(\lambda_n, 2r)} \|f'\|^p \, dm \,.$$

Hence if $w \in [\lambda_n, z]$

$$\|f(z) - f(\lambda_n)\|^p \leq \frac{C}{|E_n|} \left(\int_{E(\lambda_n, 2r)} \|f'\|^p \, dm \right) |z - \lambda_n|^p \,.$$

Let $s = \tanh r$. As $E(\lambda_n, r)$ is a disc with center $z_0 = \frac{1-s^2}{1-s^2|\lambda_n|^2}\lambda_n$ and radius $R = \frac{1-|\lambda_n|^2}{1-s^2\,|\lambda_n|^2}s$, for any z in it

$$|z - \lambda_n| \leq R + |\lambda_n - z_0| = \frac{1 - |\lambda_n|^2}{1 - s^2|\lambda_n|^2} s(1 + s\,|\lambda_n|) \leq Cs\big(1 - |\lambda_n|^2\big) \,,$$

and then

$$\|f(z) - f(\lambda_n)\|^p \leq \frac{C}{|E_n|} s^p \left(\int_{E(\lambda_n, 2r)} \|f'\|^p \, dm \right) \big(1 - |\lambda_n|^2\big)^p \,.$$

Therefore

$$\int_{E_n} \|f(z) - f(\lambda_n)\|^p \, dm(z) \leq C s^p \big(1 - |\lambda_n|^2\big)^p \int_{E(\lambda_n, 2r)} \|f'\|^p \, dm \,.$$

We use now that $\big(1 - |\lambda_n|^2\big)^p \leq C\big(1 - |z|^2\big)^p$ for each $z \in E(\lambda_n, 2r)$, and then

$$\int_{E_n} \|f(z) - f(\lambda_n)\|^p \, dm(z) \leq C s^p \int_{E(\lambda_n, 2r)} \big(1 - |z|^2\big)^p \|f'(z)\|^p \, dm(z) \,.$$

Hence

$$\sum_{n=1}^{\infty} \int_{E_n} \|f(z) - f(\lambda_n)\|^p \, dm(z) \leq C N s^p \int_{\mathbb{D}} \big(1 - |z|^2\big)^p \|f'(z)\|^p \, dm(z) \,,$$

which is bounded by $Cs^p \, \|f\|^p_{B_p(X)}$ in view of Theorem 2.5. □

Corollary 5.8 *There exist $r_0 > 0$ such that $PQ_{r,p,X} : B_p(X) \to B_p(X)$ is an isomorphism for all $r < r_0$, $1 < p < \infty$ and all Banach spaces X.*

Proof. We shall show this by noting that, if I denotes the identity in $B_p(X)$, then $\|I - PQ_r\|$ tends to zero as $r \to 0$. Recall that then, if r is such that $\|I - PQ_r\| < 1$, the inverse of PQ_r is just $\sum_{n=0}^{\infty}(I - PQ_r)^n$.

Now from Lemma 5.7 one has that $I - PQ_r \in \mathcal{L}(B_p(X), B_p(X))$ and $\|I - PQ_r\| \leq C \, \|P\| \tanh r$. □

Theorem 5.9 *For every $p > 1$ and every complex Banach space X, the Bergman space $B_p(X)$ is isomorphic to a complemented space of $\ell_p(X)$.*

Proof. We take r small enough to have that PQ_r is an isomorphism on $B_p(X)$. Then the identity in $B_p(X)$ factorizes as $I = (PQ_r)^{-1} PQ_r$. Now write $\widetilde{Q}_r : B_p(X) \to \ell_p(X)$ for the operator given by

$$\widetilde{Q}_r(f) \;=\; \big(|E_n|^{1/p} f(\lambda_n)\big)$$

and $J : \ell_p(X) \to L_p(m, X)$ for the one given by

$$J\big((x_n)\big) \;=\; \sum_{n=1}^{\infty} |E_n|^{-1/p} x_n \chi_{E_n} \,.$$

Since J is an embedding and $\widetilde{Q}_r$ is bounded due to Corollary 5.6 we can factorize the identity as $I = (PQ_r)^{-1} PJ\widetilde{Q}_r$ and therefore $B_p(X)$ is isomorphic to the image of $\widetilde{Q}_r$ in $\ell_p(X)$. □

Theorem 5.10 *Let $r < 1$, $p > 1$ and X be a Banach space. Let $P_r = P_{r,p,X}$ be the linear operator $P_r : V_p(m, X) \to B_p(X)$ defined by*

$$P_r(G) \;=\; \sum_{n=1}^{\infty} K_{\lambda_n} \otimes G\big(\overline{E}_n\big)\,.$$

Then the linear operator P_r is bounded.
Moreover $\|P_{r,p,X}\| = \|Q_{r,p',X^*}\|$.

Proof. For any polynomial $g \in \mathcal{P}(X^*)$ we have that

$$\left\langle \sum_{n=1}^{\infty} K_{\lambda_n} \otimes G\big(\overline{E}_n\big), g \right\rangle \;=\; \sum_{n=1}^{\infty} G\big(\overline{E}_n\big) g(\lambda_n) \;=\; \left\langle \sum_{n=1}^{\infty} g(\lambda_n)\chi_{E_n}, G \right\rangle \;=\; \langle Q_{r,p',X^*}(g), G\rangle\,.$$

Since $V_p(m, X)$ is isometrically embedded in $(L_{p'}(m, X^*))^*$ it follows that

$$\left| \left\langle \sum_{n=1}^{\infty} K_{\lambda_n} \otimes G(E_n), g \right\rangle \right| \;\le\; \|G\|_{V_p(m,X)} \left\| \sum_{n=1}^{\infty} g(\lambda_n)\chi_{E_n} \right\|_{L_{p'}(m,X^*)} \,.$$

Now Corollary 5.6 gives that $\|P_{r,p,X}\| \le \|Q_{r,p',X^*}\|$.
A similar argument shows that $Q^*_{r,p',X} = P_{r,p,X^*}$, giving the other inequality. □

Let us now compare $\overline{P}_r$ and the Bergman projection on $V_p(m, X)$.

Theorem 5.11 *Let $p > 1$ and X be a Banach space. Then*
(i) $\lim_{r\to 0} P_r = P$.
(ii) *The restriction of P_r to $B_p(X)$ given by*

$$P_r\big(g\big) \;=\; \sum_{n=1}^{\infty} K_{\lambda_n} \otimes \int_{\overline{E}_n} g(z)\, dm(z)$$

is an isomorphism for r close enough to zero.

Proof. An easy computation shows that $\langle P(G), g\rangle = \langle g, G\rangle$ for any $G \in V_p(m.X)$ and $g \in \mathcal{P}(X^*)$. Therefore for all $G \in V_p(m, X)$ and $g \in \mathcal{P}(X^*)$ we have

$$\langle (P - P_r)(G), g\rangle \;=\; \left\langle g - \sum_{n=1}^{\infty} g(\lambda_n)\chi_{E_n}, G \right\rangle \,.$$

Now applying Lemma 5.7 we get $\|P - P_r\| \le C \tanh r$ and (i) follows.
(ii) is proved the same way as Corollary 5.8. □

Note that $\beta(z,w)=\beta(\bar z,\overline{w})$, and then $\big(\overline{\lambda}_n\big)$ and $\big(\overline{E}_n\big)$ satisfy the same estimates and properties as (λ_n) and (E_n).

Theorem 5.12 *Let X be a Banach space and $p>1$. For each $f\in B_p(X)$ we denote*

$$S_r(f) \;=\; \sum_{n=1}^{\infty} |E_n|\, K_{\lambda_n} \otimes f\big(\overline{\lambda}_n\big).$$

Then $f=\lim_{r\to 0} S_r(f)$ in B_p.

Proof. We shall see that $S_r = S_{r,p,X} : B_p(X)\to B_p(X)$ are bounded operators and $\lim_{r\to 0} S_r = I$.

Let us denote by $\overline{Q}_r$ the operator associated to $\big(\overline{\lambda}_n\big)$ and $\big(\overline{E}_n\big)$, that is $\overline{Q}_r(f) = \sum_{n=1}^{\infty} f\big(\overline{\lambda}_n\big)\chi_{\overline{E}_n}$. We actually have $S_r = P_r\overline{Q}_r$ and

$$\big\|I-P_r\overline{Q}_r\big\| \;\le\; \big\|I-P\overline{Q}_r\big\| + \big\|P\overline{Q}_r - P_r\overline{Q}_r\big\| \;\le\; \big\|I-P\overline{Q}_r\big\| + \|P-P_r\|\,\|Q_r\|\,.$$

The result follows from Corollary 5.8 and Theorem 5.11. □

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