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CONVOLUTION OF THREE FUNCTIONS BY MEANS OF BILINEAR MAPS AND APPLICATIONS

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ABSTRACT. Let X, Y and E be complex Banach spaces, and let u: $X \times Y \to E$ be a bounded bilinear map. If f, g are analytic functions in the unit disc taking values in X and Y with Taylor coefficients x_n and y_n respectively, we define the E-valued function $f *_u g$ whose Taylor coefficients are given by $u(x_n, y_n)$. Given two bounded bilinear maps, u: $X \times Y \to E$ and v: $Z \times E \to F$, in our main theorem we prove that Young's Theorem can be improved by showing that the function $f *_v (g *_u h)$ is in the Hardy space $H^p(F)$ provided that f, g and h are in the vector valued Besov spaces corresponding to those that appear in some classical inequalities by Hardy-Littlewood and Littlewood-Paley.

We also investigate the class of Banach spaces for which these inequalities hold in the vector setting, and we give a number of applications of our theorem for these spaces and for certain bilinear maps (such as convolution, tensor products,...), obtaining results both in the scalar and the vector valued cases.

Introduction

When dealing with spaces of vector-valued analytic functions there is a natural way to understand multipliers between them. If X and Y are Banach spaces and $L(X, Y)$ stands for the space of linear and continuous operators we may consider the convolution of $L(X, Y)$-valued analytic functions, say $F(z) = \sum_{n=0}^{\infty} T_n z^n$, and X-valued polynomials, say $f(z) = \sum_{n=0}^{m} x_n z^n$, to get the Y-valued function $F * f(z) = \sum T_n(x_n) z^n$. The second author considered such a definition and studied multipliers between $H^1(X)$ and $BMOA(Y)$ in [5].

When the functions take values in a Banach algebra A then the natural extension of multiplier is simply that if $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$, then $f * g(z) = \sum a_n.b_n z^n$ where $a.b$ stands for the product in the algebra A. Of course, similarly one can consider $a_n \in L^p(\mathbb{R})$, $b_n \in L^q(\mathbb{R})$ and the convolution $a_n * b_n \in L^r(\mathbb{R})$ (where p, q, r satisfy the condition in Young's theorem). The reader is referred to [3] for results along these lines.

In this paper we shall consider a much more general notion of convolutions coming from general bilinear maps and that will extend the previous examples.

Assume X, Y, Z are Banach spaces and let u: $X \times Y \to Z$ be a bounded bilinear map. Given a X-valued polynomial $f(z) = \sum_{n=0}^{m} x_n z^n$ and given a Y-valued polynomial $g(z) = \sum_{n=0}^{k} y_n z^n$ we define the u-convolution of f and g as the polynomial

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given by

$$f *_u g(z) = \sum_{n=0}^{\min\{m,k\}} u(x_n, y_n) z^n.$$

This will make sense also for general vector valued analytic functions and we shall study this convolution for functions in certain vector valued Besov spaces.

Throughout the paper we denote by $\mathcal{P}(X)$ and $\mathcal{H}(X)$ the set of polynomials and holomorphic functions from the unit disc $\mathbb{D}$ into a Banach space X respectively. As usual, we write $M_p(f, r) = (\frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(re^{it})\|^p dt)^{\frac{1}{p}}$, and $H^p(X)$ stands for the Hardy space of X-valued functions, understood as the subspace of $L^p(\mathbb{T}, X)$ of those functions f with $\hat{f}(n) = 0$ for $n < 0$, or in other words the closure of polynomials under the norm given by $\sup_{0<r<1} M_p(f, r)$. For $1 \le p, q \le \infty$, we shall be also dealing with the spaces $\Lambda_{p,q}(X)$ given by those functions in $\mathcal{H}(X)$ such that

$$\int_0^1 (1-r)^{q-1} M_p^q(f', r) dr < \infty,$$

with the obvious modification for the case $q = \infty$ (see Section 1).

These spaces were considered first (in the scalar valued case) by Hardy-Littlewood and Flett (see [12], [9]). The main reason for their consideration comes from the following two results:

Let $2 \le p < \infty$. It was shown by Littlewood and Paley (see [14]) that there exists a constant $C > 0$ such that

$$\left(\int_0^1 (1-r)^{p-1} M_p^p(f', r) dr\right)^{\frac{1}{p}} \le C\|f\|_p. \tag{0.1}$$

Now let $1 \le p \le 2$. It was shown by Hardy and Littlewood (see [12]) that there exists a constant $C > 0$ such that

$$\left(\int_0^1 (1-r) M_p^2(f', r) dr\right)^{\frac{1}{2}} \le C\|f\|_p. \tag{0.2}$$

In other words, $H^p \subset \Lambda_{p,2}$ for $1 \le p \le 2$, and $H^p \subset \Lambda_{p,p}$ for $2 \le p < \infty$.

We shall see that some results, known for Hardy spaces, actually hold in the setting of $\Lambda_{p,q}$-spaces. The aim of this paper is to give an improvement of a Young's type theorem for convolution of three functions in the setting of vector valued analytic functions and in a very wide sense of convolution which allows to recover several known results and produces a lot of applications. Our main result is as follows:

Let $1 \leq p_1, p_2, p_3 \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 2$ and $1 \leq q_1, q_2, q_3 \leq \infty$ such that $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$.

Let u: $X \times Y \to E$ and v: $Z \times E \to F$ be bounded bilinear maps where X, Y, Z, E, F are complex Banach spaces.

If $1 \leq p \leq \infty$ is such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - 2$, then there exists a constant $C > 0$ such that

$$\left\| \sum_{n=0}^{N} v(z_n, u(x_n, y_n))z^n \right\|_p \leq C\|u\|\|v\|\|f\|_{p_1,q_1}\|g\|_{p_2,q_2}\|h\|_{p_3,q_3}$$

for any $f(z) = \sum_{n=0}^{N} x_n z^n \in \mathcal{P}(X)$, $g(z) = \sum_{n=0}^{N} y_n z^n \in \mathcal{P}(Y)$ and $h(z) = \sum_{n=0}^{N} z_n z^n \in \mathcal{P}(Z)$.

The paper is divided into six sections. In Section 1 we introduce the convolution, the spaces and the property $(H)_p$ corresponding to the vector-valued formulation of (0.1) and (0.2). We present some elementary examples and geometric properties of spaces having property $(H)_p$. In the second section we prove the main theorem and give the corresponding corollary for vector-valued Hardy spaces. Section 3 is devoted to some applications to the scalar valued case. Section 4 deals with the bilinear map between $L^p(\mathbb{R}^n)$-spaces given by convolution $u(f, g) = f * g$. In Section 5 we give some properties on the Taylor coefficients of functions in Hardy spaces with values in spaces with $(H)_p$ property. In Section 6 we take u: $X \times Y \to X\hat{\otimes}Y$ and achieve certain results for projective tensor products. Finally, we get new results on the space of multipliers between vector valued Hardy spaces in Section 7.

1. Preliminaries

Definition 1.1. Let u: $X \times Y \to Z$ be a bounded bilinear map. Let $f \in \mathcal{H}(X)$ and $g \in \mathcal{H}(Y)$ given by $f(z) = \sum_{n=0}^{\infty} x_n z^n$ and $g(z) = \sum_{n=0}^{\infty} y_n z^n$. We define the u-convolution of f an g as the function in $\mathcal{H}(Z)$ given by

$$f *_u g(z) = \sum_{n=0}^{\infty} u(x_n, y_n)z^n.$$

LEMMA 1.1. *Let $f \in \mathcal{P}(X)$ and $g \in \mathcal{P}(Y)$. Then*

$$f *_u g(r^2 e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(f(re^{i(\theta-t)}), g(re^{it}))dt; \tag{1.1}$$

$$[S(f *_u g)]'(r^2 e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(f(re^{i(\theta-t)}), g'(re^{it}))dt, \tag{1.2}$$

where $Sf(z) = zf(z)$;

$$[S^2(f *_u g)]''(r^2 e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u((Sf)'(re^{i(\theta-t)}) + f(re^{i(\theta-t)}), (Sg)'(re^{it}))dt, \tag{1.3}$$

where $S^2 f(z) = z^2 f(z)$.

Proof. (1.1) follows from the orthonormality of the system e^{int}.

(1.2) follows from the fact that

$$[S(f *_u g)]'(z) = \sum_{n=0}^{\infty} u(x_n, (n+1)y_n)z^n = f *_u (Sg)'(z).$$

(1.3) follows by writing

$$\begin{aligned}[S^2(f *_u g)]''(z) &= \sum_{n=0}^{\infty} u((n+1)x_n, (n+1)y_n)z^n + \sum_{n=0}^{\infty} u(x_n, (n+1)y_n)z^n \\ &= [(Sf)' + f] *_u (Sg)'(z). \end{aligned}$$ □

Definition 1.2. Let $1 \le p < \infty$. A complex Banach space X is said to have property $(H)_p$, to be denoted $X \in (H)_p$, if there exists a constant $C > 0$ such that

$$\left(\int_0^1 (1-r)^{\max\{2,p\}-1} M_p^{\max\{2,p\}}(f', r)dr\right)^{\frac{1}{\max\{2,p\}}} \le C\|f\|_p \tag{1.4}$$

for any polynomial $f \in \mathcal{P}(X)$.

Remark 1.1. The property $(H)_1$ was already defined and studied in [5], denoted there by (HL).

Definition 1.3. Let $1 \le p \le \infty$ and $1 \le q < \infty$. We shall denote by $\Lambda_{p,q}(X)$ the space of functions $f \in \mathcal{H}(X)$ such that

$$(1-r)M_p(f', r) \in L^q\left(\frac{dr}{1-r}\right),$$

and set $\|f\|_{p,q} = \|f(0)\| + (\int_0^1 (1-r)^{q-1} M_p^q(f', r)dr)^{\frac{1}{q}}$.

Accordingly, we shall denote by $\Lambda_{p,\infty}(X)$ the space of functions $f \in \mathcal{H}(X)$ such that

$$M_p(f', r) = O\left(\frac{1}{1-r}\right) \qquad (r \to 1),$$

and set $\|f\|_{p,\infty} = \|f(0)\| + \sup_{0<r<1}(1-r)M_p(f', r)$.

Remark 1.2. $\Lambda_{\infty,\infty}(X) = \text{Bloch}(X) = \{f \in \mathcal{H}(X)\colon \sup_{z\in D}(1-|z|)|f'(z)| < \infty\}$.

Let us point out some elementary embeddings:

PROPOSITION 1.1. *Let $1 \le p, q \le \infty$ and let X be a complex Banach space.*

(*i*) $H^p(X) \subset \Lambda_{p,\infty}(X)$.
(*ii*) $\Lambda_{p,q}(X) \subset \Lambda_{p,\infty}(X)$.
(*iii*) *If $X \in (H)_p$ then $H^p(X) \subset \Lambda_{p,q}(X)$ for $q \ge \max\{p, 2\}$.*

Proof. (i) This follows from the estimate

$$M_p(f', r^2) \le C\frac{M_p(f,r)}{1-r}.$$

(ii) Since $M_p(f,r)$ is increasing, we have

$$\frac{1}{q}M_p^q(f',r)(1-r)^q \le \int_r^1 (1-s)^{q-1}M_p^q(f',s)ds\,,$$

what actually gives that if $f \in \Lambda_{p,q}(X)$ then $M_p(f',r) = o(\frac{1}{1-r})$.
(iii) This follows from (i) and the inclusion $L^\infty(\frac{dr}{1-r}) \cap L^{\max\{p,2\}}(\frac{dr}{1-r}) \subseteq L^q(\frac{dr}{1-r})$. □

Let us now compute the norm of $f(z) = \sum_{n=0}^\infty x_n z^{2^n}$ in $\Lambda_{p,q}(X)$.

PROPOSITION 1.2. *Let $1 \le p \le \infty$, $1 \le q < \infty$ and let $f(z) = \sum_{n=0}^\infty x_n z^{2^n}$, where $x_n \in X$. Then*

$$\|f\|_{p,\infty} \approx \sup_{n\in\mathbb{N}} \|x_n\|, \tag{1.5}$$

and

$$\|f\|_{p,q} \approx \left(\sum_{n=0}^\infty \|x_n\|^q\right)^{\frac{1}{q}}. \tag{1.6}$$

Proof. Note that

$$r^{2^n-1}2^n\|x_n\| \le M_1(f',r) \tag{1.7}$$

and

$$M_\infty(f',r) \le \sum_{n=0}^\infty 2^n\|x_n\|r^{2^n-1}. \tag{1.8}$$

To get (1.5) assume first that $\sup_{n\in\mathbb{N}} \|x_n\| \le 1$; then, from (1.8), we have

$$M_p(f',r) \le M_\infty(f',r) \le \sum_{n=0}^{\infty} 2^n r^{2^n-1} \le \frac{C}{1-r}.$$

On the other hand, if $M_p(f',r) \le \frac{C}{1-r}$ then (taking $r = 1-2^{-n}$) (1.7) gives

$$(1-2^{-n})^{2^n-1} 2^n \|x_n\| \le C2^n ,$$

what shows that $\sup_{n\in\mathbb{N}} \|x_n\| \le C$.

To get (1.6) first use (1.7) to obtain

$$\begin{aligned}\left(\sum_{n=1}^{\infty} \|x_n\|^q\right)^{\frac{1}{q}} &\le C\left(\sum_{n=0}^{\infty}\left(\int_{1-2^{-n}}^{1-2^{-(n+1)}} 2^{nq}(1-r)^{q-1} r^{(2^n-1)q} dr\right) \|x_{n+1}\|^q\right)^{\frac{1}{q}}\\ &\le C\left(\int_0^1 (1-r)^{q-1} M_p^q(f',r)dr\right)^{\frac{1}{q}} = C\|f\|_{p,q}.\end{aligned}$$

To see the other inequality, consider the operator given by

$$T(\{x_n\}) = (1-r)f'(re^{it}).$$

Note that (1.5) gives, for any $1 \le p \le \infty$, the boundedness of T as an operator from $\ell^\infty(X)$ into $L^\infty(\frac{dr}{(1-r)}, L^p(\mathbb{T}, X))$ (where, as usual, $L^p(\frac{dr}{1-r}, Y)$ stands for the space of Y-valued functions on (0, 1) that are p-integrable with respect to the measure $\frac{dr}{1-r}$).

It follows from (1.8) that it is also bounded from $\ell^1(X)$ into $L^1(\frac{dr}{(1-r)}, L^p(\mathbb{T}, X))$.

Now use interpolation (see [4]) to get that

$$T\colon l^q(X) \to L^q\left(\frac{dr}{(1-r)}, L^p(\mathbb{T}, X)\right)$$

is bounded as well. □

Recall that for $2 \le q < \infty$, a Banach space is said to have cotype q (see [17]) if there exists a constant $C > 0$ such that, for any finite family $\{x_n\}_{n\ge 0}$ in X,

$$\left(\sum_{n\ge 0} \|x_n\|^q\right)^{\frac{1}{q}} \le C\left\|\sum_{n\ge 0} x_n z^{2^n}\right\|_1 .$$

Also recall that Kahane's inequalites can be stated as

$$\|\sum_{n\ge 0} x_n z^{2^n}\|_p \approx \|\sum_{n\ge 0} x_n z^{2^n}\|_1$$

for any $0 < p < \infty$.

Using this and Proposition 1.2 we get the following corollary.

COROLLARY 1.1. *Let $q = \max\{p, 2\}$. If $X \in (H)_p$ then X has cotype q.*

Let us give the L^q-spaces satisfying the property $(H)_p$.

PROPOSITION 1.3. *Let H be a complex Hilbert space. Then $H \in (H)_2$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} x_n z^n \in \mathcal{H}(H)$. From Plancherel's we get

$$\|f\|_2 \approx \left(\|x_0\|^2 + \int_0^1 (1-r) M_2^2(f', r)\,dr\right)^{\frac{1}{2}}. \qquad \square$$

PROPOSITION 1.4. *Let (Ω, Σ, μ) be a σ-finite measure space.*

(i) *If $p \geq 2$ and $p' \leq q \leq p$ then $L^q(\mu) \in (H)_p$.*
(ii) *If $1 \leq p \leq 2$ and $p \leq q \leq 2$ then $L^q(\mu) \in (H)_p$.*

Proof. Observe that the $(H)_p$ property can be stated in terms of the boundedness of the operator $T\colon H^p(X) \to L^{\max\{2,p\}}(\frac{dr}{1-r}, L^p(\mathbb{T}, X))$ given by

$$T(f)(r, t) = (1-r) f'(re^{it}).$$

Note first that

$$T\colon H^2\left(L^2(\mu)\right) \to L^2\left(\frac{dr}{1-r}, L^2\left(\mathbb{T}, L^2(\mu)\right)\right)$$

is bounded by Proposition 1.3. Both results then follow by interpolation (see [7]).

To see (i), choose $\theta = 1 - \frac{2}{p}$ and $s = \theta(\frac{1}{q} - \frac{1}{p})^{-1}$, so that $\frac{1}{p} = \frac{1-\theta}{2}$ and $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{s}$, which gives

$$\left[H^2\left(L^2(\mu)\right), BMOA\left(L^s(\mu)\right)\right]_\theta = H^p\left(L^q(\mu)\right)$$

and

$$\left[L^2\left(\frac{dr}{1-r}, L^2\left(\mathbb{T}, L^2(\mu)\right)\right), L^\infty\left(\frac{dr}{1-r}, L^\infty\left(\mathbb{T}, L^s(\mu)\right)\right)\right]_\theta$$
$$= L^p\left(\frac{dr}{1-r}, L^p\left(\mathbb{T}, L^q(\mu)\right)\right).$$

In order to interpolate, just note that $BMOA(X) \subset \text{Bloch}(X)$ for any X, so

$$T\colon BMOA\left(L^s(\mu)\right) \to L^\infty\left(\frac{dr}{1-r}, L^\infty(\mathbb{T}, L^s(\mu))\right)$$

is bounded for any value $1 \leq s \leq \infty$.

To see (ii), let θ be such that $\frac{1}{p} = 1 - \frac{\theta}{2}$ and s such that $\frac{1}{q} = \frac{1-\theta}{s} + \frac{\theta}{2}$. Then

$$\left[H^1\left(L^r(\mu)\right), H^2\left(L^2(\mu)\right)\right]_\theta = H^p\left(L^q(\mu)\right)$$

and

$$\left[L^2\left(\frac{dr}{1-r}, L^1\left(\mathbb{T}, L^s(\mu)\right)\right), L^2\left(\frac{dr}{1-r}, L^2\left(\mathbb{T}, L^2(\mu)\right)\right)\right]_\theta$$
$$= L^2\left(\frac{dr}{1-r}, L^p\left(\mathbb{T}, L^q(\mu)\right)\right).$$

It follows from our assumptions that $1 \le s \le 2$; then $L^s(\mu) \in (H)_1$ (see [5]), and by interpolation we get $L^q(\mu) \in (H)_p$. □

2. The theorem and its proof

Let us start off with the following formulation of the convolution.

LEMMA 2.1. *Let $f \in \mathcal{P}(X)$ and $g \in \mathcal{P}(Y)$. Then*

$$f *_u g(z) = u(f(0), g(0)) + \frac{3}{4\pi}\int_0^1\int_{-\pi}^{\pi}(1-s^3)^2 ze^{-it}u(f'(zse^{-it}), (S^2g)''(s^2e^{it})dtds.$$

Proof. Let us use the fact that

$$\int_0^1 (1-s^3)^2 s^{3n-1}ds = \frac{2}{3(n+2)(n+1)n}$$

and write, if $f(z) = \sum_{n\ge0} x_n z^n$ and $g(z) = \sum_{n\ge0} y_n z^n$,

$$f *_u g(z) = u(x_0, y_0) + \frac{3}{2}\sum_{n=1}^{\infty}\int_0^1 (1-s^3)^2 s^{3n-1}(n+2)(n+1)nu(x_n, y_n)z^n ds,$$

where the last sum equals

$$\frac{3}{2}\sum_{n=1}^{\infty}\int_0^1 (1-s^3)^2 zu(nz^{n-1}s^{n-1}x_n, (n+2)(n+1)s^{2n}y_n)ds$$
$$= \frac{3}{4\pi}\int_0^1\int_{-\pi}^{\pi}(1-s^3)^2 ze^{-it}u$$
$$\times\left(\sum_{n\ge1} nz^{n-1}s^{n-1}x_n e^{-i(n-1)t}, \sum_{k\ge0}(k+2)(k+1)s^{2k}y_k e^{ikt}\right)dtds$$
$$= \frac{3}{4\pi}\int_0^1\int_{-\pi}^{\pi}(1-s^3)^2 ze^{-it}u(f'(zse^{-it}), (S^2g)''(s^2e^{it})dtds.$$ □

THEOREM 2.1. *Let* $1 \leq p_1, p_2, p_3$ *be such that* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 2$ *and* $1 \leq q_1, q_2, q_3$ *be such that* $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$. *Take* p *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - 2$.

Let X, Y, Z, E, F *be complex Banach spaces and let* $u\colon X \times Y \to E$ *and* $v\colon Z \times E \to F$ *be bounded bilinear maps.*

Then there exists a constant $C > 0$ *such that*

$$\|h *_v (f *_u g)\|_p \leq C\|u\|\|v\|\|f\|_{p_1,q_1}\|g\|_{p_2,q_2}\|h\|_{p_3,q_3}$$

for any $f \in \mathcal{P}(X)$, $g \in \mathcal{P}(Y)$ *and* $h \in \mathcal{P}(Z)$.

Proof. For $f_1 \in \mathcal{P}(X)$, $f_2 \in \mathcal{P}(Y)$ and $f_3 \in \mathcal{P}(Z)$, let

$$A(f_1, f_2, f_3) = Sf_3 *_v (f_1 *_u f_2).$$

Applying (1.1) twice for $r = 1$ we get

$$A(f_1, f_2, f_3)(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\theta-t)}v(f_3(e^{i(\theta-t)}), u(f_1(e^{i(t-t')}), f_2(e^{it'})))dt'dt,$$

and by Young's theorem we have

$$\|A(f_1, f_2, f_3)\|_p \leq \|u\|\|v\|\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3}.$$

Observe now that if we write $f_r(z) = f(rz)$, then for f, g and h Lemma 2.1 and (1.3) give

$$\begin{aligned} h *_v (f *_u g)(re^{i\theta}) &= v(z_0, u(x_0, y_0)) \\ &+ \frac{3r}{2}\int_0^1 (1-s^3)^2 A((Sf)'_s, (Sg)'_s, h'_{rs})(e^{i\theta})ds \\ &+ \frac{3r}{2}\int_0^1 (1-s^3)^2 A(f_s, (Sg)'_s, h'_{rs})(e^{i\theta})ds. \end{aligned}$$

Therefore, using the vector-valued Minkowsky's inequality, we get

$$\begin{aligned} \|f *_v (g *_u h)\|_p &\leq \|v(z_0, u(x_0, y_0))\|_F \\ &+ \frac{3}{2}\|u\|\|v\|\int_0^1 (1-s^3)^2 M_{p_1}((Sf)', s)M_{p_2}((Sg)', s)M_{p_3}(h', s)ds \\ &+ \frac{3}{2}\|u\|\|v\|\int_0^1 (1-s^3)^2 M_{p_1}(f, s)M_{p_2}((Sg)', s)M_{p_3}(h', s)ds. \end{aligned}$$

Let us bound each of them separately.

On the one hand,

$$\begin{aligned} \|v(z_0, u(x_0, y_0))\|_F &\leq \|u\|\|v\|\|x_0\|_X\|y_0\|_Y\|z_0\|_Z \\ &\leq \|u\|\|v\|\|f\|_{p_1,q_1}\|g\|_{p_2,q_2}\|h\|_{p_3,q_3}. \end{aligned}$$

On the other hand, using the fact that $1-s^3 \leq 3(1-s)$ for $0 < s \leq 1$ and splitting $(1-s)^2 = (1-s)^{1-\frac{1}{q_1}}(1-s)^{1-\frac{1}{q_2}}(1-s)^{1-\frac{1}{q_3}}$, by Hölder's inequality we have

$$\int_0^1 (1-s^3)^2 M_{p_1}((Sf)', s) M_{p_2}((Sg)', s) M_{p_3}(h', s) ds \\ \leq 9\|(Sf)\|_{p_1,q_1} \|Sg\|_{p_2,q_2} \|h\|_{p_3,q_3}.$$

Since

$$f(se^{it}) - f(0) = \int_0^s e^{it} f'(re^{it}) dr,$$

it's easy to see that $M_{p_1}(f, s) \leq M_{p_1}(f', s) + \|f(0)\|$, and thus $\|(Sf)\|_{p_1,q_1} \leq C\|f\|_{p_1,q_1}$; the same is valid for g, so

$$\int_0^1 (1-s^3)^2 M_{p_1}((Sf)', s) M_{p_2}((Sg)', s) M_{p_3}(h', s) ds \\ \leq C\|f\|_{p_1,q_1} \|g\|_{p_2,q_2} \|h\|_{p_3,q_3}.$$

Dealing with the last summand is similar, and then the result follows. □

COROLLARY 2.1. *Let* $1 \leq p_1, p_2, p_3$ *be such that* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 2$ *and* $1 \leq p_1 \leq 2$.

Let X, Y, Z, E, F *be complex Banach spaces such that* $X \in (H)_{p_1}$ *and* $Y \in (H)_{p_2}$, *and let* $u: X \times Y \to E$ *and* $v: Z \times E \to F$ *be bounded bilinear maps.*

(i) *If* $1 \leq p_2 \leq 2$, *then for* p *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - 2$ *there exists a constant* $C > 0$ *such that*

$$\|h *_v (f *_u g)\|_p \leq C\|u\|\|v\|\|f\|_{p_1}\|g\|_{p_2}\|h\|_{p_3,\infty}$$

for any $f \in \mathcal{P}(X)$, $g \in \mathcal{P}(Y)$ *and* $h \in \mathcal{P}(Z)$.

(ii) *If* $2 < p_2 \leq \infty$, *then for* p *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - 2$, *and* $q = \frac{2p_2}{p_2-2}$, *there exists a constant* $C > 0$ *such that*

$$\|h *_v (f *_u g)\|_p \leq C\|u\|\|v\|\|f\|_{p_1}\|g\|_{p_2}\|h\|_{p_3,q}$$

for any $f \in \mathcal{P}(X)$, $g \in \mathcal{P}(Y)$ *and* $h \in \mathcal{P}(Z)$.

COROLLARY 2.2. *Let* $1 \leq q \leq \infty$. *There exists a constant* $C > 0$ *such that, given a bounded bilinear map* $u: X \times Y \to Z$ *between complex Banach spaces, and given two polynomials* $f(z) = \sum_{n\geq 0} x_n z^n \in \mathcal{P}(X)$ *and* $g(z) = \sum_{n\geq 0} y_n z^n \in \mathcal{P}(Y)$, *we have*

$$\sum_{n\geq 0} \|u(x_{2^n}, y_{2^n})\| \leq C\|u\|\|f\|_{1,q}\|g\|_{1,q'}.$$

Proof. Apply Theorem 2.1 for $v\colon Z^* \times Z \to \mathbb{C}$ given by the dual pairing, $p_1 = p_2 = 1$, $q_1 = q$, $q_2 = q'$, $q_3 = p_3 = \infty$ and $h(z) = \sum_{n\geq 0} z_n^* z^{2^n}$, with z_n^* of norm one and satisfying $\langle z_n^*, u(x_{2^n}, y_{2^n})\rangle = \|u(x_{2^n}, y_{2^n})\|$. Note that $\|h\|_{\infty,\infty}$ is bounded by a constant, due to Proposition 1.2. □

In the applications of Theorem 2.1 (or Corollaries 2.1 and 2.2) that follow, sometimes polynomials are replaced by functions defined by power series. In all such cases the justification for doing so requires at most easy arguments, involving density of polynomials in the corresponding function space, that will be omitted.

3. Applications to the scalar valued case

Let us consider $X = Y = Z = \mathbb{C}$, $u(\lambda, \mu) = v(\lambda, \mu) = \lambda \cdot \mu$.

The following result is known but, in particular it provides another proof of Paley's inequality for functions in H^1 (see [8]).

THEOREM 3.1. *Let $1 \leq q \leq 2$. Then, for any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^q$, we have*

$$\left(\sum_{k\in\mathbb{N}}\left(\sum_{n=2^{k-1}}^{2^k} |a_n|^{q'}\right)^{\frac{2}{q'}}\right)^{\frac{1}{2}} \leq C\|f\|_q \tag{3.1}$$

(*with the obvious modification for $q' = \infty$*).

Proof. Assume $q = 1$ and take $\lambda_n \geq 0$ such that $\sup_{k\in\mathbb{N}} \sum_{2^{k-1}\leq n<2^k} \lambda_n \leq 1$. Let $h(z) = \sum \lambda_n z^n$. One easily sees that $M_\infty(h', r) \leq \frac{C}{1-r}$ and therefore we obtain $h \in \Lambda_{\infty,\infty}$.

Now apply Corollary 2.1 to f, g and h, where $g(z) = \sum \bar{a}_n z^n$ (so that $\|g\|_1 = \|f\|_1$), and get

$$\left\|\sum_{n=0}^{\infty} \lambda_n |a_n|^2 z^n\right\|_\infty \leq C\|f\|_1^2.$$

In particular it follows that

$$\left(\sum_{n=0}^{\infty} \lambda_n |a_n|^2\right)^{\frac{1}{2}} \leq C\|f\|_1$$

for any (λ_n) satisfying $\sup_{k\in\mathbb{N}} \sum_{2^{k-1}\leq n<2^k} \lambda_n \leq 1$.

Using duality, this implies that

$$\left(\sum_k \max_{2^{k-1}\leq n<2^k} |a_n|^2\right)^{\frac{1}{2}} \leq C\|f\|_1.$$

Now using interpolation with the trivial case $q = 2$ we get (3.1). □

Our next results shows that Paley's inequality holds not only for functions in H^1 but also in the Besov class $\Lambda_{1,2}$.

THEOREM 3.2. *Let* $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=2$ *and* $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1$.
Then

$$\left(\sum_{k\in\mathbb{N}}|a_{2^n}|^{q_3'}|b_{2^n}|^{q_3'}\right)^{\frac{1}{q_3'}}\leq C\|f\|_{p_1,q_1}\|g\|_{p_2,q_2} \tag{3.2}$$

for any holomorphic functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$, $g(z)=\sum_{n=0}^{\infty}b_nz^n$.
In particular, if $2\leq q$ *then*

$$\left(\sum_{k\in\mathbb{N}}|a_{2^n}|^{q}\right)^{\frac{1}{q}}\leq C\|f\|_{p,q} \tag{3.3}$$

for any $f(z)=\sum_{n=0}^{\infty}a_nz^n$.

Proof. To see (3.2) we just have to use Hölder's inequality after Theorem 2.1 for f, g and a suitable $h(z)=\sum_{n=0}^{\infty}\alpha_nz^{2^n}$ (by (1.6) we have that $\|h\|_{p_3,q_3}\approx(\sum_{n=0}^{\infty}|\alpha_n|^{q_3})^{\frac{1}{q_3}}$).
To see (3.3), take $f=g$ in (3.2), with $p=p_1=p_2$ and $q=q_1=q_2$. □

4. Applications to L^p-spaces and convolution

In this section we let X, Y, Z be L^p-spaces, and consider the bilinear map given by Young's theorem, that is, for $\frac{1}{p}+\frac{1}{q}\geq 1$ and $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$ we have the bounded bilinear map u: $L^p(\mathbb{R})\times L^q(\mathbb{R})\to L^r(\mathbb{R})$ given by $u(f,g)=f*g$. The reader is referred to [3] for particular cases and some applications.

THEOREM 4.1. *Let* $1\leq p_i, q_i$ *be such that* $\frac{1}{p_1}+\frac{1}{p_2}>1$, $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}>2$ *and* $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1$. *If* $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}-2=\frac{1}{p}$ *then, for* $f_n\in L^{p_1}(\mathbb{R})$, $g_n\in L^{p_2}(\mathbb{R})$ *and* $h_n\in L^{p_3}(\mathbb{R})$, *we have*

$$\left\|\left(\sum_{n\geq 0}|f_n*g_n*h_n|^2\right)^{\frac{1}{2}}\right\|_p\leq C\left(\sum_{n\geq 0}\|f_n\|_{p_1}^{q_1}\right)^{\frac{1}{q_1}}\left(\sum_{n\geq 0}\|g_n\|_{p_2}^{q_2}\right)^{\frac{1}{q_2}}\left(\sum_{n\geq 0}\|h_n\|_{p_3}^{q_3}\right)^{\frac{1}{q_3}}$$

(*with the corresponding modification if* $q_i=\infty$ *for some* i).

Proof. Take u: $L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\to L^{r_1}(\mathbb{R})$, where $\frac{1}{r_1}=\frac{1}{p_1}+\frac{1}{p_2}-1$, given by $u(f,g)=f*g$, and v: $L^{p_3}(\mathbb{R})\times L^{r_1}(\mathbb{R})\to L^p(\mathbb{R})$ given by $v(h,k)=h*k$.

Now apply Theorem 2.1 to the L^{p_i}-valued functions $F(z) = \sum_{n=0}^{\infty} f_n z^{2^n}$, $G(z) = \sum_{n=0}^{\infty} g_n z^{2^n}$ and $H(z) = \sum_{n=0}^{\infty} h_n z^{2^n}$. Proposition 1.2 allows us to write

$$\left\| \sum_{n\geq 0} (f_n * g_n * h_n) z^{2^n} \right\|_{H^p(L^p)} \leq C \left(\sum_{n\geq 0} \|f_n\|_{p_1}^{q_1} \right)^{\frac{1}{q_1}} \left(\sum_{n\geq 0} \|g_n\|_{p_2}^{q_2} \right)^{\frac{1}{q_2}} \left(\sum_{n\geq 0} \|h_n\|_{p_3}^{q_3} \right)^{\frac{1}{q_3}}.$$

Now, since $p < \infty$, the proof is finished by a simple application of Khintchine's inequality. □

THEOREM 4.2. *If $1 \leq p \leq 2 \leq q < \infty$ are such that $\frac{1}{p} + \frac{1}{q} > 1$ and if $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, then there exists a constant $C > 0$ such that*

$$\sup_{\|\phi\|_{r'} \leq 1} \left(\sum_{n\geq 0} |\langle \phi, f_n * g_n \rangle|^{\frac{2q}{q+2}} \right)^{\frac{1}{2}+\frac{1}{q}} \leq C \left\| \left(\sum_{n\geq 0} |f_n|^2 \right)^{\frac{1}{2}} \right\|_p \left\| \left(\sum_{n\geq 0} |g_n|^2 \right)^{\frac{1}{2}} \right\|_q$$

for any two finite sequences $(f_n) \subset L^p(\mathbb{R}))$, $(g_n) \subset L^q(\mathbb{R})$.

Proof. Take u: $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ given by $u(f,g) = f * g$ and v: $L^r(\mathbb{R}) \times L^{r'}(\mathbb{R}) \to \mathbb{C}$ given by $v(f,g) = \langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x) dx$. Now take $p_1 = p$, $p_2 = q$ and $p_3 = r'$. Therefore (ii) in Corollary 2.1 gives

$$\left\| \sum_{n\geq 0} < \phi_n, f_n * g_n > z^{2^n} \right\|_{\infty}$$
$$\leq C \left\| \sum_{n\geq 0} f_n z^{2^n} \right\|_{H^p(L^p)} \left\| \sum_{n\geq 0} g_n z^{2^n} \right\|_{H^q(L^q)} \left\| \sum_{n\geq 0} \phi_n z^{2^n} \right\|_{\Lambda_{r', \frac{2q}{q-2}}(L^{r'})}.$$

Now Proposition 1.2 applied to $X = L^{r'}$, together with standard estimates, gives

$$\sum_{n\geq 0} |\langle \phi_n, f_n * g_n \rangle| \leq C \left\| \left(\sum_{n\geq 0} |f_n|^2 \right)^{\frac{1}{2}} \right\|_p \left\| \left(\sum_{n\geq 0} |g_n|^2 \right)^{\frac{1}{2}} \right\|_q \left(\sum_{n\geq 0} \|\phi_n\|_{r'}^{\frac{2q}{q-2}} \right)^{\frac{q-2}{2q}}.$$

Now, taking $\phi_n = \alpha_n \phi$, for $(\alpha_n) \in l^{\frac{2q}{q-2}}$ and $\phi \in L^{r'}(\mathbb{R})$, gives the result. □

5. Applications to the geometry of Banach spaces

In this section we deal with the case $Y = \mathbb{C}$, $Z = X^*$, u: $X \times \mathbb{C} \to X$ given by $u(x, \lambda) = \lambda x$ and v: $X^* \times X \to \mathbb{C}$ given by $v(x^*, x) = \langle x^*, x \rangle$.

In [6], there was an investigation into the connection of the vector-valued formulation of inequalities in the setting of Hardy spaces, such as Paley's or Hardy's inequalities, with properties in the geometry of Banach spaces such as type, cotype or Fourier type. Later in [5] it was observed that behind Paley's inequality is actually the embedding $H^1(X) \subset \Lambda_{1,2}(X)$. Let us give a brief proof of this fact.

THEOREM 5.1 (see [5]). *If $X \in (H)_1$ then X satisfies Paley's inequality, i.e., there exists a constant $C > 0$ such that*

$$\left(\sum_{n=0}^{\infty} \|x_{2^n}\|^2\right)^{\frac{1}{2}} \leq C\|f\|_1$$

for any $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$.

Proof. It follows from Corollary 2.2 and Proposition 1.2 that for any finite sequence $(\lambda_n) \in \ell^2$ we have

$$\begin{aligned}\sum_{n\geq 0} \|\lambda_n x^{2^n}\| &\leq C\|f\|_{1,2} \left\|\sum_{n=0}^{\infty} \lambda_n z^{2^n}\right\|_{1,2} \\ &\leq C\|f\|_1.\end{aligned}$$

This clearly implies the desired inequality. □

THEOREM 5.2. *Let $1 \leq q_1 \leq 2$, $X \in (H)_{q_1}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{3}{2}$. Then there exists a constant $C > 0$ such that*

$$\left(\sum_{n\geq 0} |\langle x_n, x_n^*\rangle|^2\right)^{\frac{1}{2}} \leq C\|f\|_{q_1}\|g\|_{q_2,\infty}$$

for any $f(z) = \sum_{n\geq 0} x_n z^n \in \mathcal{P}(X)$ and $g(z) = \sum_{n\geq 0} x_n^* z^n \in \mathcal{P}(X^*)$.

Proof. Assume $f(z) = \sum_{n\geq 0} x_n z^n \in \mathcal{P}(X)$ and $g(z) = \sum_{n\geq 0} x_n^* z^n \in \mathcal{P}(X^*)$. Take $p_1 = 2$, $p_2 = q_1$, $p_3 = q_2$ and $p = \infty$. Applying part (i) in Corollary 2.1 we have

$$\left\|\sum_{n=0}^{\infty} \lambda_n < x_n, x_n^* > z^n\right\|_{\infty} \leq C \left\|\sum_{n=0}^{\infty} \lambda_n z^n\right\|_2 \|f\|_{q_1}\|g\|_{q_2,\infty}.$$

Therefore

$$\left(\sum_{n=0}^{\infty} |\langle x_n, x_n^*\rangle|^2\right)^{\frac{1}{2}} \leq C\|f\|_{q_1}\|g\|_{q_2,\infty}.$$

□

6. Applications to projective tensor products

Another interesting and useful bilinear map corresponds to the embedding $X \times Y \to X\hat{\otimes}Y$. A result similar to the next one was shown in [5] under slightly different assumptions.

THEOREM 6.1 (see [5]). *Let $X, Y \in (H)_1$. There exists a constant $C > 0$ such that*

$$\int_0^1 \int_{-\pi}^{\pi} \left\| \sum_{k=1}^{n} k x_k \otimes y_k s^k e^{ikt} \right\|_{X\hat{\otimes}Y} \frac{dt}{2\pi} ds \le C \left\| \sum_{k=1}^{n} x_k z^k \right\|_1 \cdot \left\| \sum_{k=1}^{n} y_k z^k \right\|_1$$

for any $x_1, \dots, x_n \in X$ and $y_1, \dots, y_n \in Y$.

Proof. Consider u: $X \times Y \to X\hat{\otimes}Y$ given by $u(x, y) = x \otimes y$ and v: $X\hat{\otimes}Y \times (X\hat{\otimes}Y)^* \to \mathbb{C}$ given by $v(z, z^*) = \langle z, z^* \rangle$. Take $p_1 = p_2 = 1$ and $p_3 = p = \infty$.
For any $h(z) = \sum_{n=0}^{\infty} T_n z^n \in \text{Bloch}((X\hat{\otimes}Y)^*)$ we have

$$\left\| \sum_{k=1}^{n} \langle T_k, x_k \otimes y_k \rangle z^k \right\|_{\infty} \le C \|h\|_{\text{Bloch}} \left\| \sum_{k=1}^{n} x_k z^k \right\|_1 \left\| \sum_{k=1}^{n} y_k z^k \right\|_1 .$$

Now use the fact (see [1], [5]) that the predual of $\text{Bloch}(E^*)$ can be identified with the set of E-valued analytic functions on the disc such that the integral $\int_0^1 \int_{-\pi}^{\pi} \|f'(re^{it})\|_E \frac{dt}{\pi} ds$ is finite, under the pairing given by $\langle f, g \rangle = \sum_{k=1}^{n} \langle e_k^*, e_k \rangle$ for polynomials $f(z) = \sum_{k=1}^{n} e_k z^k$ and $g = \sum_{n=0}^{\infty} e_k^* z^k$. By choosing $z = 1$ we get the desired result. □

THEOREM 6.2. *Let $1 \le p_1, p_2, p_3$ be such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 2$.*
Let X, Y, Z be complex Banach spaces such that $X \in (H)_{p_1}$ and $Y \in (H)_{p_2}$. Set $q = \infty$ if $p_2 \le 2$ and $q = \frac{2p_2}{p_2 - 2}$ if $2 < p_2$. Then

$$\left\| \sum_{n \ge 0} z_n \otimes (x_n \otimes y_n) \right\|_{X\hat{\otimes}Y\hat{\otimes}Z} \le C \|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3, q},$$

for any $f(z) = \sum_{n \ge 0} x_n z^n \in \mathcal{P}(X)$, $g(z) = \sum_{n \ge 0} y_n z^n \in \mathcal{P}(Y)$ and $h(z) = \sum_{n \ge 0} z_n z^n \in \mathcal{P}(Z)$

Proof. Use u: $X \times Y \to X\hat{\otimes}Y$ given by $u(x, y) = x \otimes y$ and v: $Z \times X\hat{\otimes}Y \to Z\hat{\otimes}X\hat{\otimes}Y$ given by $v(z, w) = z \otimes w$. □

7. Applications to multipliers for vector valued functions

One of the main motivations for the new formulation of convolution comes from the study of multipliers between vector valued Hardy spaces. A sequence of operators $T_n \in L(X, Y)$ is called a multiplier between $H^p(X)$ into $H^q(Y)$, to be denoted by $(T_n) \in (H^p(X), H^q(Y))$, if $\sum_{n=0}^{\infty} T_n(x_n)z^n \in H^q(Y)$ for any $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^p(X)$. In [5] the case $(H^1(X), BMOA(Y))$ was studied.

If $\frac{1}{p} + \frac{1}{q} \geq 1$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, then a simple application of Young's theorem gives

$$H^r(L(X, Y)) \subset (H^p(X), H^q(Y)).$$

THEOREM 7.1. *Let X, Y be complex Banach spaces, and let $1 \leq p, q, r$ be such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$. Assume that $X \in (H)_p$, and let $s = \infty$ if $p \leq 2$ and $s = \frac{2p}{p-2}$ if $2 < p$. Then we have*

$$(\lambda_n T_n) \in (H^p(X), H^q(Y))$$

whenever $\sum_{n=0}^{\infty} \lambda_n z^n \in H^1$ and $\sum_{n=0}^{\infty} T_n z^n \in \Lambda_{r,s}(L(X, Y))$.

Proof. Take v: $\mathbb{C} \times X \to X$ given by $v(\lambda, x) = \lambda x$ and u: $X \times L(X, Y) \to Y$ given by $u(x, T) = T(x)$.

Let $\phi(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ and $h(z) = \sum_{n=0}^{\infty} T_n z^n$, and $f(z) = \sum_{n\geq 0} x_n z^n \in \mathcal{P}(X)$. It is clear that

$$\sum_{n\geq 0} \lambda_n (T_n x_n) z^n = \phi *_v (f *_u h)(z).$$

Thus, by Theorem 2.1, we have

$$\begin{aligned} \left\| \sum_{n\geq 0} \lambda_n (T_n x_n) z^n \right\|_q &\leq C \|\phi\|_{1,2} \|f\|_{p,\max\{p,2\}} \|h\|_{r,s} \\ &\leq C \|\phi\|_1 \|f\|_p \|h\|_{r,s}. \end{aligned}$$

□

THEOREM 7.2. *Let X, Y be complex Banach spaces, and let $1 \leq p \leq q$ be such that $X \in (H)_p$. Let Z be another complex Banach space. Then, for any $1 \leq p_1, p_2, q_1, q_2$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 2 + \frac{1}{q} - \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1 - \frac{1}{\max\{p,2\}}$, if $\sum_{n=0}^{\infty} T_n z^n \in \Lambda_{p_1,q_1}(L(X, Z))$ and $\sum_{n=0}^{\infty} S_n z^n \in \Lambda_{p_2,q_2}(L(Z, Y))$ then*

$$(S_n T_n) \in (H^p(X), H^q(Y)).$$

Proof. Take u: $X \times L(X, Z) \to Z$ given by $u(x, T) = T(x)$ and v: $Z \times L(Z, Y) \to Y$ given by $v(z, S) = S(z)$ and use Theorem 2.1, combined with the $(H)_{p_1}$ property of X. □

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