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An adaptive version of a fourth-order iterative method for quadratic equations

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Abstract

A fourth-order iterative method for quadratic equations is presented. A semilocal convergence theorem is performed. A multiresolution transform corresponding to interpolatory technique is used for fast application of the method. In designing this algorithm we apply data compression to the linear and the bilinear forms that appear on the method. Finally, some numerical results are studied.

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1. Introduction

Determining the zeros of a nonlinear equation is a classical problem. These roots cannot in general be expressed in closed form. A powerful tool to study these equations is the use of iterative processes [17]. Starting from an initial guess x_0 successive approaches (until some predetermined convergence criterion is satisfied) x_i are computed, $i = 1, 2, \ldots,$ with the help of certain iteration function $\Phi : X \to X$,

$$x_{n+1} := \Phi(x_n), \quad n = 0, 1, 2 \ldots \tag{1}$$

In general, an iterative method is of pth order if the solution x^* of $F(x) = 0$ satisfies $x^* = \Phi(x^*)$, $\Phi'(x^*) = \cdots = \Phi^{p-1}(x^*) = 0$ and $\Phi^p(x^*) \neq 0$. For such a method, the error $\|x^* - x_{n+1}\|$ is proportional to $\|x^* - x_n\|^p$ as $n \to \infty$. It can be shown that the number of significant digits is multiplied by the order of convergence (approximately) by proceeding from x_n to x_{n+1}.

Newton's method and similar second-order methods are the most used [19]. Higher-order methods require more computational cost than other simpler methods, which makes them disadvantageous to be used in general, but, in some cases, it pays to be a little more elaborated.

In this paper, we are interested in the solution of quadratic equations

$$F(x) = 0, \tag{2}$$

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where $F: X \to Y$, X, Y Banach spaces and $F''(x) = B$ is a constant bilinear form. This is an example where third-order methods are a good alternative to Newton's type methods. Some particular cases of these type of equations, which appear in many applications, as control theory, are Riccati's equations [15].

In [12] the following family of third-order methods was introduced:

$$x_{n+1} = x_n - (I + \tfrac{1}{2}L_F(x_n)[I - \beta L_F(x_n)]^{-1})F'(x_n)^{-1}F(x_n), \tag{3}$$

where

$$L_F(x_n) = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n), \quad \beta \in [0, 1].$$

This family includes the classical Chebyshev ($\beta = 0$), Halley ($\beta = \frac{1}{2}$) and Super-Halley ($\beta = 1$) methods.

We refer to [12] and its references for a general convergence analysis of this type of third-order iterative methods. For the particular case of quadratic equations, we refer to [10], where the authors use the α-theory introduced in [18,20].

In general, third-order methods [1] can be written as

$$x_{n+1} = x_n - (I + \tfrac{1}{2}L_F(x_n) + \mathrm{O}(L_F(x_n)^2))F'(x_n)^{-1}F(x_n).$$

In particular, each iteration of a third-order method is between the iterations of two C-methods,

$$x_{n+1} = x_n - (I + \tfrac{1}{2}L_F(x_n) + CL_F(x_n)^2)F'(x_n)^{-1}F(x_n). \tag{4}$$

This class of scheme has been studied in [11,2].

In this paper, we analyze a C-method with fourth order of convergence for quadratic equations. The advantage of this method is that the matrix of the different associated linear systems in each iteration is the same.

On the other hand, multiresolution representations of data, such as wavelet decompositions, are useful tools for data compression. Given a finite sequence f^L, which represents sampling of weighted-averages of a function $f(x)$ at the finest resolution level L, multiresolution algorithms connect it with its multiscale representation $\{f^0, d^1, d^2, \ldots, d^L\}$, where the f^0 corresponds to the sampling at the coarsest resolution level and each sequence d^k represents the intermediate details which are necessary to recover f^k from f^{k-1}.

We consider the framework of Harten's multiresolution [13,14]. The greatest advantage of this general framework lies in its flexibility. For instance, boundary conditions receive a simplified treatment in this framework. Different types of setting can be considered depending on the linear operator that produces the data, we refer to [5,6] for more details. For simplicity, in this paper we consider the point value setting. In this setting, interpolation plays a key role. Usually, the interpolation is performed using polynomials.

One of the applications of multiresolution is matrix compression. If a matrix represents a smooth operator, its multiresolution representation can be transformed in a sparse matrix. This is called the standard form of a matrix [4,7]. We will need also a standard form for bilinear operators [3]. We use these standard forms to solve faster the linear system and the products appearing in the method.

The paper is organized as follows: The fourth-order iterative method is described and analyzed in Section 2, where we study a semilocal convergence theorem. We improve and simplify the hypothesis used in [11]. We recall in Section 3 the discrete pointvalue framework for multiresolution introduced by Harten [13,14] and the standard forms of a matrix and of a bilinear operator. We derive, using these multiresolution transforms, an adaptive version of the method. Finally, the algorithm is tested in Section 4 on several examples.

2. A fourth-order iterative method for quadratic equations

In general, for the C-methods (4) $\Phi'''(x^*) \neq 0$, so the cubic order of convergence cannot be reached. But if we take $C = C(x) = \frac{1}{2}(1 - L_{F'}(x)/3)$, we can improve the accuracy since in this case

$$\Phi'''(x^*) = 0.$$

For quadratic equations $L_{F'}(x) = 0$, and $C = C(x)$ will be the constant $C = \frac{1}{2}$.

Let us consider now the $\frac{1}{2}$-method

$$x_{n+1} = x_n - (I + \tfrac{1}{2} L_F(x_n) + \tfrac{1}{2} L_F(x_n)^2) F'(x_n)^{-1} F(x_n). \tag{5}$$

It can be written as follows:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n),\\ z_n &= y_n + \tfrac{1}{2} L_F(x_n)(y_n - x_n),\\ x_{n+1} &= z_n + L_F(x_n)(z_n - y_n). \end{aligned}$$

We are interested in quadratic equations. In this case, $F''(x)$ is a constant bilinear operator that we denote by B. Using Taylor expansions,

$$F(y_n) = F(x_n) + F'(x_n)(y_n - x_n) + \tfrac{1}{2} F''(x_n)(y_n - x_n)^2 = \tfrac{1}{2} B(y_n - x_n)^2. \tag{6}$$

Thus, $F'(x_n)^{-1} F(y_n) = -\frac{1}{2} L_F(x_n)(y_n - x_n)$.

Similarly,

$$F(z_n) = F(x_n) + F'(x_n)(z_n - x_n) + \tfrac{1}{2} B(z_n - x_n)^2 = F'(x_n)(z_n - y_n) + \tfrac{1}{2} B(z_n - x_n)^2.$$

Then,

$$F'(x_n)^{-1}(2F(z_n) - \tfrac{1}{2} B(z_n - y_n)(z_n - x_n)) = -L_F(x_n)(z_n - y_n),$$

and the method becomes

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n),\\ z_n &= y_n - F'(x_n)^{-1} F(y_n),\\ x_{n+1} &= z_n - F'(x_n)^{-1}(2F(z_n) - \tfrac{1}{2} B(z_n - y_n)(z_n - x_n)), \end{aligned}$$

or equivalently,

$$\begin{aligned} F'(x_n)(y_n - x_n) &= -F(x_n),\\ F'(x_n)(z_n - y_n) &= -F(y_n),\\ F'(x_n)(x_{n+1} - z_n) &= -2F(z_n) + \tfrac{1}{2} B(z_n - y_n)(z_n - x_n). \end{aligned} \tag{7}$$

Remark 1. Notice that for these equations the classical third-order methods [10] can be written as

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n),\\ z_n &= x_n + \theta(y_n - x_n), \quad 0 \leqslant \theta \leqslant 1,\\ x_{n+1} &= y_n - F'(y_n)^{-1} F(y_n) \end{aligned} \tag{8}$$

($\theta = 0$ Chebyshev, $\theta = \frac{1}{2}$ Halley and $\theta = 1$ Super-Halley).

Then, only for Chebyshev's method the matrix of these linear systems is the same.

The evaluations of F in the method will be computed as follows:

Lemma 1. *Let F be a quadratic operator, x_n, y_n and z_n the sequences defined in* (7). *Then*

$$F(y_n) = \tfrac{1}{2} B(y_n - x_n)^2, \tag{9}$$

$$F(z_n) = B(z_n - y_n)(y_n - x_n) + \tfrac{1}{2} B(z_n - y_n)^2, \tag{10}$$

$$\begin{aligned} F(x_{n+1}) = &- \tfrac{1}{2} B(z_n - y_n)(y_n - x_n) + \tfrac{1}{2} B(x_{n+1} - z_n)^2\\ &+ \tfrac{1}{2} B(z_n - x_n)(x_{n+1} - z_n) + \tfrac{1}{2} B(z_n - y_n)(x_{n+1} - z_n). \end{aligned} \tag{11}$$

Proof. We have proved (see (6)) relation (9). In addition, by using Taylor's formula

$$F(z_n) = F(y_n) + F'(y_n)(z_n - y_n) + \tfrac{1}{2}B(z_n - y_n)^2 = B(z_n - y_n)(y_n - x_n) + \tfrac{1}{2}B(z_n - y_n)^2.$$

In the same way,

$$\begin{aligned} F(x_{n+1}) &= F(z_n) + F'(z_n)(x_{n+1} - z_n) + \tfrac{1}{2}\, B(x_{n+1} - z_n)^2 \\ &= -\tfrac{1}{2}B(z_n - y_n)(y_n - x_n) + \tfrac{1}{2}\, B(x_{n+1} - z_n)^2 \\ &\quad + \tfrac{1}{2}\, B(z_n - x_n)(x_{n+1} - z_n) + \tfrac{1}{2}\, B(z_n - y_n)(x_{n+1} - z_n). \qquad \square \end{aligned}$$

Now, we present a semilocal convergence theorem for the introduced method using a punctual condition that improves the result given in [11].

Theorem 1. *Given x_0 such that there exists $F'(x_0)^{-1}$ and condition*

$$\|F'(x_0)^{-1}B\|\,\|F'(x_0)^{-1}F(x_0)\| \leqslant \tfrac{1}{2} \tag{12}$$

holds, then the iterative method (7) *is well defined and converges to x^*, solution of $F(x) = 0$.*

Proof. Using Eqs. (9)–(12), we obtain

$$\|F'(x_0)^{-1}F(y_0)\| \leqslant \tfrac{1}{2}\|F'(x_0)^{-1}B\|\,\|y_0 - x_0\|^2 \leqslant \tfrac{1}{4}\,\|F'(x_0)^{-1}F(x_0)\|.$$

By Eq. (10) it follows

$$\|F'(x_0)^{-1}F(z_0)\| \leqslant \tfrac{1}{2}\,\|F'(x_0)^{-1}B\|\,\|z_0 - y_0\|^2 + \|F'(x_0)^{-1}B\|\,\|z_0 - y_0\|\,\|y_0 - x_0\| \leqslant (\tfrac{3}{8})^2\|F'(x_0)^{-1}F(x_0)\|.$$

Moreover,

$$\|I - F'(x_0)^{-1}F'(z_0)\| = \|F'(x_0)^{-1}B(z_0 - x_0)\| \leqslant \tfrac{5}{8} < 1.$$

Then by Banach's lemma $F'(z_0)^{-1}F'(x_0)$ exists and

$$\|F'(z_0)^{-1}F'(x_0)\| \leqslant \frac{1}{1 - 5/8}.$$

By Eq. (11), condition (12) and (7),

$$\|F'(x_0)^{-1}F(x_1)\| \leqslant \left(\|L_F(x_0)^2 + 2L_F(x_0) - I\|\tfrac{1}{16} + \tfrac{1}{256}\right) \times \|F'(x_0)^{-1}F(x_0)\| =: c_0\|F'(x_0)^{-1}F(x_0)\|.$$

Furthermore,

$$\|I - F'(x_0)^{-1}F'(x_1)\| = \|F'(x_0)^{-1}B(x_1 - x_0)\| \leqslant \tfrac{1}{2} \cdot \|\tfrac{1}{2}\, L_F(x_0)^2 + \tfrac{1}{2}\, L_F(x_0) + I\| =: d_0,$$

since $d_0 < 1$, by Banach lemma $F'(x_1)^{-1}F'(x_0)$ exists and

$$\|F'(x_1)^{-1}F'(x_0)\| \leqslant \frac{1}{1 - d_0}.$$

Moreover,

$$\left(\frac{1}{1 - d_0}\right)^2 (c_0)\frac{1}{2} \leqslant \frac{1}{2},$$

and we obtain

$$\|F'(x_1)^{-1}B\|\,\|F'(x_1)^{-1}F(x_1)\| \leqslant \tfrac{1}{2}. \tag{13}$$

By an induction strategy and the same arguments as before, the following estimate holds

$$\|F'(x_n)^{-1}B\| \, \|F'(x_n)^{-1}F(x_n)\| \leqslant \tfrac{1}{2}, \quad \forall n \geqslant 0. \tag{14}$$

On the other hand,

$$\begin{aligned}\|F'(x_n)^{-1}F(x_n)\| &\leqslant \|F'(x_n)^{-1}F(x_{n-1})\| \; \|F'(x_{n-1})^{-1}F(x_n)\| \\ &\leqslant \frac{c_{n-1}}{(1-d_{n-1})} \|F'(x_{n-1})^{-1}F(x_{n-1})\| \\ &=: K\|F'(x_{n-1})^{-1}F(x_{n-1})\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_{n+1}-x_n\| &\leqslant \|x_{n+1}-z_n\| + \|z_n-y_n\| + \|y_n-x_n\| \\ &\leqslant (\tfrac{1}{8}+\tfrac{1}{4}+1)\|F'(x_n)^{-1}F(x_n)\| \\ &\leqslant \tfrac{11}{8} K^n \|F'(x_0)^{-1}F(x_0)\|,\end{aligned}$$

with $K<1$, then x_n is Cauchy's sequence and converges to x^*.

Now, we see that x^* is a solution of $F(x)=0$. Since

$$\|y_n-x_n\| \leqslant \|F'(x_n)^{-1}F(x_n)\| \leqslant K^n \|F'(x_0)^{-1}F(x_0)\|$$

and

$$\|z_n-y_n\| \leqslant \tfrac{1}{2}\|L_f(x_n)\| \, \|y_n-x_n\| \leqslant \tfrac{1}{4}\|y_n-x_n\|,$$

we have that y_n and z_n converge to x^*. Using expression (10)

$$F(z_n) = B(z_n-y_n)\left(\tfrac{1}{2}(z_n-y_n)+(y_n-x_n)\right),$$

we deduce $F(x^*)=0$. □

3. An adaptive version of the scheme

3.1. *The interpolatory multiresolution setting on the interval*

Let us consider a set of nested grids:

$$X^k = \{x_j^k\}_{j=0}^{J_k}, \quad x_j^k = jh_k, \quad h_k = 2^{-k}/J_0, \quad J_k = 2^k J_0,$$

where J_0 is some fixed integer. Consider the point-value discretization

$$D_k : \begin{cases} C([0,1]) & \to V^k, \\ f & \mapsto f^k = (f_j^k)_{j=0}^{J_k} = (f(x_j^k))_{j=0}^{J_k}, \end{cases} \tag{15}$$

where V^k is the space of real sequences of length J_k+1. A reconstruction procedure for this discretization operator is any operator R_k such that

$$R_k : V^k \to C([0,1]); \quad D_k R_k f^k = f^k, \tag{16}$$

which means that

$$(R_k f^k)(x_j^k) = f_j^k = f(x_j^k). \tag{17}$$

In other words, $(R_k f^k)(x)$ is a continuous function that interpolates the data f^k on X^k.

Thus if one writes $(R_k f^k)(x) = I_k(x; f^k)$, then, following [5] one can define the so-called direct (18) and inverse (19) multiresolution transforms as

$$f^L \to Mf^L \begin{cases} \text{Do } k = L, \ldots, 1 \\ f_j^{k-1} = f_{2j}^k, & 0 \leqslant j \leqslant J_{k-1}, \\ d_j^k = f_{2j-1}^k - I_{k-1}(x_{2j-1}^k; f^{k-1}), & 1 \leqslant j \leqslant J_{k-1}, \end{cases} \tag{18}$$

and

$$Mf^L \to M^{-1}Mf^L \begin{cases} \text{Do } k = 1, \ldots, L \\ f_{2j-1}^k = I_{k-1}(x_{2j-1}^k; f^{k-1}) + d_j^k, & 1 \leqslant j \leqslant J_{k-1}, \\ f_{2j}^k = f_j^{k-1}, & 0 \leqslant j \leqslant J_{k-1}. \end{cases} \tag{19}$$

With these algorithms, a discrete sequence f^L is encoded to produce a multi-scale representation of its information contents, $(f^0, d^1, d^2, \ldots, d^L)$; this representation is then processed (if the vector represents a piecewise smooth function, then the number of significant coefficients, after truncation $\hat{d}_j^k = \mathbf{tr}(d_j^k, \varepsilon_k)$, will be $\mathrm{O}(\log J_L)$) and the end result is a modified multi-scale representation $(\hat{f}^0, \hat{d}^1, \hat{d}^2, \ldots, \hat{d}^L)$ which is *close* to the original one, i.e. such that (in some norm)

$$\|\hat{f}^0 - f^0\| \leqslant \varepsilon_0, \quad \|\hat{d}^k - d^k\| \leqslant \varepsilon_k, \quad 1 \leqslant k \leqslant L,$$

where the truncation parameters $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_L$ are chosen according to some criteria specified by the user. After decoding the processed representation, we obtain a discrete set $\hat{f}^L$ which is expected to be *close* to the original discrete set f^L. For this to be true, some form of stability is needed, i.e. we must require that

$$\|\hat{f}^L - f^L\| \leqslant \sigma(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_L),$$

where $\sigma(\cdot, \ldots, \cdot)$ satisfies

$$\lim_{\varepsilon_l \to 0,\ 0 \leqslant l \leqslant L} \sigma(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_L) = 0.$$

The stability analysis for linear multiresolution processes, that is, the interpolation operator does not depend on data, is carried out using tools coming from wavelet theory, subdivision schemes and functional analysis (see [14,6,5]).

3.2. Deslauriers–Dubuc reconstruction

The most usual interpolatory techniques are polynomials. To obtain accurate reconstruction piecewise polynomial interpolations are considered.

Let $\mathscr{S}$ denote the set

$$\mathscr{S} = \mathscr{S}(r, s) = \{-s, -s+1, \ldots, -s+r\}, \quad r \geqslant s > 0, \quad r \geqslant 1,$$

and let $\{L_m(y)\}_{m \in \mathscr{S}}$ denote the Lagrange interpolation polynomials for the stencil of grid points $\{x_{j+m}^k\}_{m \in \mathscr{S}}$,

$$L_m(y) = \prod_{l=-s, l \neq m}^{-s+r} \left(\frac{y - l}{m - l} \right), \quad L_m(j) = \delta_j^m, \quad j \in \mathscr{S}.$$

Thus, we can consider

$$\mathscr{I}_k(x, f^k) = \sum_{m=-s}^{-s+r} f_{j+m}^k L_m \left(\frac{x - x_j^k}{h_k} \right), \quad x \in [x_{j-1}^k, x_j^k], \quad 1 \leqslant j \leqslant J_k.$$

Observe that if $f(x)=P(x)$, where $P(x)$ is a polynomial of degree less than or equal to r, then $\mathscr{I}_k(x, f^k)=f(x)$ for $x\in[x^k_{j-1}, x^k_j]$. It follows that, for smooth functions

$$\mathscr{I}_k(x, f^k)=f(x)+\mathrm{O}(h_k)^{r+1}.$$

Therefore, the order of the reconstruction procedure, which characterizes its accuracy, is $p=r+1$.

The particular situation $r=2s-1$ corresponds to an interpolatory stencil which is symmetric around the given interval and was considered, in [8], by Deslauriers and Dubuc (see also [9,16]). For instance, for $s=2$ ($r=3$) we obtain the following multiresolution transform:

$$\begin{cases} \text{Do } k=L,\ldots,1 \\ f^{k-1}_j = f^k_{2j}, & 0\leqslant j\leqslant J_{k-1}, \\ d^k_j = f^k_{2j-1} - \left(\dfrac{-f^{k-1}_{j-2}+9f^{k-1}_{j-1}+9f^{k-1}_j-f^{k-1}_{j+1}}{16}\right), & 1\leqslant j\leqslant J_{k-1}. \end{cases} \tag{20}$$

$$\begin{cases} \text{Do } k=1,\ldots,L \\ f^k_{2j-1} = d^k_j + \left(\frac{-f^{k-1}_{j-2}+9f^{k-1}_{j-1}+9f^{k-1}_j-f^{k-1}_{j+1}}{16}\right), & 1\leqslant j\leqslant J_{k-1}, \\ f^k_{2j} = f^{k-1}_j, & 0\leqslant j\leqslant J_{k-1}. \end{cases} \tag{21}$$

At the boundary, we choose one-sided stencils, of $r+1=2s$ points, at intervals where the centered-stencil choice would require function values which are not available. If $s=2$

$$d^k_1 = f^k_1 - \left(\tfrac{5}{16}f^{k-1}_0 + \tfrac{15}{16}f^{k-1}_1 - \tfrac{5}{16}f^{k-1}_2 + \tfrac{1}{16}f^{k-1}_3\right).$$

Modifications at the right boundary can be found by symmetry.

3.3. *Standard forms of linear and bilinear operators*

A generalized standard form for Harten's multiresolution analysis is obtained in [4]. Given a matrix A, its standard form, using (18) and (19), is the matrix

$$A^s = M\,A\,M^{-1}. \tag{22}$$

In the linear case, where the interpolation operator I_k does not depend on the data, the multiresolution transform is a matrix M.

Eq. (22) is equivalent to applying algorithm (18) to the columns of A, and the transpose of algorithm (19) to the rows of the resulting matrix.

The standard form satisfies

$$M(Af)=A^sM(f) \quad \text{for every vector } f; \quad (A^2)^s = A^sA^s.$$

Both M and M^{-1} are continuous transformations. Thus if an $N\times N$ matrix A represents a piecewise smooth operator, each transformed row in AM^{-1} represents a piecewise smooth operator. Sparsity is obtained by truncation of each transformed column in MAM^{-1}. The matrix A^s will have $\mathrm{O}(N\log N)$ significant coefficients, $\mathrm{O}(\log N)$ for each column. The transformation of the columns leaves the significant coefficients in the typical *finger shape*.

We refer to [4] for the rest of the details.

Finally, we are interested to obtain a standard form for bilinear operators (see [3] for more details). For simplicity, we consider

$$B:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$$

the extension to dimension N is self-evident.

Given two vectors $x, y \in \mathbb{R}^2$, we have

$$B(x, y) := (x_1, x_2) * \left(\frac{B_1}{B_2}\right)\left(\frac{y_1}{y_2}\right),$$

where $*$ denotes the product by blocks,

$$(x_1, x_2) * \left(\frac{B_1}{B_2}\right) = \begin{pmatrix} b_1^{11}x_1 + b_1^{21}x_2 & b_1^{12}x_1 + b_1^{22}x_2 \\ b_2^{11}x_1 + b_2^{21}x_2 & b_2^{12}x_1 + b_2^{22}x_2 \end{pmatrix}.$$

Considering

$$H_1 := \begin{pmatrix} b_1^{11} & b_1^{21} \\ b_2^{11} & b_2^{21} \end{pmatrix},$$

$$H_2 := \begin{pmatrix} b_1^{12} & b_1^{22} \\ b_2^{12} & b_2^{22} \end{pmatrix},$$

it is easy to check that

$$(x_1, x_2) * \left(\frac{B_1}{B_2}\right) = \left(\frac{H_1}{H_2}\right) * \left(\frac{x_1}{x_2}\right).$$

Therefore,

$$\begin{aligned} B(x, y) &= M\left((x_1, x_2)\left(\frac{B_1}{B_2}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \\ &= M * \left(\frac{H_1}{H_2}\right) * M^{-1}M\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} M^{-1}M\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \end{aligned}$$

Thus to compute $MB(x, y)$ we can do

(a) $B^s := M * \left(\frac{H_1}{H_2}\right) M^{-1}$.

This equation is equivalent to applying algorithm (18) to the columns of H_1 and H_2, and the transpose of algorithm (19) to the rows of the resulting matrix block.

(b) $Mx = M\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(c) $C := (B^s Mx)M^{-1}$.

(d) $My = M\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

(e) $C\,My$.

Remark 2. In dimension N, H_i is the matrix formed for the row i of each block-matrix of the bilinear operator B^{T}, where T denotes the transposed by blocks.

We solve the nonlinear system above adaptively (after truncation the matrix of the system are sparse matrix) using the standard forms that we have defined,

$$MF'(x_n)M^{-1}M(y_n - x_n) = M(-F(x_n)),$$

$$MF'(x_n)M^{-1}M(z_n - y_n) = M(-F(y_n)),$$

$$MF'(x_n)M^{-1}M(x_{n+1} - z_n) = M(-2F(z_n) + \tfrac{1}{2}B(z_n - y_n)(z_n - x_n)).$$

Table 1
x_0 such that $\|x^* - x_0\| = 0.5$

Iteration	Chebyshev	$C = \frac{1}{2}$
1	$4.06e+00$	$6.84e+00$
2	$4.77e-01$	$1.42e-01$
3	$3.54e-04$	$1.16e-07$
4	$4.79e-13$	$0.00e+00$
5	$0.00e+00$	—

Table 2
x_0 such that $\|x^* - x_0\| = 0.1$

Iteration	Chebyshev	$C = \frac{1}{2}$
1	$8.41e-03$	$1.21e-03$
2	$1.78e-09$	$5.44e-14$
3	$2.90e-13$	$0.00e+00$
4	$0.00e+00$	—

To evaluate $M(F(x_n))$, $M(F(y_n))$ and $M(F(z_n))$ the standard forms of the linear and bilinear parts are also used for fast matrix–vector and matrix–matrix multiplication after truncation.

Using the stability of the multiresolution, bounds of F and its Fréchet derivatives ($\|F''(x)\| = \|B\|$) and the condition number of the matrix appearing in the original systems, we obtain the following theorem (more details can be found in [3]).

Theorem 2. *Assume that x_n is well defined and converges to x^*. Denoting by $\{\tilde{x}_n\}$ the sequence of the adaptive version of the method, if we consider the truncation parameters $\varepsilon_k(n) = \varepsilon(n)$ sufficiently small, then it will be well defined and there will exist $n_{\mathrm{Tol}} \in \mathbb{N}$ such that*

$$\|x^* - \tilde{x}_n\| \leqslant \mathrm{Tol}, \quad \forall n \geqslant n_{\mathrm{Tol}}.$$

Remark 3. Tol is a free parameter for the user and it is related with the desired final accuracy.

4. Numerical analysis

In order to check the performance of the above-presented iterative method, various tests on quadratic equation have been performed.

Let us consider the equation

$$F(x) = x^{\mathrm{T}} Bx + Cx + D = 0, \tag{23}$$

where

$$B = \mathrm{rand}\ (n * n * n),$$
$$C = \mathrm{rand}(n * n),$$

and D is computed in order to obtain $x_i^* = 1$ as solution.

In Tables 1 and 2 we consider $N = 30$. The $C = \frac{1}{2}$-method has fourth order of convergence and improves the results of the Chebyshev method.

We define the ratio R as the number of operations in the $C - \frac{1}{2}$ method divided by the number of operations in the adapted method (Table 3).

Since the number of significant coefficients is $\mathrm{O}(N \log N)$, the difference between the adapted C-method and the classical C-method is bigger as N grows.

Table 3
Operations number

N	Ratio R
16	6.42
32	10.46
64	22.34

5. Conclusion

We have analyzed a fourth-order iterative method for quadratic equations. We have studied its convergence and we have tested its numerical behavior with respect to the classical Chebyshev method. The method seems to work very well in our numerical experiments.

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