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Calculus of nth roots and third order iterative methods ⋆

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Abstract

We apply a family of iterative methods to the problem of extracting the nth root of a positive number R, that is, to solve the nonlinear equation $t^n - R = 0$. For each value of n we obtain the method in the family for which the highest order of convergence is reached.

Key words: Root extraction, Iterative methods, Order of convergence
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The problem of approximating the nth root of a real number $R > 1$ has been widely studied by different authors. This problem is equivalent to solve the equation $f(t) = 0$, where

$$f(t) = t^n - R. \tag{1}$$

Let $\alpha = \sqrt[n]{R}$ be the solution of this equation. There are many ways of trying to solve this problem. So, for instance, we can obtain an approach to the solution by considering Newton's method. If we apply Newton's method to the function defined in (1), we have

$$t_{k+1} = t_k - \frac{f(t_k)}{f'(t_k)} = \frac{(n-1)t_k^n + R}{nt_k^{n-1}}, \quad k \geq 0.$$

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In particular, for the calculus of square roots, we have the famous Heron's formula (75 a. C. approx.):

$$t_{k+1} = \left(t_k + \frac{R}{t_k}\right) / 2, \quad k \geq 0.$$

In accordance with some authors [1], this algorithm was known by the Mesopotamian civilizations almost two thousand years before Christ. Heron also knew formulas for the calculus of cubic and higher roots.

It is an established fact that the convergence of Newton's method is quadratic, at least for t_0 sufficiently close to the solution α.

A modern approach is the given by Dubeau [2], who applies Newton's method to functions

$$F(t) = t^{\beta-n}(t^n - R), \tag{2}$$

where $\beta \in \mathbb{R}$ is looked for in order to get cubic convergence. In fact, the value

$$\beta = \frac{n+1}{2}$$

has been obtained as the appropriate for this purpose.

Another approach is the given by Gerlach in [3]. There he considers the problem of finding a solution α of $f(t) = 0$, where f is a general function, not necessarily in the form (1). The natural extension of this idea is to modify $f(t)$ by the function $F(t) = f(t)g(t)$ and determine $g(t)$ such that the order of convergence of Newton's method applied to $F(t)$ is increased. To do that, Gerlach uses the following result:

Theorem 1 *Let F be a sufficiently differentiable function, with a simple root α, that is $F(\alpha) = 0$, $F'(\alpha) \neq 0$. In addition, let us assume that $F''(\alpha) = 0$ and $F'''(\alpha) \neq 0$. Then, for the initial approximation t_0 close enough to α, the convergence of Newton's method*

$$t_{k+1} = t_k - \frac{F(t_k)}{F'(t_k)}, \quad k \geq 0.$$

is cubic.

Applying this theorem to $F(t) = f(t)g(t)$, we have that

$$F''(\alpha) = 0 \iff f''(\alpha)g(\alpha) + 2f'(\alpha)g'(\alpha) = 0.$$

Upon integration (see [3] for details) we obtain $g(t) = C/\sqrt{f'(t)}$, and then, for $C = 1$,

$$F(t) = \frac{f(t)}{\sqrt{f'(t)}}.$$

Notice that we can assume, without loss of generality, that $f'(t) > 0$ in a neighbourhood of α. Newton's method applied to $F(t)$ yields the iteration scheme

$$t_{k+1} = t_k - \frac{F(t_k)}{F'(t_k)} = t_k - \frac{1}{1 - \frac{1}{2}L_f(t_k)} \frac{f(t_k)}{f'(t_k)}, \quad k \geq 0,$$

where $L_f(t) = f(t)f''(t)/f'(t)^2$.

But this method is a well known third order method: Halley's method. An interesting geometrical derivation of this method can be found in the paper of Scavo and Thoo [4], together with further references.

For calculating nth roots, the function considered is $f(t) = t^n - R$. So, in this case

$$g(t) = \frac{1}{\sqrt{f'(t)}} = \frac{1}{\sqrt{nt^{n-1}}}.$$

This is the same function obtained in (2) for $\beta = (n+1)/2$.

Summing up the above comments: the sequence obtained by Dubeau is the same obtained applying Halley's method to $f(t) = t^n - R$.

Halley's method is one of the most common third order methods and, maybe, the most rediscovered one. But there are other third order methods. Let us consider now the application of other third order iterative methods to the problem of the nth root extraction. In particular, let us analyse the behaviour of the methods

$$t_{k+1} = G_\lambda(t_k) = t_k - \left[1 + \frac{1}{2}\frac{L_f(t_k)}{1 - \lambda L_f(t_k)}\right] \frac{f(t_k)}{f'(t_k)}, \quad k \geq 0, \quad \lambda \in \mathbb{R}, \quad (3)$$

for $f(t) = t^n - R$.

The above uniparametric family of methods has been introduced in [5] for the study of iterative processes in Banach spaces. Notice that when $\lambda = 1/2$ the corresponding method of (3) is Halley's method. In addition, this family includes other classical third order methods, such as Chebyshev's ($\lambda = 0$) and super-Halley's ($\lambda = 1$).

As $G_\lambda(\alpha) = \alpha$, $G'_\lambda(\alpha) = G''_\lambda(\alpha) = 0$, for all $\lambda \in \mathbb{R}$, the methods of (3) are locally convergent, at least cubically. Now we have

$$G'''(\alpha) = \frac{3(1-\lambda)f''(\alpha)^2 - f'(\alpha)f'''(\alpha)}{f'(\alpha)^2}.$$

So, $G'''(\alpha) = 0$ if

$$\lambda = 1 - \frac{L_{f'}(\alpha)}{3}.$$

Then, for $f(t) = t^n - R$,

$$L_{f'}(t) = f'(t)f'''(t)/f''(t)^2 = \frac{n-2}{n-1}$$

and

$$\lambda = \frac{2n-1}{3(n-1)}.$$

Consequently, for this value of λ, we have found a method of (3) that is convergent with order four when it is applied to the problem of extracting nth roots. For other values of λ, the convergence is cubic.

As particular case, let us consider the problem of the calculus of square roots, that is, $n = 2$. The methods of (3) are now

$$t_{k+1} = \frac{R^2(1-2\lambda) - 6Rt_k^2 + (2\lambda - 3)t_k^4}{4t_k(-R\lambda + (\lambda - 2)t_k^2)}.$$

The optimum value of λ is then $\lambda = 1$ and the best method in this case is super-Halley's:

$$t_{k+1} = \frac{R^2 + 6Rt_k^2 + t_k^4}{4t_k(R + t_k^2)}.$$

This fact is not surprising because it is known that for quadratic functions (such as $f(t) = t^2 - R$) super-Halley method has order four [6]. Under a different approach, this method is also obtained in [7] taking $p = 4$ in the scheme

$$t_{k+1} = \left(\sum_{i\ even} \binom{p}{i} t_k^{p-i} R^{i/2} \right) \Big/ \left(\sum_{i\ odd} \binom{p}{i} t_k^{p-i} R^{(i-1)/2} \right).$$

These sequences converge to $\alpha = \sqrt{R}$ with order p. Super-Halley method also appears in [8], by using the method of replacement to construct new methods for solving nonlinear equations.

For the case of cubic roots ($n = 3$), the methods of (3) are the following:

$$t_{k+1} = \frac{R^2(1-2\lambda) - R(5+2\lambda)t_k^3 + (4\lambda - 5)t_k^6}{3t_k^2(-2R\lambda + (2\lambda - 3)t_k^3)}. \tag{4}$$

All these methods are cubically convergent, except for $\lambda = 5/6$. For this value of λ the convergence is of order four. The expression of the method is

$$t_{k+1} = \frac{2R^2 + 20Rt_k^3 + 5t_k^6}{3t_k^2(5R + 4t_k^3)}$$

and it is not one of the third order methods previously known. At least, it doesn't appear in the extensive lists of methods included in [7] or [8].

Now, as an example we compare some of these methods for the calculus of $\alpha = \sqrt[3]{8} = 2$. We analyse the errors $t_k - \alpha$ for different values of the parameter λ in (4). In all cases we have taken $t_0 = R = 8$.

Step	$\lambda = 0$	$\lambda = 1/2$	$\lambda = 5/6$	$\lambda = 1$
$k = 1$	2.514	2.093	1.474	0.869
$k = 2$	0.722	0.384	0.077	−0.012
$k = 3$	$6.447\ 10^{-2}$	$7.119\ 10^{-3}$	$2.343\ 10^{-6}$	$1.709\ 10^{-7}$
$k = 4$	$1.016\ 10^{-4}$	$5.98\ 10^{-8}$	$2.094\ 10^{-24}$	$-4.162\ 10^{-22}$
$k = 5$	$4.365\ 10^{-13}$	$3.564\ 10^{-23}$	$1.336\ 10^{-96}$	$6.01\ 10^{-66}$
$k = 6$	$3.465\ 10^{-38}$	$7.546\ 10^{-69}$	$2.213\ 10^{-385}$	$-1.809\ 10^{-197}$

Table 1. Error comparisons.

In table 1 we observe that the number of significant digits is multiplied by the order of convergence (approximately) by proceeding from t_k to t_{k+1}. In that way, as we know, the method obtained for $\lambda = 5/6$ is optimum.

In this paper we have been concerned with the order of convergence of different iterative methods. It would be also interesting to compare other measures for the speed of convergence, such as the informational efficiency, the efficiency index or the computational efficiency [9].

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