Celest Mech Dyn Astr (2008) 101:49–68
DOI 10.1007/s10569-008-9129-6

ORIGINAL ARTICLE

# Exact solution of a triaxial gyrostat with one rotor

**Antonio Elipe · Víctor Lanchares**

Received: 21 September 2007 / Revised: 10 February 2008 / Accepted: 3 March 2008 /
Published online: 7 May 2008


**Abstract** The problem of the attitude dynamics of a triaxial gyrostat under no external torques and one constant internal rotor, is a three degrees-of-freedom system, although thanks to the existence of integrals of motion it can be reduced to only one degree-of-freedom problem. We introduce coordinates to represent the orbits of constant angular momentum as a flow on a sphere. This representation shows that the problem is equivalent to a quadratic Hamiltonian depending on two parameters. We find the exact solution of the orbits in terms of elliptic functions. By making use of properties of elliptic functions we find the solution at each region of the parametric partition from the solution of one region. We also prove that heteroclinic orbits are planar curves.

**Keywords** Attitude motion · Gyrostat · Elliptic functions

## 1 Introduction

A gyrostat $\mathcal{G}$ is a mechanical system made of a rigid body $\mathcal{P}$ called the *platform* and other bodies $\mathcal{R}$ called the *rotors*, connected to the platform and in such a way that the motion of the rotors does not modify the distribution of mass of the gyrostat $\mathcal{G}$. Due to this double spinning, the platform on the one hand and the rotors on the other, the gyrostat is also known with the name of *dual-spin* body.

Among the first applications of this problem, we may quote the work of Volterra (1899) to study the rotation of the earth. Later on, the problem has been studied mainly from a theoretical interest and a wide list of references may be found in Leimanis' textbook (1965). In recent years, the gyrostat model has attracted the attention of aerospace engineers and it is used for controlling the attitude dynamics of spacecraft and for stabilizing their rotations

A. Elipe (✉)
Grupo de Mecánica Espacial and IUMA, Universidad de Zaragoza, 50009 Zaragoza, Spain
e-mail: elipe@posta.unizar.es

V. Lanchares
Department of Mathematics and Computation, Universidad de la Rioja, 26004 Logroño, Spain

(see, for instance, Cochran et al. 1982; Hall 1995a; Hall and Rand 1994; Elipe and Lanchares 1997b; Lanchares et al. 1998; Iñarrea and Lanchares 2000; Vera and Vigueras 2006 and also Hughes 1986 for further references).

In this paper we deal with a particular case of gyrostat; it consists of a rigid body and a symmetrical rotor whose axis of symmetry is aligned along one of the principal axes of inertia of the body. Moreover, we assume that the angular momentum of the rotor about its axis of symmetry is constant. This is one of the two cases of practical interest of a gyrostat (Hughes 1986, p. 158) named Kelvin gyrostat. The other case, the apparent gyrostat, has been considered by other authors (Hall 1995a,b; Hall and Rand 1994; Koiller 1983; Holmes and Marsden 1983). However, after an appropriate conversion of the parameters involved (Hall 1995b; Hughes 1986), the differential equations describing both cases are the same. The attitude motion of the gyrostat is described by a system of three differential equations that determine the position of the angular velocity with respect to the body frame (Euler's equations), a single equation to describe the motion of the rotor relative to the platform, and another system of equations to describe the attitude of the platform with respect to the inertial frame.

Usually the unknowns in Euler's equations are taken to be the components of the angular velocity $\boldsymbol{\omega}$, but they lead to complicated visualizations as in the model of Poinsot (1851). We prefer to operate with the components of $\boldsymbol{G}$ in the body frame, because the phase flow determined occurs on a sphere of constant radius (Hughes 1986, p. 113).

It is proved (Elipe and Lanchares 1997b) that the Hamiltonian of the gyrostat here considered is equivalent to a two-parameter quadratic Hamiltonian and that the symplectic structure is that of the rigid body. For such kind of problems, Lanchares and Elipe (1995a,b) and Lanchares et al. (1995) made a complete classification of the bifurcations by continuous variation of the parameters involved, showing that the phase flow evolves through several bifurcations. In those papers, the phase portrait was obtained graphically by plotting the contour levels of the Hamiltonian (a quadratic form) on the unit sphere, although more sophisticated methods, like the "painting Hamiltonians" technique (Coffey et al. 1990) may be used. However, and since the system can be reduced to a one degree of freedom system (hence integrable), it still remains to find the exact analytical expressions of the orbits described by the angular momentum on the unit sphere. A first attempt in a more general context was made in Lanchares' Ph.D. dissertation (1993), although Cochran et al. also considered the exact solutions for a particular case (Cochran et al. 1982). In this note we complete this work and give the solution in terms of elliptic functions for every region of the parametric plane. We show that knowing the solution in one region, it is possible to have the solution everywhere by using appropriate properties of elliptic functions.

The interest of having the exact expression of the orbits is twofold. On the one hand because it has an intrinsic interest as a mathematical problem. On the other hand, the torque free motion may be considered as a perturbed problem when other torques, such as gravitational or magnetic torques act on it; in such a case, the closer the unperturbed problem to the actual problem, the more accurate the approximate solution obtained by perturbation theory will be. Albeit dealing with elliptic functions in the context of Lie transformations is an awkward task, some solutions have been proposed. One solution proposed by Howland (1989) and developed by Henrard and Wauthier (1989) has been fruitfully applied to several non-linear problems (Miller and Coppola 1993). Another solution consists of expanding elliptic functions and integrals in power series of their nome $q = \exp(-\pi K'/K)$ (see Abad et al. 1994 for details) and it has been successfully employed in several problems (Abad et al. 1997; Abad et al. 1998; Elipe and Vallejo 2001).

The structure of the paper is the following. In Sect. 2 we present the Hamiltonian in terms of components of the angular momentum, and show that the problem is equivalent to a general quadratic Hamiltonian depending on several parameters. Bifurcations are also described. Although much of this section is already known, we present here a short description for the sake of making the most self-contained the problem we are addressing. In Sect. 3 it is shown that the equations of motion are equivalent to solving a quadrature, whose solution is an elliptic function. In Sect. 4 some fundamental solutions for a specific flow regime are found; it is shown that by using appropriate properties of elliptic functions (namely, Landen, reciprocal modulus and imaginary modulus transformations) these fundamental solutions recover the solution of other flow regimes.

## 2 Hamiltonian of the gyrostat and parametric classification

Let us assume that the gyrostat has a fixed point $O$, that we will identify with the center of mass of the gyrostat and centered on it there are two orthonormal reference frames:

- $\mathcal{S}$, the space frame $O\boldsymbol{s}_1\boldsymbol{s}_2\boldsymbol{s}_3$, fixed in inertial space.
- $\mathcal{B}$, the body frame $O\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$, fixed in the platform.

Since, by definition, the motion of the rotors does not alter the distribution of mass of the gyrostat, there is a constant inertia tensor associated to $\mathcal{G}$ and we may assume that the body frame is precisely the frame of principal axes of inertia of the gyrostat. The attitude of $\mathcal{B}$ in $\mathcal{S}$ results in three rotations by means of the Euler angles.

Let

$$\boldsymbol{G} = G_1\boldsymbol{s}_1 + G_2\boldsymbol{s}_2 + G_3\boldsymbol{s}_3 = g_1\boldsymbol{b}_1 + g_2\boldsymbol{b}_2 + g_3\boldsymbol{b}_3,$$

be the expression of the angular momentum in both reference frames. Thus, proceeding as in Deprit and Elipe (1993), it was proved (Elipe and Lanchares 1997a) that

**Theorem 1** *The symplectic structure of the gyrostat is the one of the rigid body, that is, Poisson brackets satisfy the identities*

$$\begin{aligned} &\{g_1; g_2\} = -g_3, \quad \{g_2; g_3\} = -g_1, \quad \{g_3; g_1\} = -g_2, \\ &\{G_1; G_2\} = G_3, \quad \{G_2; G_3\} = G_1, \quad \{G_3; G_1\} = G_2. \end{aligned} \tag{1}$$

Now, we proceed to compute the kinetic energy of the gyrostat. Let $\boldsymbol{\omega} = \omega_1\boldsymbol{b}_1 + \omega_2\boldsymbol{b}_2 + \omega_3\boldsymbol{b}_3$ be the angular velocity in the body frame, and let $\boldsymbol{x} = x_1\boldsymbol{b}_1 + x_2\boldsymbol{b}_2 + x_3\boldsymbol{b}_3$ be the position vector of a particle $P$ of the gyrostat with mass $d\,m$; its absolute velocity is $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{x}$, where $\boldsymbol{v} = \dot{x}_1\boldsymbol{b}_1 + \dot{x}_2\boldsymbol{b}_2 + \dot{x}_3\boldsymbol{b}_3$. If the particle belongs to the platform, then $\boldsymbol{v} = 0$. The kinetic energy of the gyrostat is obtained by computing the volume quadrature

$$\begin{aligned} T &= \frac{1}{2}\int_{\mathcal{G}} (\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{x})^2\,\mathrm{d}m \\ &= \frac{1}{2}\int_{\mathcal{G}} (\boldsymbol{\omega}\times\boldsymbol{x})^2\,\mathrm{d}m + \boldsymbol{\omega}\cdot\int_{\mathcal{R}} (\boldsymbol{x}\times\boldsymbol{v})\,\mathrm{d}m + \frac{1}{2}\int_{\mathcal{R}} \boldsymbol{v}^2\,\mathrm{d}m \\ &= \frac{1}{2}\boldsymbol{\omega}\cdot\mathbb{I}\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\boldsymbol{f} + T_{\mathcal{R}}, \end{aligned} \tag{2}$$

where $\mathbb{I}$ is the diagonal tensor of inertia of the gyrostat $\mathcal{G}$ while $\boldsymbol{f} = f_1\boldsymbol{b}_1 + f_2\boldsymbol{b}_2 + f_3\boldsymbol{b}_3$ is the angular moment of the rotors and $T_{\mathcal{R}}$ is the kinetic energy of the rotors in their relative motion. Along the paper it is supposed that $\boldsymbol{f}$ is constant; that is, we consider a "Kelvin" gyrostat.

The Hamiltonian is the Legendre transformation with respect to the velocities of the Lagrangian function. Let us call the abbreviation $\boldsymbol{q} = (\phi, \theta, \psi)$ to denote the set of Eulerian angles; the kinetic energy of the gyrostat (2) is made of the addition of a pure quadratic term $\left(\frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\boldsymbol{\omega}\right)$ in the velocities $\dot{\boldsymbol{q}}$, plus a linear part $(\boldsymbol{\omega} \cdot \boldsymbol{f})$ in the velocities $\dot{\boldsymbol{q}}$ (for $\boldsymbol{f}$ does not depend on the Euler angles), plus a function of the time $(T_{\mathcal{R}}(t))$. By virtue of Euler's theorem for homogeneous functions, the Legendre transformation of the Lagrangian (with a potential $V(\boldsymbol{q})$) is

$$\mathcal{L}_L : \dot{\boldsymbol{q}} \longrightarrow \nabla_{\dot{\boldsymbol{q}}}(T - V) \cdot \dot{\boldsymbol{q}} - (T - V) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\boldsymbol{\omega} - T_{\mathcal{R}} + V,$$

and since the relative kinetic energy is a function of $t$ only, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\boldsymbol{\omega} + V(\boldsymbol{q}).$$

Since our paper deals exclusively with a gyrostat in free rotation ($V = 0$), the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\boldsymbol{\omega}, \tag{3}$$

Rather than expressing $\mathcal{H}$ in terms of the canonical coordinates and moments, we will express it in terms of the total angular moment that, as we observed in the Introduction, plays a critical role in our work.

N.B. The independent variable $t$ does not appear explicitly in the Hamiltonian (3), hence it is an integral.

The angular momentum vector of the gyrostat in the body frame is

$$\begin{aligned}
\boldsymbol{G} &= \int_{\mathcal{G}} (\boldsymbol{x} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}))\, \mathrm{d}m \\
&= \int_{\mathcal{P}} (\boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x}))\, \mathrm{d}m + \int_{\mathcal{R}} (\boldsymbol{x} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}))\, \mathrm{d}m \\
&= \int_{\mathcal{G}} (\boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x}))\, \mathrm{d}m + \int_{\mathcal{R}} (\boldsymbol{x} \times \boldsymbol{v})\, \mathrm{d}m \\
&= \mathbb{I}\boldsymbol{\omega} + \boldsymbol{f}.
\end{aligned}$$

Let us denote by $a_j$ the inverse of the principal moment of inertia $I_j$; that is $a_j = 1/I_j$. Assume moreover that $a_1 \geq a_2 \geq a_3 > 0$. With these conventions, there is a diagonal tensor $\mathbb{A}$, such that the angular velocity is $\boldsymbol{\omega} = \mathbb{A}(\boldsymbol{G} - \boldsymbol{f})$. The Hamiltonian (3) in these notations is

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{G} - \boldsymbol{f}) \cdot \mathbb{A}(\boldsymbol{G} - \boldsymbol{f}). \tag{4}$$

From here on, we shall assume that the rotor moment is constant ($f_i$ = constant, $i$=1, 2, 3).

**Theorem 2** *The Hamiltonian of a gyrostat in torque-free motion is a quadratic form with the phase space on the $\mathcal{S}^2$ sphere*

*Proof* By expanding the expression (4), and after dropping the constant terms $\left(\sum a_i f_i^2\right)$, we find the following expression for the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(a_1 g_1^2 + a_2 g_2^2 + a_3 g_3^2) - (a_1 g_1 f_1 + a_2 g_2 f_2 + a_3 g_3 f_3). \tag{5}$$

The Poisson structure (1) gives rise to the equations of the motion

$$\begin{aligned} \dot{g}_1 &= \{g_1; \mathcal{H}\} = (a_3 - a_2) g_2 g_3 + a_2 f_2 g_3 - a_3 f_3 g_2, \\ \dot{g}_2 &= \{g_2; \mathcal{H}\} = (a_1 - a_3) g_1 g_3 + a_3 f_3 g_1 - a_1 f_1 g_3, \\ \dot{g}_3 &= \{g_3; \mathcal{H}\} = (a_2 - a_1) g_1 g_2 + a_1 f_1 g_2 - a_2 f_2 g_1. \end{aligned} \tag{6}$$

From this system, one can easily prove that the norm of the angular momentum vector $\boldsymbol{G}$ is an integral

$$\|\boldsymbol{G}\|^2 = g_1^2 + g_2^2 + g_3^2 = G^2 = \text{constant}; \tag{7}$$

thus history of the rotation of $\boldsymbol{G}$ in the body frame is represented as a curve on the $S^2$ sphere of constant radius $G$. □

The system (6) admits two integrals; the kinetic energy (5) and the norm of the total angular momentum (7). Therefore it is integrable. The phase space of (6) may be regarded as a foliation of invariant manifolds

$$S^2(G) = \{(g_1, g_2, g_3) \,|\, g_1^2 + g_2^2 + g_3^2 = G^2\}.$$

By using the angular momentum $\boldsymbol{G}$ instead of the angular velocity $\boldsymbol{\omega}$, the geometric model depicting the rotations of $\boldsymbol{G}$ is a sphere with constant radius. Most importantly, unlike Poinsot's ellipsoids, the underlying model is independent of the ellipsoid of inertia (Table 1).

The differential system (6) belongs to a general class of Hamiltonian systems, the one of the type

$$\mathcal{H} = \mathcal{T}_2 + \mathcal{T}_1, \quad \text{with} \quad \mathcal{T}_2 = \frac{1}{2} \sum_{1 \le i,j \le 3} A_{ij}\, \xi_i \xi_j \quad \text{and} \quad \mathcal{T}_1 = \frac{1}{2} \sum_{1 \le i \le 3} B\, \xi_i. \tag{8}$$

The unknowns $\xi$ have the Poisson structure

$$\{\xi_i; \xi_j\} = - \sum_{1<k<3} \epsilon_{i,j,k}\, \xi_k,$$

where $\epsilon_{i,j,k}$ stands for the Levi–Civita symbol. The six coefficients $A_{ij}$ and the three coefficients $B_i$ are parameters independent of the variables $\xi$. The class depends on nine parameters, but it is possible to reduce it to five standard classes; this is done (Frauendiener 1995) by rotations in the phase space $(\xi_1, \xi_2, \xi_3)$, additions of invariant quantities to the Hamiltonian and time reversion. As the table below shows, one can roughly divide the classes by the number of parameters they contain.

**Table 1** The five cases of parametric quadratic Hamiltonians represented by (8)

| Case | Parameters | Hamiltonian |
|---|---|---|
| A | 1 | $\mathcal{H} = \frac{1}{2}\xi_1^2 + R\xi_2$ |
| B.a | 2 | $\mathcal{H} = \frac{1}{2}\xi_1^2 + Q\xi_1 + R\xi_2$ |
| B.b | | $\mathcal{H} = \frac{1}{2}\xi_1^2 + \frac{1}{2}P\xi_2^2 + Q\xi_1$ |
| C | 3 | $\mathcal{H} = \frac{1}{2}\xi_1^2 + \frac{1}{2}P\xi_2^2 + Q\xi_1 + R\xi_2$ |
| D | 4 | $\mathcal{H} = \frac{1}{2}\xi_1^2 + \frac{1}{2}P\xi_2^2 + Q\xi_1 + R\xi_2 + S\xi_3$ |

Cases A, B.a, B.b and C were identified and analyzed extensively by Elipe and Lanchares (1994). Case D is due to Frauendiener (1995) and is yet to be analyzed.

In fact, to each class corresponds a different type of gyrostat.

**Case A** Axially symmetric gyrostat ($a_1 = a_2 > a_3$) with one rotor spinning about anyone of the principal axes of inertia ($f_i \neq 0$ and $f_j = f_k = 0$).

**Case B.a** Axially symmetric gyrostat ($a_1 = a_2 > a_3$) with three rotors spinning about each principal axis of inertia ($f_1, f_2, f_3 \neq 0$).

**Case B.b** Asymmetric gyrostat ($a_1 > a_2 > a_3$) with one rotor spinning about any one of the principal axes of inertia ($f_i \neq 0,\ f_j = f_k = 0$).

**Case C** Asymmetric gyrostat ($a_1 > a_2 > a_3$) with two rotors spinning about any two of the principal axes of inertia ($f_i \neq 0,\ f_j \neq 0$ and $f_k = 0$).

**Case D** Asymmetric gyrostat ($a_1 > a_2 > a_3$) with three rotors spinning about each principal axis of inertia ($f_1, f_2, f_3 \neq 0$).

Note that when the rotor is not aligned with any principal axis (Hall 1995b), it may be decomposed as the sum of three different rotors aligned with the principal axes and thus, it can be reduced to one of these five cases, in general to Case D.

The problem we are considering in this paper, a tri-axial gyrostat with only one rotor, corresponds to Case B.b, that is, a biparametric Hamiltonian. Elipe and Lanchares (1997b) obtained Table 2 showing the correspondence among parameters $P$ and $Q$ and parameters of the gyrostat.

For the sake of simplicity of the notation and graphics representation, let us introduce the dimensionless variables

$$(u, v, w) \longrightarrow (g_1, g_2, g_3)/G, \qquad (\varphi_1, \varphi_2, \varphi_3) \longrightarrow (f_1, f_2, f_3)/G,$$

and after a time scaling $\tau = G\,t$, the Hamiltonian considered in this paper becomes

$$\mathcal{H} = \frac{1}{2}u^2 + \frac{1}{2}Pv^2 + Qu, \tag{9}$$

and variables $(u, v, w)$ lie on the unit sphere $u^2 + v^2 + w^2 = 1$. By using the symplectic structure (1), the equations of motion are

**Table 2** Reduction of the case of one spinning rotor to the generic Hamiltonian $\mathcal{H} = \frac{1}{2}\xi_1^2 + \frac{1}{2}P\xi_2^2 + Q\xi_1$

| Axis | $P$ | $P\in$ | $Q$ | Variables |
|---|---|---|---|---|
| Biggest ($\boldsymbol{b}_1$) | $\frac{a_3 - a_2}{a_1 - a_2}$ | $(-\infty, 0)$ | $\frac{-a_1 f_1}{a_1 - a_2}$ | $(g_1, g_2, g_3) \longrightarrow (\xi_1, \xi_3, \xi_2)$ |
| Smallest ($\boldsymbol{b}_3$) | $\frac{a_2 - a_1}{a_3 - a_1}$ | $(0, 1)$ | $\frac{-a_3 f_3}{a_3 - a_1}$ | $(g_1, g_2, g_3) \longrightarrow (\xi_3, \xi_2, \xi_1)$ |
| Intermediate ($\boldsymbol{b}_2$) | $\frac{a_1 - a_3}{a_2 - a_3}$ | $(1, \infty)$ | $\frac{-a_2 f_2}{a_2 - a_3}$ | $(g_1, g_2, g_3) \longrightarrow (\xi_2, \xi_1, \xi_3)$ |

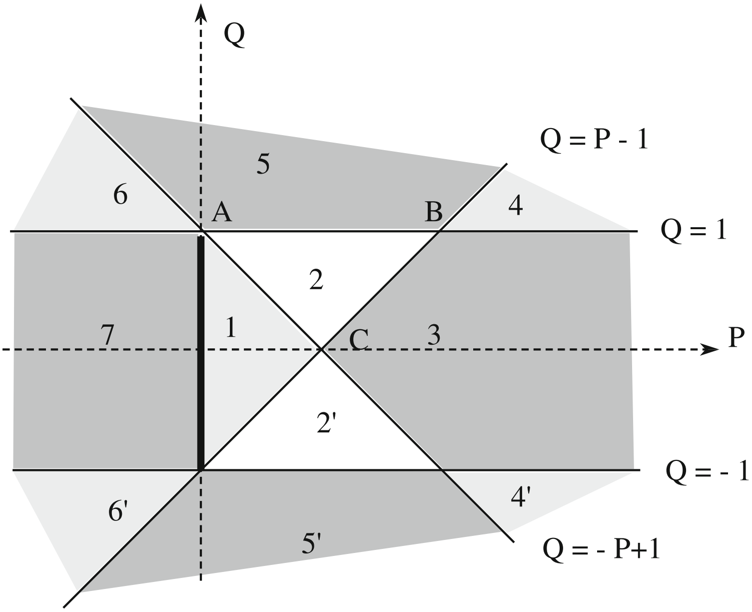


**Fig. 1** Partition of the parameter plane $PQ$. The phase flow is similar at points belonging to the same region in the partition, the number of equilibria is the same, as well as their stability. Boundary lines in the partition are bifurcation lines

$$\dot{u} = Pvw, \quad \dot{v} = -(Q+u)w, \quad \dot{w} = (Q+u-Pu)v. \tag{10}$$

The complete classification of Hamiltonians of the type (9) with parameters $P, Q \in \mathbb{R}^2$ was made by Lanchares and Elipe (1995b). The parametric plane partition is represented in Fig. 1. Differential system (10), as it was already noted by several authors (see e.g. Hall 1992), enjoys the symmetries

$$(v, t) \to (-v, -t), \quad (w, t) \to (-w, -t), \quad (u, Q) \to (-u, -Q).$$

Hence, the phase flow is symmetric with respect to the planes $v = 0$ and $w = 0$, and due to the third symmetry, it is sufficient to make the analysis for non-negative values of $Q$. Thus, in Fig. 1 regions for $Q < 0$ are labeled with the same number as their symmetric ones, since the flow is equivalent.

In the same paper it was proved that there are at most six equilibria, which will play an essential role in the integration:

$$\begin{aligned}
&E_1 = (+1, 0, 0) && \text{everywhere}\\
&E_3 = (-1, 0, 0) && \text{everywhere}\\
&E_{\mathrm{Mer}} = \left(-Q, 0, \pm\sqrt{1-Q^2}\right) && \text{when} |Q| \le 1\\
&E_{\mathrm{Eq}} = \left(\frac{Q}{P-1}, \pm\frac{\sqrt{(P-1)^2-Q^2}}{P-1}, 0\right) && \text{when} |Q| \le |P-1|
\end{aligned}$$

## 3 Integration by quadratures

From the equations of motion (10) and taking into account the first integrals (7) and (9) on the manifold $\mathcal{H} = H =$ const., it is easy to arrive at the following differential equation for the $u$ variable

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \sqrt{f(u)g(u)}, \tag{11}$$

where $f(u)$ and $g(u)$ are the following quadratic polynomials

$$f(u) = 2H - u^2 - 2Qu, \qquad g(u) = (1-P)u^2 + 2Qu + P - 2H. \tag{12}$$

In order to have a complete description of the solutions it is necessary to solve the quadrature (11) in terms of the parameters $P$ and $Q$, as well as the energy $H$ in each region of the parameter plane. However, we can take advantage of the discrete symmetries and equivalence transformations to reduce the solutions only to those regions where the type of flow is different. Indeed, on account of the symmetries of the problem it is only necessary to consider the case $Q \geq 0$ (Note that $Q = 0$ stands for a rigid body in free motion). On the other hand, replacing $v^2$ by $1 - u^2 - w^2$ in the Hamiltonian function we get

$$H' = \frac{1}{2}u^2 + P'w^2 + Q'u$$

where

$$H' = \frac{2H - P}{2(1-P)}, \quad P' = \frac{P}{P-1}, \quad Q' = \frac{Q}{1-P}.$$

This equivalence transformation maps region 1 to region 7, and region 2 to regions $4'$ and 6, and it is related to the symmetry transformation given in Hall (1992). Thus, the solution must be calculated in regions 1, 2, 3 and 5, which are representative of the four types of flow described and which may be seen in Fig. 2.

Cochran et al. (1982) noted that there are two basic types of solutions, depending on the nature of the roots of the polynomials $f(u)$ and $g(u)$: four real roots; two real roots and a pair of complex conjugate ones. However, the role played by the parameters in the nature of the roots as well as if they lie in the interval $[-1, 1]$, where $u$ belongs to, is not considered. In this way this is a simplified approach, because parameters $P$ and $Q$ play an important role in the number of equilibria as well as in the bifurcations among them, as it is highlighted in Hall and Rand (1994).

Let us consider the differential equation (11). It is possible to solve it by means of Cayley's method (Bowman 1961) to reduce it to a standard form by means of a rational transformation of the type

$$u = \frac{\alpha + z/\alpha}{1 + z}$$

with $\alpha$ a solution of the quadratic equation

$$\alpha^2 - \frac{2H - 1}{Q}\alpha + 1 = 0.$$

This transformation strongly depends on the roots of the polynomials $f(u)$ and $g(u)$. Not only it depends on their real or complex nature, but also on how they are ordered. Taking this fact into account it is important to know the conditions under which the nature or the order of the roots change. This follows from the discriminant of the quartic polynomial $f(u)g(u)$.

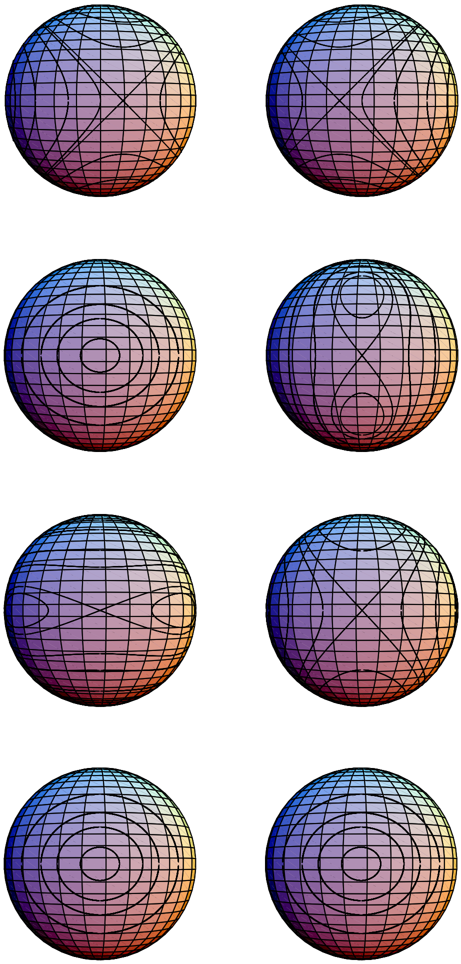

**Fig. 2** Phase portraits on the unit sphere for regions considered 1, 2, 3 and 5. First row, left figure, view from $+v$ axis; right figure, view from $-v$ axis. Remaining rows, left figure, view from $+u$ axis; right figure, view from $-u$ axis

**Proposition 1** *There is a change in the nature or the order of the roots if the energy takes values*

$$H_1 = \frac{1}{2} + Q, \quad H_3 = \frac{1}{2} - Q, \quad H_{\text{Mer}} = -\frac{Q^2}{2}, \quad H_{\text{Eq}} = \frac{P}{2} + \frac{Q^2}{2(P-1)},$$

or if $P = 1$ or $P = 0$.

The above result tells us that different solutions are obtained when the critical values $H_1$, $H_3$, $H_{\text{Mer}}$, $H_{\text{Eq}}$, $P = 0$ and $P = 1$ are crossed. Even more, it also tells us that the solution depends on the energy, as they are the energy levels on the sphere $u^2+v^2+w^2 = 1$. The cases $P = 0$ and $P = 1$ correspond to the limit situation when the gyrostat is axially symmetric.

It is worthy to notice that the values of the energy correspond to those of the equilibrium points of the problem, what confirms the important role of the equilibrium points in the phase flow of a dynamical system. Besides, when a change in the relative values of the energy takes place the solution also changes. But a change in the relative values of the energy implies bifurcation among equilibrium points. As a consequence we can obtain the partition of the parameter plane $PQ$ where different kind of flow is expected, recovering Fig. 1.

## 4 Fundamental solutions

From the preceding section it seems that many different solutions are needed to completely solve the problem. However, we can compute a solution and trace it through the different values of the parameters and the energy to produce the rest of solutions. In this way from a *fundamental solution* we can obtain a collection of solutions.

To begin with, we consider the flow in region 1, defined by $P \in [0, 1]$, $(P + Q - 1)(P - Q - 1) > 0$, restricted to the case $Q > 0$ due to the discrete symmetry. In this case, we have six equilibrium points and the critical energy values satisfy

$$H_{\text{Mer}} < H_{\text{Eq}} < H_3 < H_1.$$

Let $r_1$ and $r_2$ be the roots of $f(u)$ and $r_3$ and $r_4$ the roots of $g(u)$ given by

$$\begin{aligned}
r_1 &= -Q + \sqrt{Q^2 + 2H} = -Q + \sqrt{2(H - H_{\text{Mer}})},\\
r_2 &= -Q - \sqrt{Q^2 + 2H} = -Q - \sqrt{2(H - H_{\text{Mer}})},\\
r_3 &= \frac{-Q + \sqrt{Q^2 - (1-P)(P-2H)}}{1-P} = \frac{-Q + \sqrt{2(1-P)(H - H_{\text{Eq}})}}{1-P},\\
r_4 &= \frac{-Q - \sqrt{Q^2 - (1-P)(P-2H)}}{1-P} = \frac{-Q - \sqrt{2(1-P)(H - H_{\text{Eq}})}}{1-P}.
\end{aligned}$$

Then, there follows

**Proposition 2** *If* $P \in [0, 1]$, $(P + Q - 1)(P - Q - 1) > 0$ *and* $Q > 0$ *then*

(*i*) *If* $H \in [H_{\text{Mer}}, H_{\text{Eq}}]$, *then* $-1 \le r_2 \le r_1 \le 1$ *and* $r_3$, $r_4$ *are complex conjugate.*
(*ii*) *If* $H \in [H_{\text{Eq}}, H_3]$, *then* $-1 \le r_2 \le r_4 \le r_3 \le r_1 \le 1$.
(*iii*) *If* $H \in [H_3, H_1]$, *then* $r_4 \le r_2 \le -1 \le r_3 \le r_1 \le 1$.

*Proof* The proof is based on computing the values of functions $r_j(H) - r_k(H)$. For instance, let us consider the function $r_1(H) - r_3(H)$ in the case $H \in [H_{\text{Eq}}, H_3]$. Both $r_1$ and $r_3$ are increasing functions of $H$. On the other hand $r_1(H_{\text{Eq}}) > r_3(H_{\text{Eq}})$, whereas $r_1(E_3) = r_3(E_3)$. Thus, from elementary calculus, it must be $r_1 \ge r_3$ for $H \in [H_{\text{Eq}}, H_3]$. □

Once the nature and the order of the roots has been established we can proceed to the integration of the differential equation (11), but we only do so when the energy belongs to the interval $[H_{\text{Eq}}, H_3]$. It is in this interval where we have to consider two different solutions, as the variable $u$ can vary in two disjoint intervals, namely $[r_2, r_4]$ and $[r_3, r_1]$. These two solutions correspond to the orbits filling the bright areas in Fig. 3. Furthermore, as all roots are real, a bilinear transformation yields directly to the solution (Bowman 1961; Byrd and Friedman 1954).

In case $u \in [r_2, r_4]$, the bilinear transformation

$$z = \frac{r_1 - r_4}{r_4 - r_2} \frac{u - r_2}{r_1 - u} \tag{13}$$

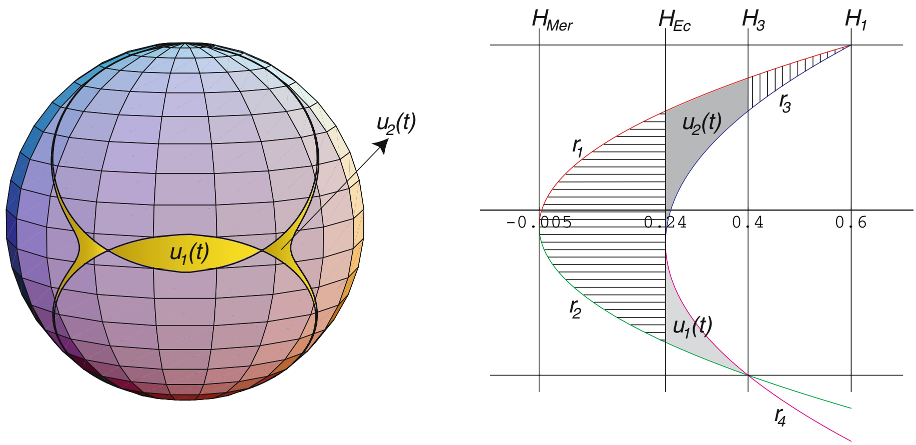


**Fig. 3** Area where fundamental solutions are calculated. On the left as viewed in phase space from point $E_3$. On the right as a function of the energy and the roots of the quartic

converts the differential equation (11) into

$$h\mathrm{d}t = \frac{\mathrm{d}z}{\sqrt{4z(1-z)(1-k^2 z)}}$$

where

$$k^2 = \frac{(r_1 - r_3)(r_4 - r_2)}{(r_1 - r_4)(r_3 - r_2)}, \quad h = \frac{1}{2}\sqrt{(1-P)(r_1 - r_4)(r_3 - r_2)}. \tag{14}$$

Thus, in this case we obtain the first fundamental solution

$$u_1(t) = \frac{A_1 + A_2 \operatorname{sn}^2(ht, k)}{A_3 + A_4 \operatorname{sn}^2(ht, k)}, \tag{15}$$

where

$$A_1 = r_2(r_1 - r_4), \quad A_2 = r_1(r_4 - r_2),$$
$$A_3 = r_1 - r_4, \quad A_4 = r_4 - r_2.$$

It corresponds to the orbits surrounding the equilibrium point $E_3$, located at $(u, v, w) = (-1, 0, 0)$. Besides, taking into account the first integrals

$$H = \frac{1}{2}u^2 + \frac{1}{2}Pv^2 + Qu, \quad u^2 + v^2 + w^2 = 1,$$

we obtain, after some algebraic manipulations,

$$v_1^2 = \frac{(r_1 - r_2)^2 (r_1 - r_4)(r_4 - r_2) \operatorname{sn}^2(ht, k)}{P(A_3 + A_4 \operatorname{sn}^2(ht, k))^2},$$

$$w_1^2 = \frac{(r_1 - r_4)^2 (1 - r_2^2) \operatorname{cn}^2(ht, k) \operatorname{dn}^2(ht, k)}{(A_3 + A_4 \operatorname{sn}^2(ht, k))^2}.$$

In the case $u \in [r_3, r_1]$, the bilinear transformation

$$z = \frac{r_3 - r_2}{r_1 - r_3} \frac{r_1 - u}{u - r_2}$$

converts the differential equation (11) into

$$h\mathrm{d}t = \frac{\mathrm{d}z}{\sqrt{4z(1-z)(1-k^2z)}}$$

where $h$ and $k^2$ are given by (14). Now, we obtain the second fundamental solution

$$u_2(t) = \frac{B_1 + B_2\,\mathrm{sn}^2(ht,k)}{B_3 + B_4\,\mathrm{sn}^2(ht,k)}, \tag{16}$$

where

$$B_1 = r_1(r_3 - r_2), \quad B_2 = r_2(r_1 - r_3),$$
$$B_3 = r_3 - r_2, \qquad B_4 = r_1 - r_3.$$

It corresponds to the orbits surrounding the heteroclinic loop connecting the unstable points $E_c$. Besides

$$v_2^2 = \frac{(r_1-r_2)^2(r_1-r_3)(r_3-r_2)\,\mathrm{sn}^2(ht,k)}{P(B_3 + B_4\,\mathrm{sn}^2(ht,k))^2},$$

$$w_2^2 = \frac{(r_3-r_2)^2(1-r_1^2)\,\mathrm{cn}^2(ht,k)\,\mathrm{dn}^2(ht,k)}{(B_3 + B_4\,\mathrm{sn}^2(ht,k))^2}.$$

Considering the fundamental solutions as functions of complex variable, it is possible to extend them for values of the energy, and the parameters, other than those required to obtain them. First of all, let us extend the solutions for different values of the energy in the region 1. To this end, the theory of transformation of elliptic functions provides a basic tool to demonstrate that the two fundamental solutions are valid for the whole range of energies in region 1.

### 4.1 $H \in [H_{\mathrm{Mer}}, H_{\mathrm{Eq}}]$

From Proposition 2 we know that $r_3$ and $r_4$ are complex conjugate, while $r_1$ and $r_2$ are real. It is not difficult to prove that now the modulus $k^2$, the same for the two fundamental solutions, becomes a complex number on the unit circle. In fact, if

$$r_3 = \alpha + i\beta, \qquad r_4 = \alpha - i\beta,$$

we obtain

$$k^2 = \frac{(r_1-r_3)(r_4-r_2)}{(r_1-r_4)(r_3-r_2)} = \frac{(r_1-\alpha-i\beta)(\alpha-r_2-i\beta)}{(r_1-\alpha+i\beta)(\alpha-r_2+i\beta)},$$

and then

$$k^2 = \frac{(r_1-\alpha)(\alpha-r_2)-\beta^2-i\beta(r_1-r_2)}{(r_1-\alpha)(\alpha-r_2)-\beta^2+i\beta(r_1-r_2)},$$

a complex number such that $|k^2| = 1$, as numerator and denominator are complex conjugate. On the other hand, the argument of the elliptic function also becomes a complex number, as well as the coefficients appearing in $u_1(t), u_2(t)$. Now, we look for a transformation mapping the modulus $k^2$ into the real interval [0, 1].

**Proposition 3** *Let be* $\mathrm{sn}(z,k)$ *such that* $k^2 \in \mathbb{C}$ *and* $k^2 = \cos\theta + i\sin\theta$. *The composition of Landen's and inverse modulus transformations yields*

$$\mathrm{sn}(z,k) = \frac{\mathrm{sn}(\sqrt{k}z,\lambda)\,\mathrm{dn}(\sqrt{k}z,\lambda)}{\sqrt{k}\,\mathrm{cn}(\sqrt{k}z,\lambda)},$$

*where* $\lambda^2 = \dfrac{(1+k)^2}{4k} \in [0,1]$.

*Proof* It is only necessary to prove that the square of the new modulus belongs to the interval [0, 1], but,

$$\lambda^2 = \left(\frac{1+k}{2\sqrt{k}}\right)^2 = \left(\frac{1+\cos\theta/2 + i\sin\theta/2}{2(\cos\theta/4 + i\sin\theta/4)}\right)^2 = \cos^2\theta/2 \in [0,1].$$

Applying Proposition 3 to the fundamental solutions we obtain

$$\mathrm{sn}^2(ht,k) = \frac{\mathrm{sn}^2(Gt,\lambda)\,\mathrm{dn}^2(Gt,\lambda)}{k\,\mathrm{cn}^2(Gt,\lambda)} = \frac{1-\mathrm{cn}(2Gt,\lambda)}{k(1+\mathrm{cn}(2Gt,\lambda))},$$

where

$$G = \frac{1}{2}\sqrt{(1-P)\sqrt{(r_1-r_3)(r_1-r_4)(r_3-r_2)(r_4-r_2)}} \in \mathbb{R}$$

and

$$\lambda^2 = \frac{\left(\sqrt{(r_3-r_2)(r_1-r_4)} + \sqrt{(r_1-r_3)(r_4-r_2)}\right)^2}{4\sqrt{(r_1-r_3)(r_1-r_4)(r_3-r_2)(r_4-r_2)}}.$$

Therefore, the first fundamental solution can be written as

$$u_1 = C_1\frac{1+C_2\,\mathrm{cn}(2Gt,\lambda)}{1+C_3\,\mathrm{cn}(2Gt,\lambda)},$$

where

$$C_1 = \frac{r_2\sqrt{(r_1-r_3)(r_1-r_4)} + r_1\sqrt{(r_3-r_2)(r_4-r_2)}}{\sqrt{(r_1-r_3)(r_1-r_4)} + \sqrt{(r_3-r_2)(r_4-r_2)}},$$

$$C_2 = \frac{r_2\sqrt{(r_1-r_3)(r_1-r_4)} - r_1\sqrt{(r_3-r_2)(r_4-r_2)}}{r_2\sqrt{(r_1-r_3)(r_1-r_4)} + r_1\sqrt{(r_3-r_2)(r_4-r_2)}},$$

$$C_3 = \frac{\sqrt{(r_1-r_3)(r_1-r_4)} - \sqrt{(r_3-r_2)(r_4-r_2)}}{\sqrt{(r_1-r_3)(r_1-r_4)} + \sqrt{(r_3-r_2)(r_4-r_2)}}$$

are real numbers. On the other hand, the second fundamental solution becomes

$$u_2 = C_1\frac{1-C_2\,\mathrm{cn}(2Gt,\lambda)}{1-C_3\,\mathrm{cn}(2Gt,\lambda)},$$

which it is no more than the first one shifted half a period (see Fig. 4).

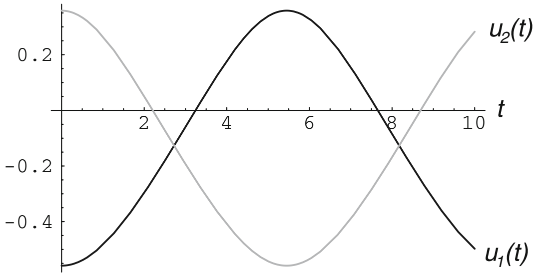


**Fig. 4** The two fundamental solutions for $H \in [H_{\rm Mer}, H_{\rm Eq}]$

### 4.2 $H \in [H_3, H_1]$

In this case, from Fig. 3, it is clear that only the second fundamental solution can be extended (to the right to the vertical ruled lines area), as the first one will give rise to values of the variable $u$ outside the interval $[-1, 1]$.

In order to have the new solution we look at the modulus $k^2$, which becomes negative, as it is deduced from Proposition 2. Then, the imaginary modulus transformation,

$$\mathrm{sn}(z, ik) = \frac{1}{\sqrt{1+k^2}}\,\mathrm{sd}\left(\sqrt{1+k^2}z, k/\sqrt{1+k^2}\right),$$

is the appropriate one to obtain the desired solution, given by

$$u_2(t) = \frac{\bar{B}_1 + \bar{B}_2\,\mathrm{sn}^2(Gt, \lambda)}{\bar{B}_3 + \bar{B}_4\,\mathrm{sn}^2(Gt, \lambda)}$$

where

$$\lambda^2 = \frac{(r_1 - r_3)(r_2 - r_4)}{(r_1 - r_2)(r_3 - r_4)}, \quad G = \frac{1}{2}\sqrt{(1-P)(r_1 - r_2)(r_3 - r_4)},$$

$$\bar{B}_1 = r_1(r_3 - r_4), \quad \bar{B}_2 = r_4(r_1 - r_3),$$
$$\bar{B}_3 = r_3 - r_4, \qquad \bar{B}_4 = r_1 - r_3.$$

### 4.3 Asymptotic solutions

When $H = H_{\rm Eq}$, the two roots $r_3$ and $r_4$ are the same and the modulus $k^2$ becomes 1, and the fundamental solutions are now expressed in terms of hyperbolic functions. They give rise to the four heteroclinic orbits connecting the unstable points $E_{\rm Ec}$. In this way, we obtain

$$u_1(t) = \frac{B_1 + B_2\cosh ht}{B_3 + B_4\cosh ht}, \qquad u_2(t) = \frac{-B_1 + B_2\cosh ht}{-B_3 + B_4\cosh ht}, \tag{17}$$

where

$$\begin{aligned} h &= \sqrt{(1-P)(r_1 - r_3)(r_3 - r_2)}, \\ B_1 &= (r_1 - r_3)r_2 - (r_3 - r_2)r_1, \quad B_2 = (r_1 - r_3)r_2 + (r_3 - r_2)r_1, \\ B_3 &= (r_1 - r_3) - (r_3 - r_2), \qquad B_4 = (r_1 - r_3) + (r_3 - r_2). \end{aligned} \tag{18}$$

By computing $v_{1,2}(t)$ and $w_{1,2}(t)$ from the first integrals it follows that the heteroclinic loops lie into planes. However, this can be proved in a different way.

**Theorem 3** *Heteroclinic orbits are planar curves on the sphere.*

*Proof* Let us prove this result for the heteroclinic orbits connecting points $E_{\rm Eq}$ and surrounding points $E_{\rm Mer}$, $E_3$ in the region 1 ($0 < Q < 1,\ Q < |P-1| < 1$). The proof for the remaining case (heteroclinic orbits in region 7) is similar.

We take advantage of the symmetric character of the orbits with respect to the plane $v = 0$ and we consider the plane $-u + \alpha w + \delta = 0$, with $\alpha$ a tuning parameter to be determined. As the plane must contain the points $E_{\rm Eq}$, it must be $\delta = Q/(P-1)$. Thus, we have the equation of the heteroclinic in terms of $w$

$$u = \alpha w + Q/(P-1), \quad v = \sqrt{1 - w^2 - (\alpha w + Q/(P-1))^2}, \quad w = w.$$

By replacing these expressions in the Hamiltonian (9), we get

$$H(\text{curve}) = \frac{\left[P(P-1) + Q^2\right] + \left[-\alpha^2 + (1+2\alpha^2)P - (1+\alpha^2)P^2\right] w^2}{2(P-1)}.$$

But this energy must be constant and equal to the energy (1) at the equilibrium $E_{\rm Eq}$, hence the coefficient of $w^2$ must vanish, which happens for $\alpha = \pm\sqrt{P/(1-P)}$. Whence the four heteroclinics surrounding the equilibria $E_{Mer}$ are on the planes

$$-u \pm \sqrt{P/(1-P)}\, w + Q/(P-1) = 0.$$

□

N.B. The quotient $P/(1-P)$ cannot be negative, which happens if and only if $P \in [0, 1]$.

Special attention deserves the limit case $P + Q - 1 = 0$ in the upper boundary of region 1. Now the three equilibrium points $E_{Eq}$ and $E_3$ collide. This has a counterpart in the roots of polynomials $f(u)$ and $g(u)$. Indeed, in that case $r_2 = r_3 = r_4 = -1$ and the solution becomes a rational function of $t$. This solution can be obtained directly by integrating the differential equation (11) and it reads

$$u(t) = \frac{2P - 1 - P^2(1-P)t^2}{1 + P^2(1-P)t^2}.$$

Nevertheless, it also appears as a limit of the second fundamental solution, given in (17), by applying twice L'Hôpital rule when $Q$ approaches $1 - P$. On the other hand, the first fundamental solution reduces to the equilibrium point $E_3$, as it is deduced from (17) and (18). Even more, in this case the solutions are two homoclinic loops, which are minor circles resulting from the intersection of the unit sphere and the planes

$$u \pm \sqrt{P/(1-P)} w + 1 = 0.$$

These two planes intersect at $E_3$ and then, the two orbits are tangent at this point.

### 4.4 Limit solutions

Let us consider now those solutions corresponding to limit cases, when parameters $P$ and $Q$ reach the boundary lines of region 1. Firstly we will also consider $Q = 0$ as a limit situation, in which we recover the rigid body in torque free motion.

Let us take $Q = 0$ and $0 < P < 1$. In this situation $r_2 = -r_1$ and $r_4 = -r_3$. Then, from (14), (15) and (16), we can write

$$u_1(t) = -u_2(t) = -r_1 \frac{1 - k\,\text{sn}^2(ht, k)}{1 + k\,\text{sn}^2(ht, k)},$$

where

$$k^2 = \frac{(r_1 - r_3)^2}{(r_1 + r_3)^2}, \qquad h = \frac{1}{2}\sqrt{(1-P)(r_1+r_3)^2}.$$

The Gauss transformation (Byrd and Friedman 1954), yields

$$u_1(t) = -r_1\,\mathrm{dn}(Gt, \lambda),$$

with

$$G = (1+k)h = \sqrt{8H(1-P)} \quad \text{and} \quad \lambda^2 = \frac{4k}{(1+k)^2} = \frac{P(1-2H)}{2H(1-P)}.$$

By appropriate conversion of the parameters $H$ and $P$ (see Table 2) we recover the solution for a triaxial rigid body given in Deprit and Elipe (1993) in the case of circulations around the axis of biggest moment of inertia.

An extreme limit situation takes place at $P = 1$ and $Q = 0$. Note that in this case, $g(u)$ it is no longer a polynomial in $u$ as it is just a constant. However, taking limits in (14) we arrive to

$$\lim_{(P,Q)\to(1,0)} k^2 = 1, \qquad \lim_{(P,Q)\to(1,0)} = \frac{1}{2}\sqrt{2H-1}, \quad \left(0 < H < \frac{1}{2}\right).$$

Similarly, for the ratios of the coefficients in (15), we get

$$\lim_{(P,Q)\to(1,0)} \frac{A_1}{A_2} = 1, \quad \lim_{(P,Q)\to(1,0)} \frac{A_3}{A_4} = -1, \quad \lim_{(P,Q)\to(1,0)} \frac{A_2}{A_4} = \sqrt{2H}.$$

Taking into account that $\mathrm{sn}(z, 1) = \tanh z$ and that $h$ is purely imaginary, we obtain

$$u_1(t) = \sqrt{2H}\cos\sqrt{1-2H}t.$$

Besides, it can be seen that

$$v_1(t) = \sqrt{2H}\sin\sqrt{1-2H}t, \qquad w_1(t) = \pm\sqrt{1-2H}.$$

That is, the orbits are circles parallel to the plane $w = 0$, i.e. orbits are the intersection of the family of cylinders $u^2 + v^2 = H$ with the sphere $u^2 + v^2 + w^2 = 1$.

Moreover, when $w = 0$, or $H = 1/2$, the frequency of the solutions is 0, and there is no movement; that is, every point on the equator of the sphere is an equilibrium.

We also obtain circular solutions for $P = 0$. Indeed, since now $r_1 = r_3$ and $r_2 = r_4$, there follows

$$k^2 = 0, \qquad h = \sqrt{2H + Q^2}, \quad (-Q^2/2 < H < 1/2 + Q).$$

On the other hand, $A_2 = A_4 = 0$, $B_2 = B_4 = 0$ and then $u_1(t) = r_2$ and $u_2(t) = r_1$ are constant. In addition,

$$v_{1,2}(t) = \sqrt{1 - u_{1,2}^2}\sin ht, \qquad w_{1,2}(t) = \sqrt{1 - u_{1,2}^2}\cos ht.$$

It is worth noting that in the case $H = -Q^2/2$, the frequency of the circular solutions becomes 0, and the corresponding circle (in general a minor circle parallel to the plane $u = 0$) is made of equilibria. This situation takes place when the four roots come into coincidence and they belong to the interval $(-1, 1)$, that is to say, $|Q| < 1$. If $|Q| \geq 1$ all the solutions remain circular, but there is not a set of non-isolated equilibria.

## 5 Extended solutions

In the previous section, the fundamental solutions were extended throughout the whole region 1, including the boundary. Now we will do a similar analysis in the remaining regions.

For the sake of conciseness we will do this only in region 3, because for regions 2 and 5 it is clear that solutions evolve in a natural way from those computed in region 1. Anyway, the procedure is completely similar.

First of all, we need to determine the nature and order of the roots in region 3, defined by the conditions $|Q| < 1$ and $(P + Q)^2 - 1 > 0$. In this way we have

**Proposition 4** *If* $|Q| < 1$ *and* $(P + Q)^2 - 1 > 0$ *then*

(*i*) *If* $H \in [H_{Mer}, H_3]$*, then* $r_3 \le -1 \le r_2 \le r_1 \le 1 \le r_4$.
(*ii*) *If* $H \in [H_3, H_1]$*, then* $r_2 \le -1 \le r_3 \le r_1 \le 1 \le r_4$.
(*iii*) *If* $H \in [H_1, H_{Eq}]$*, then* $r_2 \le -1 \le r_3 \le r_4 \le 1 \le r_1$.

Now we will extend the fundamental solutions for all the cases described in Proposition 4.

### 5.1 $H \in [H_{Mer}, H_3]$

In this situation the modulus $k^2 > 1$ and the *frequency* $h$ is purely imaginary. Two transformations, the reciprocal modulus and the imaginary argument transformation yields

$$\mathrm{sn}^2(z, k) = -\frac{1}{k^2}\frac{\mathrm{sn}^2(kz, \lambda)}{\mathrm{cn}^2(kz, \lambda)}, \qquad \lambda^2 = 1 - \frac{1}{k^2}.$$

Thus, the first fundamental solution can be rewritten now as

$$u_1(t) = \frac{r_2(r_1 - r_3) + r_3(r_1 - r_2)\,\mathrm{sn}^2(ht, k)}{(r_1 - r_3) + (r_1 - r_2)\,\mathrm{sn}^2(ht, k)},$$

where

$$h = \frac{1}{2}\sqrt{(P-1)(r_4 - r_2)(r_1 - r_3)} \qquad k^2 = \frac{(r_1 - r_2)(r_4 - r_3)}{(r_4 - r_2)(r_1 - r_3)}. \tag{19}$$

Similarly, the second fundamental solution is transformed into

$$u_2(t) = \frac{r_1(r_4 - r_2) + r_4(r_2 - r_1)\,\mathrm{sn}^2(ht, k)}{(r_4 - r_2) + (r_2 - r_1)\,\mathrm{sn}^2(ht, k)},$$

with $h$ and $k^2$ given by (19). In fact, $u_2(t)$ is nothing but $u_1(t)$ displaced half a period. Figure 5 shows the two solutions for values of the parameters in the range considered and they correspond to the orbits surrounding the stable points $E_{\mathrm{Mer}}$.

### 5.2 $H \in [H_3, H_1]$

Now, $k^2 < 0$, while $h$ remains real. The imaginary modulus transformation given by

$$\mathrm{sn}^2(z, ik) = \frac{1}{1+k^2}\frac{\mathrm{sn}^2\left(\sqrt{1+k^2}z, \lambda\right)}{\mathrm{dn}^2\left(\sqrt{1+k^2}z, \lambda\right)}, \qquad \lambda^2 = \frac{k^2}{1+k^2},$$

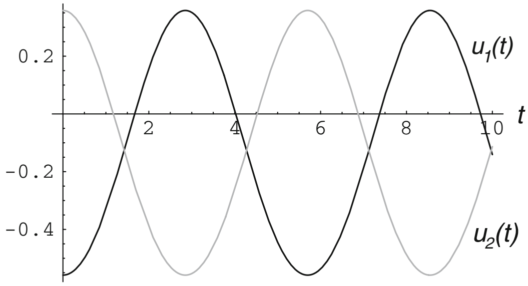


**Fig. 5** The two fundamental solutions for $H \in [H_{\text{Mer}}, H_3]$ in region 3

maps a modulus $k^2 \in (-\infty, 0)$ into the interval $(0, 1)$. While the first fundamental solution does not produce a valid orbit, the second one is transformed into

$$u_2(t) = \frac{r_1(r_4 - r_3) + r_4(r_3 - r_1)\,\text{sn}^2(ht, k)}{(r_4 - r_3) + (r_3 - r_1)\,\text{sn}^2(ht, k)},$$

where $h$ and $k$ are given by

$$h = \frac{1}{2}\sqrt{(P-1)(r_4 - r_3)(r_1 - r_2)}, \qquad k^2 = \frac{(r_1 - r_3)(r_4 - r_2)}{(r_4 - r_3)(r_1 - r_2)}.$$

This solution corresponds to the orbits around the homoclinic loops attached to the unstable points $E_1$ and $E_3$.

## 5.3 $H \in [H_1, H_{\text{Ec}}]$

In this case, the modulus $k^2$ becomes greater than one, whereas $h$ is now purely imaginary. However, the sequence of inverse modulus and imaginary argument transformations does not produce a valid solution. In fact, the two fundamental solutions give rise to real values of $u$, but outside the interval $[-1, 1]$, as it can be seen in Fig. 6.

This situation seems to require a new fundamental solution. Nevertheless, it can be obtained from the first one by means of the *translation property* for autonomous dynamical systems. Let us consider the *new* fundamental solution

$$\bar{u}_1(t) = u_1(t + K),$$

with $K$ the complete integral of first kind associated to the modulus $k^2$, that is to say

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}.$$

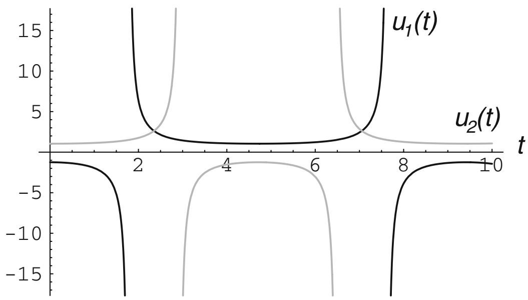


**Fig. 6** The two fundamental solutions for $H \in [H_1, H_{\text{Ec}}]$ in region 3

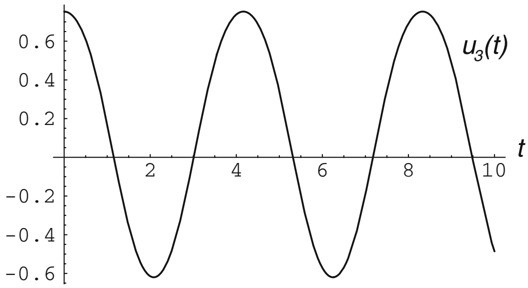


**Fig. 7** The first fundamental solution shifted half a period produces a valid solution for $H \in [H_1, H_{Ec}]$ in region 3

Taking into account that $\mathrm{sn}(z + K) = \mathrm{cd}(z)$, we obtain for $\bar{u}_1$ the following expression

$$\bar{u}_1(t) = \frac{A_1 + A_2 - (A_2 + k^2 A_1)\,\mathrm{sn}^2(ht, k)}{A_3 + A_4 - (A_4 + k^2 A_3)\,\mathrm{sn}^2(ht, k)},$$

but with these new coefficients, the two transformations mentioned above produce the desired solution, which is depicted in Fig. 7.

It is worth noting that $\bar{u}_1(t)$ is the result of an alternative bilinear transformation that converts the differential equation (11) into the Jacobi standard form. While (13) maps the roots $r_2, r_4, r_1$ and $r_3$ into 0, 1, $+\infty$ and $1/k^2$, respectively, the bilinear transformation associated to $\bar{u}_1(t)$ maps the roots $r_2, r_4, r_1$ and $r_3$ into 1, 0, $1/k^2$, $+\infty$. In this way, $u_1(t)$ runs from $r_2$ to $r_4$, whereas $\bar{u}_1(t)$ runs from $r_4$ to $r_2$. This is the reason why $\bar{u}_1(t)$ can be extended in this case, but not $u_1(t)$. Indeed, when $H \in [H_1, H_{Ec}]$, variable $u$ belongs to the interval $[r_3, r_4]$. On the other hand, from Proposition 4, $r_2 < -1 < r_3 < r_4 < 1 < r_1$. Thus, if we run from $r_2$ to $r_4$ we are out of the limits for $u$. On the contrary, if we run in the opposite direction, we reach the interval $[r_3, r_4]$, where $u$ belongs.

## 6 Conclusions

We find exact analytical solutions for the problem of the attitude dynamics of a gyrostat in torque free motion. As expected, the trajectories described by the angular momentum on the $S^2$ sphere are given in terms of elliptic functions. We prove that those solutions do not depend on the character of the roots of the quartic polynomial which determine the elliptic function, but the solution at one region can be extended for all values of the parameters and hence we are able to find the solution at each region from the one already computed.

**Acknowledgements** We are much indebted to the anonymous referees for their comments and suggestions which improved the quality of the manuscript. This paper has been supported by the Spanish Ministry of Science and Technology (Projects # ESP2005-07107 and # MTM 2005-08595).

## References

Abad, A., Elipe, A., Vallejo, M.: Automated Fourier series expansions for elliptic functions. Mech. Res. Commun. **21**, 361–366 (1994)

Abad, A., Elipe, A., Vallejo, M.: Normalization of perturbed non-linear Hamiltonian oscillators. Int. J. Nonlinear Mech. **32**, 443–453 (1997)

Abad, A., Elipe, A., Palacián, J., San-Juan, J.F.: ATESAT: A symbolic processor for artificial satellite theory. Math. Comput. Simul. **45**, 497–510 (1998)

Bowman, F.: Introduction to Elliptic Functions with Applications. Dover Pub., New York (1961)

Byrd, P.F., Friedman, M.D.: Handbook of Elliptic Integrals for Engineers an Physicists. Springer Verlag, Berlin (1954)
Coffey, S.L., Deprit, A., Deprit, E., Healy, L.: Painting the phase space portrait of an integrable dynamical system. Science **247**, 833–836 (1990)
Cochran, J.E., Shu, P.H., Rew, S.D.: Attitude motion of asymmetric dual-spin spacecraft. J. Guid. Control Dyn. **5**, 37–42 (1982)
Deprit, A., Elipe, A.: Complete reduction of the Euler–Poinsot problem. J. Astronaut. Sci. **41**, 603–628 (1993)
Elipe, A., Lanchares, V.: Biparametric quadratic Hamiltonians on the unit sphere: complete classification. Mech. Res. Commun. **21**, 209–214 (1994)
Elipe, A., Lanchares, V.: Phase flow of an axially symmetrical gyrostat with one constant rotor. J. Math. Phys. **38**, 3533–3544 (1997a)
Elipe, A., Lanchares, V.: Two equivalent problems: Gyrostats in free motion and parametric quadratic Hamiltonians. Mech. Res. Commun. **24**, 583–590 (1997b)
Elipe, A., Vallejo, M.: On the attitude dynamics of perturbed triaxial rigid bodies. Celest. Mech. Dyn. Astron. **81**, 3–12 (2001)
Frauendiener, J.: Quadratic Hamiltonians on the unit sphere. Mech. Res. Commun. **22**, 313–317 (1995)
Hall, C.D.: Equivalence of two classes of dual-spin spacecraft spinup dynamics. J. Guid. Control Dyn. **15**, 1032–1034 (1992)
Hall, C.D.: Spinup dynamics of biaxialgyrostats. J. Astronaut. Sci. **43**, 263–275 (1995a)
Hall, C.D.: Spinup dynamics of gyrostats. J. Guid. Control Dyn. **18**, 1177–1183 (1995b)
Hall, C.D., Rand, R.H.: Spinup dynamics of axial dual-spin spacecraft. J. Guid. Control Dyn. **17**, 30–37 (1994)
Henrard, J., Wauthier, P.: A geometric approach to the ideal resonance problem. Celest. Mech. **44**, 227–238 (1989)
Holmes, P.J., Marsden, J.E.: Horseshoes and Arnold diffusion for Hamiltonian systems on Lie groups. Indiana Univ. Math. J. **32**, 273–309 (1983)
Howland, R.A.: A note on the application of a generalized canonical approach to non-linear Hamiltonian systems. Celest. Mech. **45**, 407–412 (1989)
Hughes, P.C.: Spacecraft Attitude Dynamics. Wiley, New York (1986)
Iñarrea, M., Lanchares, V.: Chaos in the reorientation process of a dual-spin spacecraft with time dependent moments of inertia. Int. J. Bifurcat Chaos **10**, 997–1018 (2000)
Koiller, J.: A mechanical system with a "wild" horseshoe. J. Math. Phys. **25**, 1599–1604 (1983)
Lanchares, V.: Sistemas dinámicos bajo la acción del grupo SO(3): El caso de un Hamiltoniano cuadrático. Ph.D. dissertation, University of Zaragoza (in Spanish) (1993)
Lanchares, V., Elipe, A.: Bifurcations in biparametric quadratic potentials. Chaos **5**, 367–373 (1995a)
Lanchares, V., Elipe, A.: Bifurcations in biparametric quadratic potentials II. Chaos **5**, 531–535 (1995b)
Lanchares, V., et al.: Surfaces of bifurcation in a triparametric quadratic Hamiltonian. Phys. Rev. E **52**, 5540–5548 (1995)
Lanchares, V., Iñarrea, M., Salas, J.P.: Spin rotor stabilization of a dual-spin spacecraft with time dependent moments of inertia. Int. J. Bifurcat. Chaos **8**, 609–617 (1998)
Leimanis, E.: The General Problem of the Motion of Coupled Rigid Bodies About a Fixed Point. Springer Verlag, Berlin (1965)
Miller, B.R., Coppola, V.T.: Synchronization of perturbed non-linear Hamiltonians. Celest. Mech. Dyn. Astron. **55**, 331–350 (1993)
Poinsot, L.: Théorie nouvelle de la rotation des corps. J. Math. Pures Appl. **16**, 289–336 (1851)
Vera, J.A., Vigueras, A.: Hamiltonian dynamics of a gyrostat in the $n$-body problem: relative equilibria. Celest. Mech. Dyn. Astron. **94**, 289–315 (2006)
Volterra, V.: Sur la théorie des variations des latitudes. Acta Math. **22**, 201–358 (1899)