Multiperfect numbers on lines of the Pascal triangle

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Abstract

A number n is said to be multiperfect (or multiply perfect) if n divides its sum of divisors $\sigma(n)$. In this paper, we study the multiperfect numbers on straight lines through the Pascal triangle. Except for the lines parallel to the edges, we show that all other lines through the Pascal triangle contain at most finitely many multiperfect numbers. We also study the distribution of the numbers $\sigma(n)/n$ whenever the positive integer n ranges through the binomial coefficients on a fixed line through the Pascal triangle.

Keywords: Multiperfect; multiply perfect; perfect; sum of divisors; Pascal triangle; binomial coefficients; Catalan numbers.

MSC: 11A25; 11B65.

1 Introduction and main results

For a positive integer n we put $\sigma(n)$ for the sum of its divisors. Given an integer k, the number n is said to be multiperfect, multiply perfect, or k-fold perfect if $\sigma(n) = kn$. Of course, ordinary perfect numbers are 2-fold perfect. The single 1-fold perfect is the trivial case $n = 1$. The 3-fold perfect numbers are also called triperfect, and only six of them are known: they are 120, 672, 523776, 459818240, 1476304896, 51001180160. All of them were already known by the seventeenth century. Several multiperfect numbers are also known for every $k \leq 11$. Their number varies from thousands for $k = 8, 9, 10$, to only one for $k = 11$ which has more than a thousand decimal digits and was discovered in 2001. Descartes discovered the first 4-fold number, and Fermat the first 5-fold number, respectively. Dickson's *History of the Theory of Numbers* [1, p. 33–38] records a long interest in such numbers. See also [5, Section B2], or the web page [23] for more details and references.

Except for the well-known Euclid-Euler rule for $k = 2$, no formula to generate multiperfect numbers is known. Lehmer [6] proved that if n is odd, then n is perfect just if $2n$ is 3-fold perfect. Moreover, no odd multiperfect number is known. There are several conjectures on the size of k in what relates to the size of n. For example, from the maximal order

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of the sum of divisors function, it is known that there exists a positive constant c such that the inequality $\sigma(n)/n > c \log\log n$ holds for infinitely many positive integers n, where here and from now on we use log for the natural logarithm. Contrary to this inequality, Erdős conjectured that if there were infinitely many multiperfect numbers, then $k = o(\log\log n)$ as $n \to \infty$ through multiperfect numbers. It has even been suggested there may be only finitely many k-fold perfect numbers altogether with $k \geq 3$, and it is further believed that all multiperfect numbers with $k = 3$, 4, 5, 6, and 7 are known.

There are several results in the literature addressing perfect and multiperfect numbers of various shapes. For example, Pomerance [18] proposed as a problem to find all positive integers n such that $n!$ is multiperfect. In the solution [2] to the above problem, it is shown that this happens only for $n = 1$, 3, 5. In [7], it is shown that there is no Fibonacci number which is perfect and in [9] it was shown that there are at most finitely many Fibonacci numbers which are multiperfect. In [8], it is shown that no Fermat number; i.e., number of the form $2^{2^n}+1$ for some non-negative integer n, is perfect, and the method of proof shows easily that such numbers are not multiperfect either.

There are various extensions of perfect and multiperfect numbers in the literature. For example, a number n is called superperfect if $\sigma(\sigma(n)) = 2n$. More generally, a number n is called (m,k)-perfect if $\sigma^{(m)}(n) = kn$, where $\sigma^{(m)}$ denotes the mth fold iterate of the sum of divisors function.

It has been shown that there are only finitely many multiperfect numbers of the form $\binom{2n}{n}$ (see [13,14]). In this paper, we take a closer look at this problem and we study the multiperfect as well as $(2,k)$-perfect numbers (for any positive integer k) on straight lines on the Pascal triangle. Our theorem here is the following:

Theorem 1. *Let L be a line passing through the Pascal triangle and not parallel to any of its sides. Then the following hold:*

(i) *There are only finitely many binomial coefficients x belonging to L such that $x \mid \sigma(x)$.*
(ii) *There are only finitely many binomial coefficients x belonging to L such that $x \mid \sigma(\sigma(x))$.*

Our theorem above does not say anything about the lines parallel to the edges; they are $\{\binom{n}{m} : n \geq m\}$ with a fixed value of m or their symmetries with respect to the middle axis $\{\binom{n}{n-m} : n \geq m\}$. When $m = 1$, we have $\binom{n}{1} = n$, so saying something about these numbers amounts to saying something about all the numbers which are multiperfect. When $m = 2$, we have $\binom{n}{2} = \frac{n(n-1)}{2}$. All even perfect numbers are of this form with $n = 2^r$, where $2^r - 1$ is a prime number. However, for $m \geq 3$, and limiting ourselves to perfect numbers only, we have the following result.

Theorem 2. *There is no perfect number of the form $\binom{n}{m}$ with $n \geq 2m \geq 6$.*

The proof of Theorem 1 is effective and relies on the Prime Number Theorem and the Primitive Divisor Theorem for the Mersenne numbers, which have the form $M_n = 2^n - 1$ for all $n \geq 1$. That is, given the line L, one can write down an explicit upper bound for the size of the largest possible multiperfect number on L. Below we give a result identifying all the multiperfect numbers as well as the $(2,k)$-perfect numbers (for any positive integer k) on the central line of the Pascal triangle. For a positive integer n let $B_n = \binom{2n}{n}$ be the nth middle binomial coefficient and $C_n = \frac{1}{n+1}B_n$ be the nth Catalan number.

Theorem 3.

(i) *The only n such that B_n is multiperfect is $n = 2$. The only n such that C_n is multiperfect is $n = 1$.*
(ii) *The only n such that $B_n \mid \sigma(\sigma(B_n))$ is $n = 1$. Furthermore, the only n such that $C_n \mid \sigma(\sigma(C_n))$ are $n = 1$, 2 and 5.*

The occurrence of members of specific sequences on lines in the Pascal triangle is a question that has been studied before. For example, in [15] it is shown that if $(u_n)_{n\geq 0}$ is any fixed binary recurrent sequence of integers and L is any line of the Pascal triangle, except for an edge or a line of the form $\{\binom{n}{1} : n \geq 1\}$ or its symmetric $\{\binom{n}{n-1} : n \geq 1\}$, there are at most finitely many members of the sequence $(u_n)_{n\geq 0}$ belonging to L. The problem [10] asks to show that the largest Fibonacci number which is also a Catalan number is 5.

Divisor sums of binomial coefficients have been studied also in the recent paper [12] in the following context. It is known that the average value of $\sigma(n)/n$ when n ranges in the interval $[1, x]$ approaches $\pi^2/6$ as x tends to infinity. When $(u_n)_{n\geq 0}$ is a nondegenerate linearly recurrent sequence of integers, then the average value of $\sigma(|u_n|)/|u_n|$ when n ranges in the interval $[1, x]$ through values such that $u_n \neq 0$ approaches a finite limit when $x \to \infty$. This is a result of Shparlinski from [22] (see also [17]). For binomial coefficients, the situation is different. Indeed, it is shown in [12] that there exists a positive constant c such that the inequality

$$\frac{1}{n+1} \sum_{0\leq m\leq n} \frac{\sigma\left(\binom{n}{m}\right)}{\binom{n}{m}} > c \log\log\log n$$

holds for all sufficiently large n and further that

$$\frac{1}{2^n} \sum_{0\leq m\leq n} \sigma\left(\binom{n}{m}\right) \to \infty \quad \text{as } n \to \infty.$$

The above estimates should be consistent with our intuition that most binomial coefficients are divisible by many small primes, therefore their divisor sums should be quite a bit larger than the numbers themselves.

In the present paper, we take again a line L passing through the Pascal triangle and study the distribution of the numbers $\sigma(x)/x$, when x ranges through members of L. For the purpose of the next result, we shall assume that L contains infinitely many members of the Pascal triangle. The next result shows that numbers $\sigma(x)/x$ tend to behave differently when x ranges through a line L nonparallel to the sides than when x ranges through one line that is parallel to the sides. Informally speaking, $\sigma(x)/x$ tends to infinity when L is not parallel to the sides, while these numbers form a dense set in $[1, \infty)$ when the line is parallel to the sides.

Theorem 4. *Let L be a fixed line through the Pascal triangle containing infinitely many binomial coefficients and which is not one of the two edges.*

(i) *If L is not parallel to any of the edges, then there exists a positive constant c_L depending on the line such that the inequality $\sigma(x)/x > (\log\log x)^{c_L}$ holds for all binomial coefficients x on L which are sufficiently large.*
(ii) *If $L = \{\binom{n}{m} : n \geq m\}$, where $m \geq 1$ is a fixed positive integer, then $\{\sigma(x)/x : x \in L\}$ is dense in $[1, \infty)$.*

Density questions for the set $\{\sigma(u_n)/u_n : n \geq 1\}$, where $(u_n)_{n\geq 1}$ is some interesting sequence of positive integers have been studied before. For example, it was shown in [11] that $\{\sigma(M_n)/M_n : n \geq 1\}$ is dense in $[1, \infty)$, where again $M_n = 2^n - 1$ is the nth Mersenne number. The proof of this result applies also when the sequence of Mersenne numbers is replaced by the sequence of Fibonacci numbers. In [16], it is shown that $\{\sigma(\phi(n))/n : n \geq 1\}$ is dense in $[\frac{1}{2} + \frac{1}{2^{34}-1}, \infty)$, where $\phi(n)$ is the Euler totient function. It is not very hard to prove that $\{\sigma(\sigma(n))/n : n \geq 1\}$ is dense in $[1, \infty)$, but it is not known whether $\{\sigma(\sigma(\sigma(n)))/n : n \geq 1\}$ is dense in $[1, \infty)$.

As usual, we will use the Landau symbols O and o as well as the Vinogradov symbols $\gg$, $\ll$ and $\asymp$ with their regular meaning. Recall that the statements $A = O(B)$, $A \ll B$ and $B \gg A$ are all equivalent to the fact that $|A| < cB$ holds for some positive constant c, $A \asymp B$ means that both $A \ll B$ and $B \ll A$ hold, and $A = o(B)$ means that A/B approaches zero. We use $\pi(t)$ for the number of primes $p \leq t$.

2 Proof of Theorem 1

Let us start by presenting a parametrization of the lines in the Pascal triangle.

Lemma 1. Let $L \subseteq \{(n,m) : n \geq m\}$ be a line of the Pascal triangle not parallel to any of its sides and containing infinitely many elements. Then, L can be parametrized as

$$\begin{cases} n = at + b, \\ m = ct + d, \end{cases} \qquad t \in \mathbb{Z}, \quad t \geq t_0, \tag{1}$$

for some integers a, b, c, d with a and c coprime and $0 < c < a$.

Proof. If the line L is parallel to one of the sides, then either m is a constant or $n - m$ is a constant. Otherwise, there exist rational numbers A, B such that $n = Am + B$. Writing $A = a/c$ with coprime integers a and $c > 0$, and writing $m = ct+r$, where $r \in \{0, 1, \dots, c-1\}$ and t is an integer, we get that

$$n = at + \left(\frac{ar}{c} + \frac{e}{f}\right),$$

where we put $B = e/f$ with coprime integers e and $f > 0$. Since r can take only finitely many values, we may assume that r is fixed. Since $ar/c+e/f$ is an integer, we get that $f \mid c$, therefore $c = fc_1$ holds with some positive integer c_1, and further $r = c_1 r_1$ holds with some nonzero integer r_1. Furthermore, $(ar_1 + e)/f$ is an integer. Thus, putting $b := (ar_1 + e)/f$ and $d := r$, we get (1). Since $c > 0$ and m is positive, it follows that $t > 0$ if m is sufficiently large. The first relation above together with the fact that $n \geq m$ can be arbitrarily large implies first that $a \geq 1$. Next, again since $n \geq m$, it follows by looking at large values of t that $0 < c \leq a$. Finally, we can assume that $0 < c < a$, otherwise if $a = c$ we get that $n - m$ remains constant.

Let us now prove Theorem 1. Let $L \subseteq \{(n,m) : n \geq m\}$ be the line of the Pascal triangle; by Lemma 1 above, it can be parametrized as shown in (1). Furthermore, by the symmetry $\binom{n}{m} = \binom{n}{n-m}$, and by replacing m with $n-m$ if needed, it follows that we may assume that $0 < c \leq a/2$.

Now let us look at the binomial coefficient:

$$x := \binom{n}{m} = \binom{at+b}{ct+d} = \frac{(at+b)(at+b-1)\cdots((a-c)t+b-d+1)}{(ct+d)!}.$$

Since $a - c \geq c$, it follows that for large t, all primes p in the interval $\mathcal{I} = [(a-c)t+b-d+1, at+b]$ divide x with at most $O(1)$ exceptions. These exceptional values can occur only when $a - c = c$ (i.e., $a = 2c$, therefore $c = 1$ and $a = 2$, since we have already assumed that a and c are coprime), and $b - d + 1 < d$, in which case we can take the constant implied by the above $O(1)$ to be $\max\{|d|+1, |b|+1\}$. Let $\mathcal{A}$ be the above set of primes. For large t, we have that $2 \notin \mathcal{A}$.

Let $p \in \mathcal{A}$. Clearly, $2p \geq 2(a-c)t + 2(b-d+1)$. Since $a \geq 2c$, it follows, by an argument similar to the previous one, that the inequality $2p > at + b$ holds for large values of t unless $a = 2,\ c = 1$, in which case the inequality $2p > at + b$ also holds for all primes $p \in \mathcal{A}$ with $O(1)$ exceptions.

The above argument shows that there exists a positive constant c_1 (depending on $a,\ b,\ c,\ d$), such that if t is sufficiently large, then all primes $p \in \mathcal{A}$ except for at most c_1 of them divide exactly x (namely, such primes are divisors of the above number but the squares of these primes are not). Thus, writing $\mathcal{B}$ for the subset of $\mathcal{A}$ such that $p \parallel x$, and putting $u = \prod_{p\in\mathcal{B}} p$, it follows that $x = uv$ holds with some positive integer v coprime to u. Hence,

$$\sigma(x) = \sigma(u)\sigma(v) = \sigma(v)\prod_{p\in\mathcal{B}}(p+1).$$

The factor $p+1$ is even for all $p \in \mathcal{B}$, and there are

$$\#\mathcal{B} \geq \pi(at+b) - \pi((a-c)t+b-d) - c_1$$

such factors $p+1$. By the Prime Number Theorem, we have that the inequality $\#\mathcal{B} \geq ct/(2\log t)$ holds for large values of t. Hence, writing α for the exponent of the prime 2 in $\sigma(x)$, we get that the inequality $\alpha \geq ct/(2\log t)$ holds for large values of t.

By a well-known theorem of Kummer (see, for instance, Exercise 5.36, p. 245, of the chapter "Binomial Coefficients" in [3]), the exponent β of 2 in $\binom{n}{m}$ equals the number of carries when adding m and $n-m$ in base 2. In particular, $\beta < c_2 \log t$ for large values of t, where c_2 is any fixed constant in $(0, 1/\log 2)$. We are now ready to prove the theorem.

For (i), let us suppose that x is multiperfect. Then the relation

$$\sigma(x) = kx$$

holds for some positive integer k. Letting γ be the exponent of 2 in the factorization of k, we get, by the previous arguments, that

$$\gamma \geq \alpha - \beta \geq \frac{ct}{2\log t} - c_2\log t > \frac{ct}{3\log t} \tag{2}$$

for large values of t.

On the other hand, since the inequality $x \leq 2^n < 2^{2at}$ holds for large values of t, we have, by the maximal order of the sum of divisors function, that

$$k = \frac{\sigma(x)}{x} \ll \log\log x \leq \log\log(2^{2at}) \leq 2\log t \tag{3}$$

if t is sufficiently large. Comparing inequalities (2) and (3), we get that

$$2^{ct/(3\log t)} \leq 2^\gamma \leq k \ll \log t,$$

which shows that t is, in fact, bounded. This completes the proof of part (i) of this theorem.

For (ii), let us suppose that x is $(2,k)$-perfect for some positive integer k. Then

$$\sigma(\sigma(x)) = kx.$$

Since $2^\alpha \parallel \sigma(x)$, we get that $2^{\alpha+1} - 1 \mid \sigma(\sigma(x))$. Recall that if $u \geq 1$ is any positive integer, then a *primitive divisor* of the number $2^u - 1$ is a prime factor of $2^u - 1$ which does not divide $2^v - 1$ for any positive integer $1 \leq v < u$. It is known that a primitive divisor p of $2^u - 1$, whenever it exists, is congruent to $1 \pmod u$. It is also known that if we write

$$2^u - 1 = AB,$$

where A and B are positive integers such that every prime factor of A is primitive for $2^u - 1$ and no prime factor of B is primitive for $2^u - 1$, then $A > 2^{\phi(u)/2}$ holds for all large values of u. For us, writing

$$2^{\alpha+1} - 1 = AB$$

as described previously, then $A > 2^{\phi(\alpha+1)/2}$ provided that α is large. Since $\alpha > ct/(2\log t)$, and the inequality $\phi(u) \gg u/\log\log u$ holds for all large positive integers u, it follows that there exists a positive constant c_3 such that

$$A > \exp\left(\frac{c_3 t}{\log t \log\log t}\right) \tag{4}$$

provided that t is sufficiently large. On the other hand, all prime factors dividing A divide also kx. Note that

$$k = \frac{\sigma(\sigma(x))}{x} = \frac{\sigma(\sigma(x))}{\sigma(x)} \cdot \frac{\sigma(x)}{x} \ll (\log\log\sigma(x))\log\log x \ll (\log t)^2, \tag{5}$$

where the last inequalities above follow as in part (i) because $x < 2^{2at}$. If $p \mid A$, then $p \equiv 1 \pmod{\alpha+1}$. In particular, $p > \alpha+1 > ct/(2\log t)$, therefore, by inequality (5), we have that $p > k$ for large values of t. Thus, $p \mid x$. The exponent of p in x is, by Kummer's theorem, at most $\log t$ for large values of t. Furthermore, $p \le at+b$ and $p \equiv 1 \pmod{\alpha+1}$, therefore the number of possibilities for t is at most (even discarding the condition that p is a prime) $\lfloor (at+b)/(\alpha+1) \rfloor + 1 < c_4 \log t$ for large t, where we can take $c_4 = 3a/c$. Hence, we have just showed that

$$A \le \prod_{\substack{p \mid x \\ p \equiv 1 \pmod{\alpha+1}}} p^{\log t} < (at+b)^{c_4(\log t)^2} < \exp(2c_4(\log t)^3) \tag{6}$$

provided that y is sufficiently large. Comparing inequalities (4) and (6), we get that

$$\frac{c_3 t}{\log t \log\log t} < 2c_4(\log t)^3,$$

which obviously implies that t is bounded. This finishes the proof of part (ii) of this theorem.

3 Proof of Theorem 2

Assume that $x := \binom{n}{m}$ is perfect, where $n \ge 2m \ge 6$. Assume that $n \ge 30$. Corollary 3 in [20] shows that

$$n(n-1)\cdots(n-m+1) \tag{7}$$

is divisible by at least two primes exceeding m. Since x is perfect, it then follows by a result of Euler that $x = pu^2$ for some odd prime p and positive integer u (here, $p = 2^r - 1$ and $u = 2^{(r-1)/2}$ when the perfect number is even). Now u is divisible by a prime q exceeding m. Hence, q^2 divides the number shown at (7), and since q exceeds m, we get that $q^2 \mid n-i$ for some $i = 0, \dots, m-1$. In particular, $n \ge (m+1)^2 = m^2+2m+1$, therefore $n-m+1 > m^2$. Now let j be an integer in $\{0, 1, \dots, m-1\}$ such that $n-m+1+j$ is a multiple of p (note that j is unique if $p > m$). Since the number shown at (7) is of the form $k!\,pu^2$, it follows easily that if we remove from this product the number $n-m+1+j$ we get an equation of the form

$$(n-m+1)\cdots(n-m+1+(j-1))(n-m+1+(j+1))\cdots n = bw^2,$$

where b is a positive integer whose largest prime factor is at most m. This is impossible for $m \ge 3$ by Theorem 2 in [20]. Thus, $n \le 30$, and now a short calculation reveals no perfect number of the form $\binom{n}{m}$ with $3 \le m \le n/2$ and $n \le 30$.

4 Proof of Theorem 3

We follow the proof of Theorem 1 but make it effective. By Corollary 2 (p. 69) in [19], we have

$$\alpha \geq \pi(2n) - \pi(n) \geq \frac{3n}{5 \log n}$$

for $n \geq 21$. Now, let us observe that $\sigma(\binom{2n}{n})$ is a multiple of $\prod_{n<p\leq 2n}(p+1)$, therefore also of $2^{\pi(2n)-\pi(n)}$. Letting β be the power of 2 in the prime factorization of $\binom{2n}{n}$, Kummer's theorem gives us $2^\beta \leq 2n$.

We now prove (i). Assume that $\binom{2n}{n}$ is multiperfect. Let k be an integer such that $\sigma(\binom{2n}{n}) = k\binom{2n}{n}$. Then, $2^\gamma \mid k$, where

$$2^\gamma \geq 2^{\pi(2n)-\pi(n)-\beta} \geq \frac{2^{\frac{3n}{5\log n}}}{2n}. \tag{8}$$

On the other hand,

$$k = \frac{\sigma(\binom{2n}{n})}{\binom{2n}{n}} < \sum_{d \leq 2^{2n}} \frac{1}{d} \leq 1 + \int_1^{2^{2n}} \frac{dt}{t} \leq \log(2^{2n}) + 1 = 2n \log 2 + 1.$$

Thus, when $n > 1$,

$$\frac{2^{\frac{3n}{5\log n}}}{2n} \leq 2n \log 2 + 1 < 2n,$$

and then

$$3n \log 2 < 10(\log n) \log(2n),$$

whose largest solution is $n = 130$. In this way, we have proved that $\binom{2n}{n}$ is not multiperfect when $n > 130$. For $n \leq 130$, a quick computer search confirmed that $\binom{2n}{n}$ is multiperfect only when $n = 2$.

Assume now that the Catalan number C_n is multiply perfect. Then the exponent of 2 in $\sigma(C_n)$ is at least $\pi(2n) - \pi(n) - 1$ (by 1 smaller than before since $n + 1$ might be a prime), while the power of 2 in C_n does not exceed the power of 2 in $\binom{2n}{n}$. Hence, the lower bound on γ, the power of 2 in k defined analogously as $\sigma(C_n) = kC_n$, satisfies the inequality

$$2^\gamma \geq \frac{2^{\frac{3n}{5\log n}}}{4n} > \frac{2^{\frac{3n}{5\log n}}}{(2n)^2}. \tag{9}$$

Thus, following the same arguments as before, we arrive at

$$3n \log 2 < 15(\log n) \log(2n),$$

whose largest solution is $n = 247$. A quick computer check confirmed that the only $n \leq 247$ for which C_n is multiply perfect is $n = 1$.

We now look at (ii). Let $x = B_n$ or C_n and let α be the power of 2 in either $\sigma(B_n)$ or $\sigma(C_n)$. Then the arguments from part (i) show that

$$\alpha \geq \pi(2n) - \pi(n) - 1 \geq \frac{3n}{5 \log n} - 1 \qquad \text{for } n \geq 1.$$

Now $2^{\alpha+1} - 1$ divides $\sigma(\sigma(x)) = kx$. Clearly, $\sigma(x) < x^2 < 2^{4n}$, so the arguments from (i) show that

$$k = \frac{\sigma(\sigma(x))}{\sigma(x)} \cdot \frac{\sigma(x)}{x} < \left(1 + \int_1^{2^{2n}} \frac{dt}{t}\right)\left(1 + \int_1^{2^{4n}} \frac{dt}{t}\right)$$
$$< (1 + 2n\log 2)(1 + 2n\log 4) < (2n)^2,$$

where the last inequality holds for $n > 30$, which we are assuming from now on. Now write

$$2^{\alpha+1} - 1 = AB,$$

where all prime factors of A are congruent to 1 modulo $\alpha + 1$. It is well-known, from the proof of the Primitive Divisor Theorem (see, for example, [24]), that

$$A > \frac{2^{\phi(\alpha+1) - 2^{\omega(\alpha+1)-1}}}{\alpha + 1},$$

where for a positive integer m we put $\omega(m)$ for the number of distinct prime factors of m. We now assume that $n > 135$. Then

$$\alpha + 1 > \frac{3n}{5\log n} > 12$$

(where the last inequality holds for $n > 90$), therefore, by a result of Schinzel, [21], we have that A has a prime factor at least as large as $2(\alpha + 1) + 1$. This prime factor either divides x (so it is $< 2n$), or it divides $k < 4n^2$, therefore $\alpha + 1 < 2n^2$. Thus,

$$A > \frac{2^{\phi(\alpha+1)} - 2^{\omega(\alpha+1)-1}}{2n^2}. \tag{10}$$

Now every prime factor dividing A, either divides k, or divides x. Let p be a prime factor dividing both A and x. Then

$$p^2 > (\alpha + 1)^2 > \left(\frac{3n}{5\log n}\right)^2 > 2n,$$

where the last inequality holds for all $n > 135$, which we are assuming. Thus, $p \parallel x$. Furthermore, the number of such primes is $\le 2n/(\alpha + 1)$ and every one of such primes is $< 2n$. This shows that

$$A \le k(2n)^{2n/(\alpha+1)} < (2n)^{2/(\alpha+1)+2}.$$

Thus, comparing the above upper bound on A with the lower bound (10), we get

$$\phi(\alpha + 1) - 2^{\omega(\alpha+1)-1} < \left(\frac{2n}{\alpha + 1} + 4\right)\frac{\log(2n)}{\log 2}. \tag{11}$$

This is the master inequality. Since $\alpha + 1 > 3n/(5\log n)$, we get that

$$\phi(\alpha + 1) - 2^{\omega(\alpha+1)-1} < \left(\frac{10\log n}{3} + 4\right)\frac{\log(2n)}{\log 2}. \tag{12}$$

Let s be the number of prime factors of $\alpha + 1$. Clearly, $2^s \le \alpha + 1$, so $s \le (\log(\alpha + 1))/\log 2$, therefore

$$\phi(\alpha + 1) \ge (\alpha + 1)\prod_{i=2}^{s+1}\left(1 - \frac{1}{i}\right) \ge \frac{\alpha + 1}{s + 1} \ge \frac{(\alpha + 1)\log 2}{\log(2(\alpha + 1))}.$$

Finally, writing $\tau(m)$ for the number of divisors of m, and observing that

$$2^{\omega(\alpha+1)-1} \le \frac{\tau(\alpha+1)}{2} \le \frac{\sqrt{3(\alpha+1)}}{2},$$

we get that

$$\phi(\alpha+1) - 2^{\omega(\alpha+1)-1} \ge \frac{(\alpha+1)\log 2}{\log(2(\alpha+1))} - \frac{\sqrt{3(\alpha+1)}}{2}.$$

The function

$$t \mapsto \frac{t\log 2}{\log(2t)} - \frac{\sqrt{3t}}{2}$$

is increasing for $t \ge 8$, and since for us $\alpha+1 \ge 3n/(5\log n) > 12$, we have that

$$\phi(\alpha+1) - 2^{\omega(\alpha+1)-1} \ge \frac{3n\log 2}{5(\log n)\log(6n/(5\log n))} - \frac{3}{2}\left(\frac{n\log 2}{5}\right)^{1/2}. \tag{13}$$

From inequalities (12) and (13), we get that

$$\frac{3n\log 2}{5(\log n)\log(6n/(5\log n))} - \frac{3}{2}\left(\frac{n\log 2}{5\log n}\right)^{1/2} < \left(\frac{10\log n}{3} + 4\right)\frac{\log(2n)}{\log 2},$$

giving $n \le 310000$. Thus, there is no solution with $n > 310000$.

Now we start lowering the upper bound on n. Assume still $n > 135$ but $n \le 310000$. Here, we have that $\alpha+1 \le \pi(2n)+1 < 51000$. The product of the first 7 primes exceeds 51000, therefore

$$\frac{\phi(\alpha+1)}{\alpha+1} \ge \prod_{2\le q\le 13}\left(1-\frac{1}{q}\right) = \frac{192}{1001}.$$

Moreover, for $\alpha+1 < 51000$, we have $\tau(\alpha+1) \le 108$, therefore the largest power of 2 which is $\le \tau(\alpha+1)/2$ is at most 32. Hence, we get the inequality

$$\phi(\alpha+1) - 2^{\omega(\alpha+1)-1} \ge \frac{192(\alpha+1)}{1001} - 32 \ge \frac{594n}{5005\log n}.$$

Comparing the last inequality above with inequality (12), we get

$$\frac{594n}{5005\log n} - 32 < \left(\frac{10\log n}{\log 3} + 4\right)\frac{\log(2n)}{\log 2},$$

giving $n \le 72000$. Thus, there is no solution with $n > 72000$. Assume now that $n > 20000$. We have checked with Mathematica that $\pi(2n) - \pi(n) > 0.95n/\log n$ for all $n \in [20000, 80000]$. Thus, instead of the inequality $\alpha+1 > 3n/(5\log n)$, we can use the inequality $\alpha+1 > 0.95n/\log n$, in which case we also have $2n/(\alpha+1) < 40(\log n)/19$. Furthermore, since $2n < 144000$ and $\alpha+1 < \pi(144000) < 14000$, but the product of the first 6 primes exceeds 14000, it follows that

$$\frac{\phi(\alpha+1)}{\alpha+1} \ge \prod_{2\le q\le 11}\left(1-\frac{1}{q}\right) = \frac{16}{77}.$$

Thus,

$$\phi(\alpha+1) - 2^{\omega(\alpha+1)-1} \ge \frac{15.2n}{77\log n} - 32,$$

and comparing this with the bound (11), we get

$$\frac{15.2n}{5\log n} - 32 < \left(\frac{40\log n}{19} + 4\right)\frac{\log(2n)}{\log 2},$$

giving $n < 20000$. Thus, there is no solution with $n \geq 20000$.

Next we checked that there is no solution with $n > 1100$ in the following way. For each $n \in [1100, 20000]$, we used Mathematica to find the exact value of α. In order to do so, for each n we factored the integer B_n and C_n (which are large integers all whose prime factors are smaller than $2n$), and then

$$\alpha = \sum_{\substack{p^{\alpha_p} \| B_n \\ \alpha_p \text{ odd}}} \operatorname{ord}_2\left(\frac{p^{\alpha_p+1}-1}{p-1}\right),$$

where for a positive integer m we write $\operatorname{ord}_2(m)$ for the exponent of 2 in the factorization of m. We then checked whether the master inequality (11) holds. No solution for either $x = B_n$ or $x = C_n$ was found with $n > 1100$. We then checked that the only $n \leq 1100$ for which $x \mid \sigma(\sigma(x))$ are the claimed ones.

5 Proof of Theorem 4

We start with part (i) and assume, as in the proof of Theorem 1, that we are looking at pairs (n, m) parameterized as in Lemma 1. Let t be large and put $z_1 = \exp((\log t)^{1/2})$ and $z_2 = t^{1/4}$. Lemma 5.1 with $J = 1$ and $\varepsilon = 1/10$ in [4] shows that for any two constants $0 < \sigma_0 < \sigma_1 < 1$ there exists some positive constant c_1 such that uniformly in $y \leq x$ we have that the set of prime numbers $p \leq y$ such that $\{x/p\} \in (\sigma_0, \sigma_1)$ has cardinality

$$(\sigma_1 - \sigma_0)\pi(y) + O\left((y^{1-c_1(\log y)^2/\log x} + y^{8/5}x^{-1/2})(\log x)^4\right).$$

From the way we have chosen z_1 and z_2, we deduct that for large t, there are

$$(\sigma_1 - \sigma_0)\pi(y)\left(1 + O\left(\frac{1}{\log y}\right)\right)$$

primes $p \leq y$ such that $\{t/p\} \in (\sigma_0, \sigma_1)$ and this holds for all $y \in [z_1, z_2]$. We now put $\sigma_0 = 1 - 1/(4c(a-c))$, $\sigma_1 = 1 - 1/(5c(a-c))$, and deduct that if we put $\mathcal{P}_y$ for the set of primes $p \in (y, 2y)$ with $\{t/p\}$ in the above interval, then the inequality

$$\#\mathcal{P}_y \geq \frac{y}{30c(a-c)\log y}$$

holds uniformly for all $y \in [z_1, z_2/2]$ once t is sufficiently large. Since all such primes are at most as large as y, we get that

$$\sum_{p \in \mathcal{P}_y} \frac{1}{p} \geq \frac{c_1}{\log y}, \tag{14}$$

where we put $c_1 = 1/(30c(a-c))$. Observe that if p is such a prime, then

$$\left\{\frac{(a-c)t + (b-d)}{p}\right\} \in \left(1 - \frac{1}{3c}, 1 - \frac{1}{6c}\right),$$

and

$$\left\{\frac{ct+d}{p}\right\} \in \left(1 - \frac{1}{3(a-c)}, 1 - \frac{1}{6(a-c)}\right),$$

provided that t is large. In particular, since

$$\left(1-\frac{1}{3c}\right)+\left(1-\frac{1}{3(a-c)}\right)>1$$

for all $c\ge 1$ and $a-c\ge 1$, we get that for large t, the binomial coefficient $x=\binom{n}{m}$ is a multiple of such a prime p since there is at least one carry when adding $(a-c)t+(b-d)$ with $ct+d$ in base p.

Now give y the values 2^{ℓ}, where ℓ is an integer. Since 2^{ℓ} must belong to the interval $[z_1, z_2/2]$, it follows that we can take $\ell\in[L_1,L_2]$, where

$$L_1=\left\lfloor\frac{(\log t)^{1/2}}{\log 2}\right\rfloor+1 \qquad \text{and} \qquad L_2=\left\lfloor\frac{\log t}{4\log 2}\right\rfloor-1.$$

Thus, x is divisible by all primes in $\bigcup_{L_1\le\ell\le L_2}\mathcal{P}_{2^{\ell}}$, and the sets making up the above union are disjoint. Hence, using (14), we get

$$\begin{aligned}\frac{\sigma(x)}{x} &\ge \prod_{L_1\le\ell\le L_2}\prod_{p\in\mathcal{P}_{2^\ell}}\left(1+\frac{1}{p}\right)=\exp\left(\sum_{\substack{L_1\le\ell\le L_2\\ p\in\mathcal{P}_{2^\ell}}}\frac{1}{p}+O\left(\sum_{p\ge 2}\frac{1}{p^2}\right)\right)\\ &\ge \exp\left(c_2\sum_{L_1\le\ell\le L_2}\frac{1}{\ell}+O(1)\right)\gg\exp\left(c_2\int_{L_1}^{L_2}\frac{ds}{s}\right)\gg\exp\left(c_2\log\frac{L_1}{L_2}\right)\gg(\log t)^{c_L},\end{aligned}$$

where we put $c_2=c_1/\log 2$ and $c_L=c_2/2$. Since $x\le 2^{2at}$ for large values of t, we get that

$$\frac{\sigma(x)}{x}\gg(\log\log x)^{c_L},$$

where c_L depends only on the line L, which is what we wanted to prove.

Finally, for (ii) let m be fixed, $\alpha\in[1,\infty)$, and $\varepsilon>0$ be arbitrary. Choose m disjoint finite sets of primes $\mathcal{P}_i$ for $i=1,\ldots,m$ all exceeding m such that

$$\prod_{p\in\mathcal{P}_i}\left(1+\frac{1}{p}\right)\in[\alpha^{1/m}(1-\varepsilon),\alpha^{1/m}(1+\varepsilon)].$$

Let also t be some positive real number whose natural logarithm exceeds all members of $\bigcup_{1\le i\le m}\mathcal{P}_i$ and let $\mathcal{Q}$ be the set of primes $p\in(m,\log t)$ not belonging to any of the sets $\mathcal{P}_i$ for $i=1,\ldots,m$. Put $M=(m!)^2$, $P_i=\prod_{p\in\mathcal{P}_i}p$, $P=\prod_{i=1}^m P_i$, $Q=\prod_{q\in\mathcal{Q}}q$ and consider the following system of congruences

$$n\equiv 0 \pmod{MQ} \qquad \text{and} \qquad n+i\equiv P_i \pmod{P_i^2} \quad \text{for } i=1,\ldots,m.$$

By the Chinese Remainder Theorem, the above system of congruences puts n into an arithmetic progression $n\equiv n_0 \pmod N$, where $N=MQP^2$. The size of the modulus is, by the Prime Number Theorem,

$$N\le M\left(\prod_{p\le\log t}p\right)^2=Me^{2(1+o(1))\log t}$$

as $t \to \infty$, therefore $N < t^3$ for large values of t. Hence, there is a number $n \in [t^3, 2t^3)$ in the above progression once t is sufficiently large. Observe that

$$x = \binom{n+m}{m} = \prod_{i=1}^{m} \left(1 + \frac{n}{i}\right).$$

Putting $m_i = 1 + n/i$, it is easy to see that $m_i \equiv 1 \pmod{m!}$, that m_i is divisible by all primes p in $\mathcal{P}_i$ but not by their squares, and m_i is not divisible by any member of $\mathcal{Q}$ for any $i = 1, \ldots, m$. Hence, $m_i = P_i M_i$, where M_i is a positive integer whose prime factors exceed $\log t$. Since $M_i < 2t^3 + 1$, we get that $\omega(M_i) = O(\log t / \log\log t)$ for all $i = 1, \ldots, m$. Thus,

$$x = \binom{n+m}{m} = \prod_{i=1}^{m} P_i \prod_{i=1}^{m} M_i,$$

where the two products above are coprime. Thus,

$$\frac{\sigma(x)}{x} = \frac{\sigma\left(\prod_{i=1}^m P_i\right)}{\prod_{i=1}^m P_i} \, \frac{\sigma\left(\prod_{i=1}^m M_i\right)}{\prod_{i=1}^m M_i}.$$

Note that, from the way we have chosen our sets $\mathcal{P}_i$, we have

$$\frac{\sigma\left(\prod_{i=1}^m P_i\right)}{\prod_{i=1}^m P_i} = \prod_{i=1}^{m} \prod_{p \in \mathcal{P}_i} \left(1 + \frac{1}{p}\right) \in [\alpha(1-\varepsilon)^m, \alpha(1+\varepsilon)^m],$$

while

$$1 \le \frac{\sigma\left(\prod_{i=1}^m M_i\right)}{\prod_{i=1}^m M_i} \le \left(1 + \frac{1}{\log t}\right)^{O(m \log t/\log\log t)} = 1 + O\left(\frac{m}{\log\log t}\right),$$

so this last number is in $[1, 1+\varepsilon)$ for sufficiently large values of t. Since α and ε were arbitrary and m was fixed, we get that α can be approximated arbitrarily well by rational numbers of the form $\sigma(x)/x$ with $x = \binom{n+m}{m} \in L$, which is what we wanted to prove.

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