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Symplectic coordinates on $S^2 \times S^2$ for perturbed Keplerian problems: Application to the dynamics of a generalised Størmer problem

Manuel Iñarrea [a], Víctor Lanchares [b], Jesús F. Palacián [c,*], Ana I. Pascual [b], J. Pablo Salas [a], Patricia Yanguas [c]

[a] Área de Física Aplicada, Universidad de La Rioja, 26006 Logroño, Spain
[b] Departamento de Matemáticas y Computación, Universidad de La Rioja, 26006 Logroño, Spain
[c] Departamento de Ingeniería Matemática e Informática, Universidad Pública de Navarra, 31006 Pamplona, Spain

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ABSTRACT

In order to analyse the dynamics of a given Hamiltonian system in the space defined as the Cartesian product of two spheres, we propose to combine Delaunay coordinates with Poincaré-like coordinates. The coordinates are of local character and have to be selected accordingly with the type of motions one has to take into consideration, so we distinguish the following types: (i) rectilinear motions; (ii) circular and equatorial motions; (iii) circular non-equatorial motions; (iv) non-circular equatorial motions; and (v) non-circular and non-equatorial motions. We apply the theory to study the dynamics of the reduced flow of a generalised Størmer problem that is modelled as a perturbation of the Kepler problem. After using averaging and reduction theories, the corresponding flow is analysed on the manifold $S^2 \times S^2$, calculating the occurring equilibria and their stability. Finally, the flow of the original problem is reconstructed, concluding the existence of some families of periodic solutions and KAM tori.

* Corresponding author.
E-mail addresses: manuel.inarrea@unirioja.es (M. Iñarrea), vlancha@unirioja.es (V. Lanchares), palacian@unavarra.es (J.F. Palacián), aipasc@unirioja.es (A.I. Pascual), josepablo.salas@unirioja.es (J.P. Salas), yanguas@unavarra.es (P. Yanguas).

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1. Introduction

By averaging the perturbation of a Hamiltonian over the fibres of the circle bundle, Reeb [35] and Moser [28] obtained a Hamiltonian vector field on the so-called base or reduced (phase) space, see also the papers [25,26]. They were able to give sufficient conditions for the existence of periodic solutions by looking at the system on the base alone.

In the setting of perturbed Keplerian problems, we start with a small parameter which is a measure of the perturbation of the Kepler Hamiltonian (which is an example of an integrable system where all solutions are periodic). Then one normalises (or averages) the perturbation. After a finite number of terms have been normalised, the higher-order perturbations are truncated, giving an approximation of the full system. This approximation is well defined on the lower-dimensional reduced space, which in the case of fully spatial perturbations is $S^2 \times S^2$, that is, the Cartesian product of two spheres [28].

This space is a four-dimensional symplectic manifold, compared to the five-dimensional manifold of the original system (when the value of the Hamiltonian is fixed). Being lower-dimensional, the system on the reduced space is easier understandable. However we remark that not all the features of the full system are accurately reflected by the reduced system; it typically does not display the breakdown of invariant tori or ergodic regions.

The plan of the paper is as follows. In Section 2 we detail how the reduction of a perturbed Keplerian system to $S^2 \times S^2$ is performed, characterising this space in terms of the angular momentum and the Laplace–Runge–Lenz vectors. The purpose of Section 3 is to discuss how the different types of coordinates have to be constructed to classify the type of motions that can occur. Previous work on this appeared in [34,36] where rectilinear, circular and equatorial solutions occurred. In Section 4 we deal with the generalised Størmer problem, averaging and reducing it to the space $S^2 \times S^2$. The purpose of Section 5 is the detailed analysis of the relative equilibria of the system using the coordinates introduced in Section 3. These relative equilibria are associated with families of periodic solutions of the original Hamiltonian vector field and this is proved rigorously in Section 6. Besides, the existence of various KAM tori is also established in Section 6.

Preliminary studies on the topic we bring to the paper appeared in [36], where we dealt with coordinates on $S^2 \times S^2$ for rectilinear as well as circular equatorial motions. Related work is due to Cordani [7], who dealt with the dynamics of perturbed Keplerian Hamiltonians on $S^2 \times S^2$. He introduced global coordinates that parameterise $S^2 \times S^2$ and local symplectic coordinates to treat circular equatorial motions. The local coordinates are equivalent to those proposed in [36]. Cordani also established the existence of KAM tori around the relative equilibria he computed for several examples. In this paper we generalise the treatments of [7,36] dealing with other types of motions. We also propose a different set of coordinates to study circular equatorial motions which usually yields a more straightforward approach of this class of motions.

Other related papers are due to Cushman and collaborators [10,15,16], where they analyse with great detail the dynamics of the hydrogen atom perturbed by sufficiently small homogeneous static electric and magnetic fields. This problem is modelled as a perturbation of the Kepler problem. Concretely these authors deal with the reduction of the three-degree-of-freedom problem to a Hamiltonian of one degree of freedom through two successive averages. The main purpose of the papers [10,15] is to analyse the monodromy of the system, an obstruction feature to define global action-variables and the relationship of monodromy with non-integrability and with the existence of Hamiltonian–Hopf bifurcations. In [10], they also deal with the dynamics on $S^2 \times S^2$, defining a set of local symplectic coordinates, but they do not look very practical as the change of coordinates is only defined through a Taylor series. The reconstruction of the flow of the original problem is done in the three papers, leading to very interesting features, especially in the review paper [16], about the dynamics of the hydrogen atom influenced by electric and magnetic fields. However, the existence of KAM tori is only mentioned but not established rigorously in any of the three papers. Moreover, the authors of [16] criticise the use of the Delaunay coordinates but they do not realise how to combine them in an efficient way in order to cope with the singular cases in a right way as we show in the paper.

We try to give a systematic approach, defining local symplectic coordinates covering all possible types of elliptic motions, classifying the motions in five different classes: (i) near rectilinear; (ii) near circular equatorial; (iii) circular non-equatorial; (iv) non-circular equatorial and; (v) non-circular non-equatorial. We also reconstruct with rigour the flow of the original system through the dynamics of the reduced flow on $S^2 \times S^2$ where the non-degeneracy of the relative equilibria holds. Besides, when these equilibria are parametrically stable, the corresponding families of periodic solutions are linearly stable (elliptic). Finally, we search for KAM tori introducing action-angles coordinates from the local coordinates we have defined previously. Our conclusion is that the combination of global coordinates of $S^2 \times S^2$ with local coordinates yields a methodological and convenient way of approaching a perturbed Keplerian Hamiltonian.

Another motivation of the present work was the local analysis of two points of the plane of parametric bifurcations corresponding with the analysis of the twice-reduced space of the generalised Størmer problem. Indeed, this plane of bifurcations contains two special points related to circular equatorial motions where all the bifurcation curves are coincident—see [24], and also [21]—where we analysed a simplified version of the generalised Størmer problem and only one of the two points was present in the study. However, these points are limit situations of the behaviour of the flow corresponding with the twice-reduced space as this space gets reduced to points for circular equatorial trajectories. So, the right analysis has to be achieved properly on $S^2 \times S^2$ and we have carried out this study in Section 5.2. Our conclusion is that there is no bifurcation on the space $S^2 \times S^2$ related to circular equatorial motion, but these two points have the feature that the two-degrees-of-freedom normal forms in the neighbourhoods around them are in resonance 1:1. This also implies the impossibility of obtaining KAM tori in these circumstances as we shall see in Section 6.

All the computations have been carried out with Mathematica, Version 7.0.1, and many of the calculations cannot be printed down in the paper as they are too big. However, the interested reader can obtain the Mathematica files upon request.

2. $S^2 \times S^2$ as the reduced space of perturbed Keplerian problems

In the six-dimensional space fixed at a certain frame centred at a point O and spanned by a vector denoting the position, $\mathbf{x} = (x, y, z)$, and another vector designating the momenta, $\mathbf{X} = (X, Y, Z)$, we write down a Hamiltonian function of the form

$$\mathcal{H} = \mathcal{H}_K + \varepsilon \mathcal{P}(\mathbf{x}, \mathbf{X}) = \frac{1}{2}\left(X^2 + Y^2 + Z^2\right) - \frac{1}{\sqrt{x^2 + y^2 + z^2}} + \varepsilon \mathcal{P}(\mathbf{x}, \mathbf{X}) \tag{1}$$

where ε stands by a small parameter and $\mathcal{P}$ a regular function denoting the perturbation of the problem. The Hamiltonian $\mathcal{H}_K$ represents the Hamiltonian of the two-body problem (or, equivalently, the Hamiltonian of the Kepler problem).

We introduce a couple of sets of coordinates, suitable for dealing with perturbed Keplerian systems. The set of orbital coordinates is given by the six-tuple $(r, \vartheta, \nu, R, G, N)$ where r stands for the radial distance from the origin of reference to the particle, ϑ represents the argument of latitude, ν is the right ascension of the node whereas R, G and N are the conjugate momenta of r, ϑ and ν respectively, see more details in [12]. Besides the action G represents the modulus of the angular momentum vector, i.e. $G = |\mathbf{G}| = |\mathbf{x} \times \mathbf{X}|$ and $N = xY - yX$ stands for the third component of the angular momentum; see more details in [32]. Notice that $0 \leqslant |N| \leqslant G$. The explicit relation between polar-nodal and Cartesian coordinates is obtained through the following transformation: $\varrho : (r, \vartheta, \nu, R, G, N) \to (x, y, z, X, Y, Z)$, where

$$\begin{aligned} x &= x' \cos\nu - y' \cos I \sin\nu, & X &= X' \cos\nu - Y' \cos I \sin\nu, \\ y &= x' \sin\nu + y' \cos I \cos\nu, & Y &= X' \sin\nu + Y' \cos I \cos\nu, \\ z &= y' \sin I, & Z &= Y' \sin I, \end{aligned} \tag{2}$$

with $\cos I = N/G$ (I is the inclination between the orbital and the equatorial planes) and x', y', X' and Y' are given by

$$x' = r\cos\vartheta, \qquad X' = R\cos\vartheta - \frac{G}{r}\sin\vartheta,$$
$$y' = r\sin\vartheta, \qquad Y' = R\sin\vartheta + \frac{G}{r}\cos\vartheta. \tag{3}$$

We have to take into account that the transformation ϱ is singular for $r = 0$, $G = 0$ and $G = |N|$ as I is an angle defined on $(0, \pi)$. The condition $|N| < G$ ensures that $\mathbf{G}$ is not parallel to the z axis, so ν is well defined. Hence, polar-nodal coordinates are not valid for rectilinear and equatorial trajectories. The name of polar-nodal variables is due to the fact that they are constructed as the composition of transformations (2) and (3).

Delaunay coordinates are given by (ℓ, g, ν, L, G, N). The angle ℓ stands for the mean anomaly, g is the argument of the pericentre, L the square of the semimajor axis, hence $0 \leqslant |N| \leqslant G \leqslant L$. The condition $G < L$ ensures that the ellipse does not degenerate to a circle, thus g and ℓ are well defined and $|N| < G$ ensures that ν is well defined. Thus, Delaunay coordinates are not valid for rectilinear, circular and equatorial trajectories.

Both polar-nodal and Delaunay coordinates are symplectic. Typically, the Hamiltonian $\mathcal{H}$ is expressed as a combination of polar-nodal and Delaunay elements [13,29]. In particular $\mathcal{H}_K = -1/(2L^2)$.

The first task consists in transforming Hamiltonian (1) with the aim of introducing a symmetry to the vector field related to $\mathcal{H}$. The transformation is usually performed in the setting of averaging theory, that is, the mean anomaly ℓ is averaged over one period. The theory is accomplished using Delaunay normalisation techniques [13,31–33]. This strategy lies in the context of a perturbation theory as we are considering our Hamiltonian $\mathcal{H}$ a perturbation of the Kepler Hamiltonian $\mathcal{H}_K$. If higher-orders of the averaging process are needed one has to resort to the combination of averaging theory with Lie transformations [11].

The transformed Hamiltonian, that we also call $\mathcal{H}$ after truncating higher-order terms, depends on the two angles g and ν and their associated momenta G and N, respectively, whereas L is an integral of motion as its associated angle has been removed from the transformed Hamiltonian.

Applying reduction theory, the Hamiltonian $\mathcal{H}$ is defined on the orbit space, or base space, which is the four-dimensional space $S^2 \times S^2$.

We can use the set of variables given by $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ with the constraints $a_1^2 + a_2^2 + a_3^2 = L^2$ and $b_1^2 + b_2^2 + b_3^2 = L^2$ to parameterise $S^2 \times S^2$, where $\mathbf{a} = \mathbf{G} + L\mathbf{A}$ and $\mathbf{b} = \mathbf{G} - L\mathbf{A}$, with $\mathbf{A}$ the Laplace–Runge–Lenz vector, i.e. the vector

$$\mathbf{A} = \mathbf{X} \times \mathbf{G} - \frac{\mathbf{x}}{|\mathbf{x}|}.$$

Thus, fixing a value of $L > 0$, the product of the two-sphere

$$S^2 \times S^2 = \left\{ (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^6 \;\middle|\; a_1^2 + a_2^2 + a_3^2 = L^2,\ b_1^2 + b_2^2 + b_3^2 = L^2 \right\} \tag{4}$$

is the phase space for Hamiltonian systems of Kepler type independent of ℓ, that is, for Hamiltonians for which L is an integral. This result was first reported by Moser [28] using a regularisation technique based on stereographic projections. Observe that $S^2 \times S^2$ is a smooth space (a symplectic manifold) and therefore the reduction is symplectic and regular [26,25]. Note that for planar perturbations of the planar Kepler Hamiltonian, the corresponding reduced space is S^2.

We need to know the Poisson structure on $S^2 \times S^2$ in $\mathbf{a}$ and $\mathbf{b}$ as we will use it in the application. We have

$$\{a_1, a_2\} = 2a_3, \qquad \{a_2, a_3\} = 2a_1, \qquad \{a_3, a_1\} = 2a_2,$$
$$\{b_1, b_2\} = 2b_3, \qquad \{b_2, b_3\} = 2b_1, \qquad \{b_3, b_1\} = 2b_2, \qquad \{a_i, b_j\} = 0.$$

Explicitly, the functions a_i's and b_i's can be given in terms of the coordinates $\mathbf{x}$ and $\mathbf{X}$ as $\mathbf{G}$, $\mathbf{A}$ and L may be written in terms of the position and momenta vectors. However, we prefer to give the expressions of a_i's and b_i's in terms of the Delaunay variables [6,8,9] as they are more practical. We follow the approach introduced by Cushman [8,9]. If $e=\sqrt{1-G^2/L^2}$ represents the eccentricity of the orbit, one has

$$\begin{aligned}
a_1 &= G\sin\nu\sin I + Le\cos g\cos\nu - Le\sin g\sin\nu\cos I,\\
a_2 &= -G\cos\nu\sin I + Le\cos g\sin\nu + Le\sin g\cos\nu\cos I,\\
a_3 &= G\cos I + Le\sin g\sin I,\\
b_1 &= G\sin\nu\sin I - Le\cos g\cos\nu + Le\sin g\sin\nu\cos I,\\
b_2 &= -G\cos\nu\sin I - Le\cos g\sin\nu - Le\sin g\cos\nu\cos I,\\
b_3 &= G\cos I - Le\sin g\sin I.
\end{aligned} \tag{5}$$

The variables a_i's and b_i's—also called generators or coordinates of the reduced space—are indeed the (global) coordinates used to describe the reduced phase space as they are the functions associated to the vector fields generating the $SO(4)$ symmetry of $-1/(2L^2)$.

Now we account for the subsets of $S^2\times S^2$ related to special motions. Observe that

$$2G=\sqrt{(a_1+b_1)^2+(a_2+b_2)^2+(a_3+b_3)^2},$$

so $G=0$ on $S^2\times S^2$ if and only if $a_1+b_1=a_2+b_2=a_3+b_3\equiv 0$, $a_1^2+a_2^2+a_3^2=L^2$, and $b_1^2+b_2^2+b_3^2=L^2$. Thus, the subset of $S^2\times S^2$ given by

$$\mathcal{R}=\left\{(\mathbf{a},-\mathbf{a})\in\mathbb{R}^6 \;\middle|\; a_1^2+a_2^2+a_3^2=L^2\right\}$$

is a two-dimensional set homeomorphic to S^2 consisting of the rectilinear trajectories. In Delaunay elements, the circular orbits satisfy the condition $G=L$, and in terms of **a** and **b** this implies that $a_1=b_1$, $a_2=b_2$, and $a_3=b_3$. So the circular orbits define the two-dimensional set homeomorphic to S^2 given by

$$\mathcal{C}=\left\{(\mathbf{a},\mathbf{a})\in\mathbb{R}^6 \;\middle|\; a_1^2+a_2^2+a_3^2=L^2\right\}.$$

Similarly, equatorial trajectories satisfy $G=|N|$ and are given by the two-dimensional set

$$\mathcal{E}=\left\{(\mathbf{a},\mathbf{b})\in\mathbb{R}^6 \;\middle|\; a_1^2+a_2^2+a_3^2=L^2,\ b_1=-a_1,\ b_2=-a_2,\ b_3=a_3\right\},$$

which is again homeomorphic to S^2. Notice that the intersection $\mathcal{C}\cap\mathcal{E}$ yields the subset of equatorial and circular solutions, and it is given by zero-dimensional set composed of two points of $S^2\times S^2$, i.e.

$$\mathcal{C}\cap\mathcal{E}=\left\{(0,0,\pm L,0,0,\pm L)\right\}.$$

Just as in the planar case, the introduction of these invariants extends the use of the Delaunay variables, as we can include equatorial, circular, and rectilinear solutions [32,34,36]. The other points on $S^2\times S^2$ correspond to elliptic orbits of the Kepler problem.

From (5) it is easy to deduce that

$$
\begin{aligned}
2G \sin \nu \sin I &= a_1 + b_1, \\
2Le(\cos g \cos \nu - \sin g \sin \nu \cos I) &= a_1 - b_1, \\
-2G \cos \nu \sin I &= a_2 + b_2, \\
2Le(\cos g \sin \nu + \sin g \cos \nu \cos I) &= a_2 - b_2, \\
2N &= a_3 + b_3, \\
2Le \sin g \sin I &= a_3 - b_3,
\end{aligned} \tag{6}
$$

which allows us to express the functions G, N, $\cos g$, $\sin g$, $\cos \nu$ and $\sin \nu$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Now, the constant term $\mathcal{H}_K$ can be dropped and the reduced Hamiltonian $\bar{\mathcal{H}}$, which is independent of ℓ, can be written as a function of $\mathbf{a}$ and $\mathbf{b}$ and the constant $L > 0$, i.e. $\bar{\mathcal{H}} \equiv \bar{\mathcal{H}}(\mathbf{a}, \mathbf{b}; L)$. Note that the way in which the coordinates a_i's and b_i's appear in the Hamiltonian $\bar{\mathcal{H}}$ depends on the manner that the position $\mathbf{x}$ and the momentum $\mathbf{X}$ appear in $\mathcal{P}$, so, it depends on each specific problem. Typically, $\mathcal{P}(\mathbf{a}, \mathbf{b}; L)$ is a rational or even a polynomial function.

3. Symplectic coordinates on $S^2 \times S^2$

3.1. Global versus local coordinates

Coordinates a_i's and b_i's together with the constraints are the natural set of variables to use when one analyses a Hamiltonian vector field related to a perturbed Keplerian system averaged over the mean anomaly. However, when dealing with concrete solutions they are, in general, not very useful as the expressions become in general cumbersome as one needs to handle six coordinates plus two independent constraints.

Related work was initiated in [34,36], where the full analysis of the reduced flow in the context of the spatial restricted three-body problem for the Lunar case is obtained using the variables a_i's and b_i's. In particular we encountered four equilibria, classified in two different types, the points $(0, 0, \pm L, 0, 0, \pm L)$ and the points $(0, 0, \pm L, 0, 0, \mp L)$. The points $(0, 0, \pm L, 0, 0, \pm L)$ correspond with motions whose projections in the configuration space $Oxyz$ are of circular equatorial type (prograde and retrograde) whereas the points $(0, 0, \pm L, 0, 0, \mp L)$ correspond with motions whose projections in the space $Oxyz$ are of the class of rectilinear motions in the vertical z axis, also prograde and retrograde. Thus, reconstructing the flow, one concludes that the spatial circular restricted three-body problem in the Lunar case has two families of periodic solutions near circular equatorial and two families of periodic solutions near rectilinear type in the z axis. In this example, the local analysis to obtain the linear and non-linear stability of the relative equilibria was performed introducing ad-hoc symplectic coordinates for rectilinear or circular equatorial coordinates.

In this section we propose to combine the global coordinates a_i's and b_i's with other local symplectic coordinates in order to facilitate the analysis of the possible relative equilibria, their stability and bifurcations on $S^2 \times S^2$. Thus, the point is to give a practical point of view of how to proceed when one faces with the analysis of the reduced flow of a Hamiltonian vector field related with a perturbed Keplerian system.

3.2. Near rectilinear trajectories

An equilibrium of this type has coordinates $(a_1, a_2, a_3, -a_1, -a_2, -a_3)$ with $a_1^2 + a_2^2 + a_3^2 = L^2$. Thus, the analysis of near rectilinear trajectories may be done within the two-sphere $a_1^2 + a_2^2 + a_3^2 = L^2$. Thus, once the equilibrium (a_1^0, a_2^0, a_3^0) is given, we need to make the translation

$$a_1 = \bar{a}_1 + a_1^0, \qquad a_2 = \bar{a}_2 + a_2^0, \qquad a_3 = \bar{a}_3 + a_3^0,$$
$$b_1 = \bar{b}_1 - a_1^0, \qquad b_2 = \bar{b}_2 - a_2^0, \qquad b_3 = \bar{b}_3 - a_3^0,$$

so that we move the origin to the point of interest. Then, we introduce the coordinates x_i's and y_i's (called Q_i's and P_i's in Section 4.3.1 of [36])

$$x_1 = \frac{\bar{a}_2}{\sqrt{2L + \bar{a}_3}}, \qquad x_2 = \frac{\bar{b}_2}{\sqrt{2L - \bar{b}_3}},$$
$$y_1 = -\frac{\bar{a}_1}{\sqrt{2L + \bar{a}_3}}, \qquad y_2 = \frac{\bar{b}_1}{\sqrt{2L - \bar{b}_3}}, \tag{7}$$

with inverse

$$\bar{a}_1 = -y_1\sqrt{2L - y_1^2 - x_1^2}, \qquad \bar{a}_2 = x_1\sqrt{2L - y_1^2 - x_1^2}, \qquad \bar{a}_3 = -y_1^2 - x_1^2,$$
$$\bar{b}_1 = y_2\sqrt{2L - y_2^2 - x_2^2}, \qquad \bar{b}_2 = x_2\sqrt{2L - y_2^2 - x_2^2}, \qquad \bar{b}_3 = y_2^2 + x_2^2. \tag{8}$$

The local variables (x_1, x_2, y_1, y_2) constitute a symplectic set for which x_1, x_2 can be interpreted as coordinates, and y_1 and y_2 represent their associated momenta, respectively.

3.3. Near circular equatorial trajectories

This time, instead of working with the coordinates Q_i's and P_i's defined in Section 4.3.2 of [36] we resort to Poincaré-like coordinates [3,20]. For practical examples where the averaged Hamiltonian is obtained in terms of the Delaunay elements, the resulting expressions are shorter and easier to deal with compared to the formulae derived from the coordinates of [36].

In particular, for near circular equatorial prograde solutions ($N \approx G \approx L > 0$, i.e., $I \approx 0$ and $e \approx 0$) we introduce

$$x_1 = -\sqrt{2(L-G)}\sin(g+\nu), \qquad x_2 = -\sqrt{2(G-N)}\sin\nu,$$
$$y_1 = \sqrt{2(L-G)}\cos(g+\nu), \qquad y_2 = \sqrt{2(G-N)}\cos\nu. \tag{9}$$

For near circular equatorial retrograde orbits ($N \approx -G \approx -L < 0$, i.e., $I \approx \pi$ and $e \approx 0$) we take

$$x_1 = -\sqrt{2(L-G)}\sin(g-\nu), \qquad x_2 = \sqrt{2(G+N)}\sin\nu,$$
$$y_1 = \sqrt{2(L-G)}\cos(g-\nu), \qquad y_2 = \sqrt{2(G+N)}\cos\nu. \tag{10}$$

We remark that the angle g is not well defined when $G = L$, besides ν is not well defined when $|N| = G$, however the angle $g + \nu$ is properly defined [18] when $N = G$ and $g - \nu$ is properly defined for $N = -G$, thus x_1 and y_1 are defined properly for (9) and (10). The coordinates x_2 and y_2 are a bit more problematic but when $N = G$, then $x_2 = y_2 = 0$ in (9) whereas if $N = -G$, then $x_2 = y_2 = 0$ in (10) and therefore the coordinates x_i's and y_i's are well defined near circular equatorial motions, including true circular equatorial motions.

If x_1 and x_2 are taken as coordinates whereas y_1 and y_2 are their associated momenta, it is an easy exercise to prove that the sets of coordinates given through (9) and (10) are both symplectic on $S^2 \times S^2$.

Note that the point $(0, 0, L, 0, 0, L)$ of $S^2 \times S^2$ accounting for circular equatorial prograde solutions, corresponds with $x_1 = x_2 = y_1 = y_2 = 0$ in (9) whereas $(0, 0, -L, 0, 0, -L)$ corresponds with $x_1 = x_2 = y_1 = y_2 = 0$ in (10).

The inverses of (9) and (10) are, respectively,

$$
\begin{aligned}
g &= \arccos\left(\pm\frac{y_1}{\sqrt{x_1^2+y_1^2}}\right) - \arccos\left(\pm\frac{y_2}{\sqrt{x_2^2+y_2^2}}\right),\\
\nu &= \pm\arccos\left(\pm\frac{y_2}{\sqrt{x_2^2+y_2^2}}\right),\\
G &= \frac{1}{2}\left(2L - x_1^2 - y_1^2\right),\\
N &= \frac{1}{2}\left(2L - x_1^2 - y_1^2 - x_2^2 - y_2^2\right),
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
g &= \arccos\left(\pm\frac{y_1}{\sqrt{x_1^2+y_1^2}}\right) + \arccos\left(\pm\frac{y_2}{\sqrt{x_2^2+y_2^2}}\right),\\
\nu &= \pm\arccos\left(\pm\frac{y_2}{\sqrt{x_2^2+y_2^2}}\right),\\
G &= \frac{1}{2}\left(2L - x_1^2 - y_1^2\right),\\
N &= \frac{1}{2}\left(x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2L\right).
\end{aligned}
\tag{12}
$$

In both formulae, all possible combinations of the signs have to be taken into account, depending on the different relative positions of the angles g and ν in $[0, 2\pi)$, therefore there are eight different combinations. We have to exclude the point $x_1 = x_2 = y_1 = y_2 = 0$ as the g and ν are not well defined in this case.

We have defined the sets of coordinates in terms of Delaunay elements as this is usually more practical than the specific expressions of x_i's and y_i's as functions of $\mathbf{a}$ and $\mathbf{b}$. The reason is that, in general, the averaged Hamiltonian is written in terms of Delaunay elements, so these specific orbits are analysed straightforwardly using the coordinates (11) and (12). The same happens with the other coordinates that we introduce in the next subsections.

3.4. Near circular non-equatorial trajectories

We introduce

$$
\begin{aligned}
x_1 &= -\sqrt{2(L-G)}\sin g, \qquad x_2 = \nu - \nu_0,\\
y_1 &= \sqrt{2(L-G)}\cos g, \qquad y_2 = N - N_0,
\end{aligned}
\tag{13}
$$

where ν_0 and N_0 are supposed to be the concrete values that the relative equilibrium takes on $S^2 \times S^2$. Taking x_1, x_2 as coordinates and y_1, y_2 as their associated momenta, the set (13) is symplectic. Notice that the point $x_1 = x_2 = y_1 = y_2 = 0$ corresponds with the set of circular non-equatorial motions on $S^2 \times S^2$ (i.e., the set $\mathcal{C} \setminus \mathcal{E}$).

Let us remark that x_2 and y_2 are not problematic. Nevertheless, when $G = L$ the pericentre is not well defined, but similarly to the case of circular equatorial motions, precisely in this situation one has $x_1 = y_1 = 0$, so, these coordinates make sense near circular non-equatorial motions, including true circular non-equatorial orbits.

The inverse of (13) reads as

$$g = \pm \arccos\left(\pm \frac{y_1}{\sqrt{x_1^2 + y_1^2}}\right), \qquad \nu = x_2 + \nu_0,$$
$$G = \frac{1}{2}\left(2L - x_1^2 - y_1^2\right), \qquad N = y_2 + N_0. \tag{14}$$

There are four different combinations that depend on the position of g in $[0, 2\pi)$. The inverse change (14) is not well defined on the plane $x_1 = y_1 = 0$.

3.5. Near non-circular equatorial trajectories

We distinguish between prograde and retrograde motions.

For prograde orbits we take

$$x_1 = g + \nu - k_0, \qquad x_2 = -\sqrt{2(G - N)}\sin\nu,$$
$$y_1 = G - G_0, \qquad y_2 = \sqrt{2(G - N)}\cos\nu, \tag{15}$$

where k_0 is the value of the angle $g + \nu$ (which is well defined for equatorial prograde motions) at the equilibrium and G_0 is the value of G at the equilibrium. Considering x_1 and x_2 the coordinates and y_1 and y_2 their associated momenta, the set of coordinates (15) is symplectic.

Notice that the coordinates x_1 and y_1 are always well defined. Besides, when $N = G$, although the argument of the node is not well defined, one has that $x_2 = y_2 = 0$ so, these coordinates are properly defined near circular non-equatorial prograde motions as well as in exactly non-circular equatorial prograde orbits.

In the case of retrograde trajectories we define

$$x_1 = g - \nu - k_0, \qquad x_2 = \sqrt{2(G + N)}\sin\nu,$$
$$y_1 = G - G_0, \qquad y_2 = \sqrt{2(G + N)}\cos\nu, \tag{16}$$

where k_0 is the value of the angle $g - \nu$ (which is well defined for equatorial retrograde motions) at the equilibrium whereas G_0 corresponds with the value of G at the equilibrium. If we interpret x_1 and x_2 as the coordinates and y_1 and y_2 as their associated momenta, the set of coordinates (16) is symplectic.

The coordinates x_1 and y_1 are well defined. When $N = -G$, the argument of the node is not well defined, but then $x_2 = y_2 = 0$, hence, x_i's and y_i's are properly defined near circular non-equatorial motions, even in true non-circular equatorial retrograde motions.

The points $x_1 = x_2 = y_1 = y_2 = 0$ of (15) and (16) correspond with the set $\mathcal{E} \setminus \mathcal{C}$.

This time the inverses of (15) and (16) are, respectively,

$$g = x_1 \pm \arccos\left(\pm \frac{y_2}{\sqrt{x_2^2 + y_2^2}}\right) + k_0,$$
$$\nu = \pm \arccos\left(\pm \frac{y_2}{\sqrt{x_2^2 + y_2^2}}\right),$$
$$G = y_1 + G_0,$$
$$N = \frac{1}{2}\left(2G_0 + 2y_1 - x_2^2 - y_2^2\right), \tag{17}$$

and

$$\begin{aligned}
g &= x_1 \pm \arccos\left(\pm \frac{y_2}{\sqrt{x_2^2 + y_2^2}}\right) + k_0, \\
\nu &= \pm \arccos\left(\pm \frac{y_2}{\sqrt{x_2^2 + y_2^2}}\right), \\
G &= y_1 + G_0, \\
N &= \frac{1}{2}\left(x_2^2 + y_2^2 - 2y_1 - 2G_0\right). \qquad (18)
\end{aligned}$$

In both formulae, all possible combinations of the signs have to be taken into account, so there are eight different possibilities. The change (18) is not well defined on the plane $x_2 = y_2 = 0$.

3.6. Near non-circular non-equatorial trajectories

Now we define

$$\begin{aligned}
x_1 &= g - g_0, \qquad x_2 = \nu - \nu_0, \\
y_1 &= G - G_0, \qquad y_2 = N - N_0, \qquad (19)
\end{aligned}$$

where we have taken the values (g_0, ν_0, G_0, N_0) as those corresponding with a certain equilibrium point on $S^2 \times S^2$. It is obvious that (19) is symplectic.

For this type of motions the angles g and ν are well defined, therefore x_i's and y_i's are properly defined. Moreover, for particular values g_0, ν_0, G_0 and N_0, the point $x_1 = x_2 = y_1 = y_2 = 0$ yields the relative equilibrium on $S^2 \times S^2$ that corresponds with non-circular non-equatorial solutions.

The trivial inverse of (19) is

$$\begin{aligned}
g &= x_1 + g_0, \qquad \nu = x_2 + \nu_0, \\
G &= y_1 + G_0, \qquad N = y_2 + N_0. \qquad (20)
\end{aligned}$$

4. The generalised Størmer problem and its reduction to $S^2 \times S^2$

4.1. Hamiltonian model

This problem studies the dynamics of charged particles around rotating magnetic planets. Specifically, the generalised Størmer problem we analyse here, describes the dynamics of a dust particle of mass m and charge q orbiting a rotating magnetic planet of mass M. The magnetic field of the planet is supposed to be a perfect magnetic dipole of strength μ aligned along the north-south poles of the planet. Moreover, the planet's magnetosphere is taken as a rigid conducting plasma which rotates with the same angular velocity Ω as the planet, in such a way that the charge q is subject to a corotational electric field. Furthermore, the gravitational interaction in our model takes into account the non-sphericity of a planet given by means of the so-called J_2 term [30]. Introducing cylindrical coordinates and momenta $(\rho, z, \phi, P_\rho, P_z, P_\phi)$, the generalised Størmer problem is given through the following two-degree-of-freedom Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(P_\rho^2 + P_z^2 + \frac{P_\phi^2}{\rho^2}\right) - \frac{1}{r} - \delta\,\frac{P_\phi}{r^3} + \frac{\delta^2}{2}\,\frac{\rho^2}{r^6} + \delta\beta\frac{\rho^2}{r^3} + 3J_2\frac{z^2}{2r^5} - \frac{J_2}{2r^3}, \qquad (21)$$

where J_2 is a positive dimensionless parameter for an oblate planet, whereas it is negative for a prolate one. For example, in the case of Saturn, $J_2 = 0.016298$ [5]. Lengths and time are expressed, respectively, in units of the radius of the planet, $\mathcal{R}$, and the Keplerian frequency $w_K = \sqrt{M/\mathcal{R}^3}$ (Gaussian units). The variable $r = \sqrt{\rho^2 + z^2}$ stands for the distance of the charged particle to the centre of mass of the planet. Cylindrical variables are natural to formulate the problem, as the system is invariant under rotations around the z axis, P_ϕ is an integral of the system, indeed it corresponds with the action N. Furthermore, Hamiltonian (21) depends also on three external parameters, namely, δ, β and J_2, which indicate respectively the ratio between the magnetic and the Keplerian interaction (i.e., the charge-mass ratio q/m of the particle), the ratio between the electrostatic and the Keplerian interactions (i.e., the ratio Ω/w_K, thus $\beta > 0$ and for Saturn $\beta \approx 2/5$), and the oblateness of the planet taken into consideration. The value of δ can be positive or negative, depending on the charge of the particle, but we can locate it in the interval $[-10^{-2}, 10^{-2}]$, see the details in [21]. Notice also that for a given planet, β and J_2 are fixed while δ varies. We also assume that $J_2 > 0$, therefore our analysis is valid for oblate planets. On the other hand, the system depends on the two internal parameters P_ϕ and $\mathcal{H} = E$ (the energy).

Hamiltonian (21) generalises that considered in the papers by Dullin and collaborators [19,14] and Iñarrea et al. [21,22]. In particular, assuming that $J_2 = 0$, Dullin et al. [14] and Howard et al. [19] proved the existence and stability of orbits around the planet lying on the equatorial plane (equatorial orbits), and orbits that do not intersect the equatorial plane, i.e. the so-called halo orbits. The studies carried out in [21,22] also assume that $J_2 = 0$ and are based on the twice-reduced flow corresponding with the second reduction. Indeed, after averaging over ℓ, as P_ϕ is an integral of motion one can reduce again the Hamiltonian and work in the twice-reduced phase space, a two-dimensional space homeomorphic to a sphere. In this setting, the occurrence of relative equilibria, together with their stability analysis and bifurcations, have been achieved with detail.

We have used this model to establish the existence of equatorial and halo solutions [23]. However, our purpose here is the analysis of the reduced flow related to the average of Hamiltonian (21) over the mean anomaly. This analysis has to be performed on $S^2 \times S^2$, combining the global coordinates **a** and **b** with the local coordinates given by x_i's and y_i's in Section 3. In [24], using the Hamiltonian function of (21) we have extended the previous research of [21,22], as we take into account the oblateness coefficient of the planet. Moreover, the reconstruction of the flow of the original system from the flow in the twice-reduced space is rigorously done, concluding the existence of some invariant tori together with different types of bifurcations of them. Furthermore, the existence of some KAM tori is also established in [24]. However, not all the information about the dynamics of Hamiltonian (21) can be recovered from the flow in the twice-reduced phase space. For instance, the analysis of circular equatorial motions cannot be carried out in the twice-reduce space as this space gets reduced to a unique point. Furthermore, there are other features of the system that need to be analysed in the first-reduced phase space, that is, on $S^2 \times S^2$. Hence, the study we perform along the paper must be understood complementary to that of [24].

4.2. Averaging

The first step consists in passing from cylindrical coordinates to a mixed of polar-nodal and Delaunay elements. We stress that the terms that are not factorised by δ, β or J_2 in (21) correspond with the Kepler Hamiltonian $\mathcal{H}_K$ in cylindrical coordinates. Considering that the effect of the magnetic and electric fields and that the perturbation caused by the presence of J_2 are all small compared with the pure Kepler attraction, we can think of our model as a perturbed Keplerian Hamiltonian in the space and that δ, β and J_2 are small parameters such that the terms of the Hamiltonian (21) which do not correspond with the Keplerian Hamiltonian are of the same asymptotic order in the small parameter. Thus, we can apply the Delaunay normalisation to average the perturbation with respect to ℓ. We give no details on the process of averaging as the reader is addressed to [13], however we remark that both the averaged Hamiltonian and the generating function, that we do not write down explicitly as it is too large, are computed in closed form, allowing the analysis of any type of elliptical motion [31,33].

Thus, we arrive at the Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{O}(2) \tag{22}$$

with

$$\begin{aligned}
\mathcal{H}_0 &= -\frac{1}{2L^2}, \\
\mathcal{H}_1 &= \frac{\delta}{16L^5G^7(L+G)}\big(2(L+G)\big(4\beta L^3G^7 + 4\beta L^3G^5N^2 - \delta G^4 \\
&\quad - 8L^2G^4N - \delta G^2N^2 + 3\delta L^2G^2 + 3\delta L^2N^2\big) \\
&\quad + (L-G)\big(G^2-N^2\big)\big(8\beta L^3G^5 + \delta G^2 + 2\delta LG + \delta L^2\big)\cos(2g)\big) \\
&\quad + \frac{J_2(G^2-3N^2)}{4L^3G^5}.
\end{aligned}$$

By putting $\mathcal{O}(2)$ we assume terms of order two, that have been also obtained explicitly and that play a certain role in the analysis of the problem as we will illustrate in the next sections.

Note that $\mathcal{H}_1$ has a pole at $G = 0$, which implies that rectilinear motions have to be excluded from our analysis. This is related with the fact that the charged particle cannot enter the planet, therefore we have to fix a minimum value for G, say G_m, such that we study the values of G with $0 < G_m \leqslant G \leqslant L$, hence collision with the planet is avoided.

4.3. Reduction

Now, our goal is to perform the reduction of $\mathcal{H}$ given by (22) to the orbit space. This may be achieved once we have truncated higher-order terms. Although we will need the expression of terms of order two, $\mathcal{O}(2)$, later, for the moment we content ourselves to reduce only $\mathcal{H}_1$. We can also drop the zeroth-order term $\mathcal{H}_0$ as it is a constant of motion. Using the expressions of g, G and N in terms of a_i's and b_i's given by (6), and after doing some algebra calculations to simplify the formulae, we arrive at

$$\begin{aligned}
\bar{\mathcal{H}} &= \frac{\delta\beta(4L(2L^2-a_3^2-b_3^2) + 2(4L^2-(a_3+b_3)^2)G)}{8L^3(L+G)^2} \\
&\quad + \frac{1}{128L^5G^7}\big(32\delta\beta L^2(a_3+b_3)^2G^6 - 8\big(3\delta^2 - 4J_2L^2 + 8\delta L^2(a_3+b_3)\big)G^4 \\
&\quad + \big(\delta^2\big(28L^2 - 3a_3^2 + 2a_3b_3 - 3b_3^2\big) - 24J_2L^2(a_3+b_3)^2\big)G^2 \\
&\quad + 10\delta^2L^2(a_3+b_3)^2\big),
\end{aligned} \tag{23}$$

with $G = \sqrt{(L^2 + a_1b_1 + a_2b_2 + a_3b_3)/2}$. We remark that $\bar{\mathcal{H}}$ is a rational function in a_i's and b_i's, with a singularity at $a_1b_1 + a_2b_2 + a_3b_3 = -L^2$. Thus, the Hamiltonian normal form is not bounded on $S^2 \times S^2$, that is, $\bar{\mathcal{H}}$ is not a Morse function in the whole space $S^2 \times S^2$, consequently we cannot apply the bounds on the number of critical points nor the Morse inequalities, those given in [35,27,28,36].

Notice that the reduction to $S^2 \times S^2$ is regular. Moreover, by the introduction of the coordinates a_i's and b_i's we have extended the Hamiltonian expressed in Delaunay coordinates and can consider equatorial or circular motions. We need to exclude rectilinear motions as the original Hamiltonian was not well defined for $G = 0$.

The next step is to obtain the Hamiltonian vector field associated with $\bar{\mathcal{H}}$, yielding explicit values $\dot{a}_i$'s and $\dot{b}_i$'s as follows: $\dot{a}_i = \{a_i, \bar{\mathcal{H}}\}$, $\dot{b}_i = \{b_i, \bar{\mathcal{H}}\}$, for $i = 1, 2, 3$.

5. Relative equilibria and stability

5.1. Circular equatorial solutions

We exclude the searching of rectilinear motions, so we start with the case of near circular equatorial solutions as relative equilibria of the Hamiltonian vector field related with $\bar{\mathcal{H}}$.

Replacing $(a_1, a_2, a_3, b_1, b_2, b_3)$ by $(0, 0, \pm L, 0, 0, \pm L)$ in the vector field we get $\dot{a}_i = \dot{b}_i = 0$ for $i = 1, 2, 3$. Thus, we conclude that circular equatorial motions are solutions of the Hamiltonian. In order to study the normal form and stability of the equilibria we need to make the changes defined in (9) and in (10) to Hamiltonian $\mathcal{H}_1$ of (22). Then, we make a Taylor-expansion around $x_1 = x_2 = y_1 = y_2 = 0$, yielding after dropping constant terms

$$\begin{aligned}\bar{\mathcal{H}} &= c_1^{\pm}(\delta, J_2, L)\left(x_1^2 + y_1^2\right) + c_2^{\pm}(\delta, \beta, J_2, L)\left(x_2^2 + y_2^2\right) + \cdots \\ &= \frac{6\delta^2 - 3J_2L^2 \mp 4\delta L^3}{4L^9}\left(x_1^2 + y_1^2\right) \\ &\quad + \frac{-\delta^2 + 3J_2L^2 + 2\delta L^3(\pm 1 - \beta L^3)}{4L^9}\left(x_2^2 + y_2^2\right) + \cdots,\end{aligned} \tag{24}$$

where $c_1^{\pm}$ and $c_2^{\pm}$ take the explicit values of the factors of $x_1^2 + y_1^2$ and $x_2^2 + y_2^2$, respectively. Moreover, the sign "+" refers to prograde solutions whereas "−" is related with retrograde solutions. The same convention holds for the other cases where the coefficients $c_i^{\pm}$'s arise.

We notice that the quadratic part of $\bar{\mathcal{H}}$ is already diagonalised, which shows the convenience of using the sets of coordinates defined in the paper. This feature is in contrast with the approach followed in [36], where a long process was done in order to diagonalise the quadratic part of a normal form.

As the linearisations around the two points are both of the form centre × centre (generically when the eigenvalues do not vanish), it is immediate to deduce that the points $(0, 0, \pm L, 0, 0, \pm L)$ are, in general, linearly stable on $S^2 \times S^2$. However, there are different situations depending on the combinations among the parameters.

Let us start the analysis of (24) for the prograde solutions, and positive-charged particles ($\delta > 0$). We do not take into account in our study the pure Kepler case, i.e., the case $\delta = \beta = J_2 = 0$.

By inspection of the normal form, the coefficient of $x_1^2 + y_1^2$ remains negative for the allowed values of all the parameters; we can ensure it if we restrict the value of L to the interval $(1, 3]$, which is a right interval to consider the generalised Størmer problem as a perturbed Kepler problem [21]. Even when β and J_2 are fixed while δ and L vary (δ accordingly with the particle and L accounting for the distance from the particle to the centre of the planet) it is easier but equivalent to interpret the results in terms of β, thus, we set

$$\beta_c = \frac{-\delta^2 + 3J_2L^2 + 2\delta L^3}{2\delta L^6}. \tag{25}$$

We conclude that if $\beta < \beta_c$ the coefficient of $x_2^2 + y_2^2$ is positive, hence the quadratic part of $\bar{\mathcal{H}}$ is undefined while if $\beta > \beta_c$ the quadratic part of $\bar{\mathcal{H}}$ is negative definite. Finally, if $\beta = \beta_c$ the quadratic form is negative semidefinite, however there is still a set of four eigenvectors which implies that in all the cases the quadratic form is semisimple.

Resonances between the x_1–y_1 and x_2–y_2 directions can occur for various combinations of the parameters, either if $\beta < \beta_c$ or $\beta > \beta_c$. For instance, if $\beta = (-7\delta^2 + 6J_2L^2 + 6\delta L^3)/(2\delta L^6)$, the quadratic part of $\bar{\mathcal{H}}$ is in resonance 1:1. This is the precise value of β where most of the bifurcation lines of the two-reduced space are confluent, see [24], although no bifurcation occurs on $S^2 \times S^2$. If $\beta = (4\delta^2 + 3J_2L^2)/(4\delta L^6)$ the system is in resonance 1:−2. However, for the allowed values of the parameters, the resonance 1:−1 cannot occur. The resonance 1:1 is connected with the confluence of the bifurcation curves for prograde orbits appearing in the study of the twice-reduced space achieved in [24], see also [21]. Nevertheless, no bifurcation occurs on $S^2 \times S^2$ for the value of β leading to the resonance 1:1 for prograde motions.

When $\beta > \beta_c$ the quadratic form is negative definite, thus, applying the Dirichlet theorem, we conclude that the equilibrium becomes also parametrically and non-linearly stable. If $\beta < \beta_c$, the equilibrium is also parametrically stable (the only chance of parametric instability is the resonance 1:−1) whereas the non-linear stability should be studied using Arnold's theorem [2,3] or other analog results for resonant Hamiltonians [4,17]. When $\beta = \beta_c$ the equilibrium is not parametrically stable and the non-linear stability is an open question. Even more, after computing the determinant of the Jacobian matrix associated to the quadratic part of $\bar{\mathcal{H}}$, it is deduced that the equilibrium is not isolated when $\beta = \beta_c$ as the determinant vanishes at β_c, whereas it is always isolated when $\beta \neq \beta_c$.

When $\delta < 0$ the analysis can be carried out similarly to the previous paragraphs. However, since the coefficient of $x_1^2 + y_1^2$ can also vanish, more cases arise. Thus, when $J_2 = J_{2c}$ with

$$J_{2c} = \frac{2\delta(3\delta - 2L^3)}{3L^2}, \tag{26}$$

the quadratic form of $\bar{\mathcal{H}}$ becomes negative semidefinite. Anyway the two eigenvalues c_1^+ and c_2^+ cannot vanish at the same time. We have not pursued further research on this as the results are of the same type as before. The resonance 1:1 can occur, but the resonance 1:−1 is not allowed for the parameters with physical significance.

Now we deal with retrograde orbits, starting with $\delta > 0$.

This time the quadratic form can be undefined, positive definite or negative definite. If $J_2 = J_{2c}$ with

$$J_{2c} = \frac{2\delta(3\delta + 2L^3)}{3L^2}, \tag{27}$$

the coefficient of $x_1^2 + y_1^2$ vanishes while if $\beta = \beta_c$ with

$$\beta_c = \frac{-\delta^2 + 3J_2L^2 - 2\delta L^3}{2\delta L^6}, \tag{28}$$

the coefficient of $x_2^2 + y_2^2$ vanishes. Both coefficients are allowed to vanish at the same time, leading therefore to a very degenerate situation. Besides, other combinations allow the quadratic Hamiltonian to be in resonance 1:1 (when $\beta = (-7\delta^2 + 6J_2L^2 - 6\delta L^3)/(2\delta L^6)$) and 1:−1 (when $\beta = (5\delta + 2L^3)/(2L^6)$) and other higher-order resonances. The case of resonance 1:1 corresponds with the confluence of almost all bifurcation curves in the analysis of the twice-reduced space performed in [24]. However, we stress that no bifurcation occurs on $S^2 \times S^2$ for the value of β leading to the resonance 1:1 for retrograde motions.

Linear, parametric and non-linear stabilities depend on the different situations: leaving apart the degenerate case ($J_2 = J_{2c}$ and $\beta = \beta_c$, with the value of J_{2c} inserted inside β_c), the rest of situations are of semisimple character, therefore the linear stability of the equilibrium is ensured. Parametric stability is concluded when $J_2 \neq J_{2c}$, $\beta \neq \beta_c$ and, moreover, the system is not in resonance 1:−1. Finally, non-linear stability of the equilibrium is satisfied when the quadratic form is either positive or negative definite. The equilibrium is isolated excepting for $J_2 = J_{2c}$ or $\beta = \beta_c$.

Finally, for retrograde motions but with $\delta < 0$, it is straightforward to prove that the coefficients of (24) do not change sign for the values of the parameters with physical interest. Concretely, the sign of $x_1^2 + y_1^2$ remains negative while the sign of $x_2^2 + y_2^2$ is positive, therefore the quadratic part of the normal form is undefined. The resonance 1:−1 may happen. Thus, the equilibrium is linearly stable and its parametric stability is assured excepting for the resonance 1:−1, i.e., excepting for $\beta = (5\delta + 2L^3)/(2L^6)$. The analysis of the non-linear stability would deserve further development based on Arnold's theorem [2] and its variants for resonant cases [4,17], but we do not cope with it. This time $(0, 0, -L, 0, 0, -L)$ is always an isolated equilibrium on $S^2 \times S^2$.

5.2. Circular non-equatorial solutions

We seek points of the type $(a_1, a_2, a_3, a_1, a_2, a_3)$ with $a_1^2 + a_2^2 + a_3^2 = L^2$. Using directly the Hamiltonian vector field in **a** and **b**, we get the circle

$$a_1^2 + a_2^2 = L^2 - \left(\frac{2\delta L^4}{\delta^2 - 3J_2L^2 + 2\delta\beta L^6}\right)^2, \qquad a_3 = \frac{2\delta L^4}{\delta^2 - 3J_2L^2 + 2\delta\beta L^6}, \tag{29}$$

together with $b_1 = a_1$, $b_2 = a_2$ and $b_3 = a_3$. Moreover, in order that these equilibria make sense as circular non-equatorial motions one needs that $|a_3| < L$ which implies that

$$\begin{aligned} &\mathrm{Min}\left\{\left|\frac{-\delta^2 + 3J_2L^2 - 2\delta L^3}{2\delta L^6}\right|, \left|\frac{-\delta^2 + 3J_2L^2 + 2\delta L^3}{2\delta L^6}\right|\right\} \\ &\quad < \beta < \mathrm{Max}\left\{\left|\frac{-\delta^2 + 3J_2L^2 - 2\delta L^3}{2\delta L^6}\right|, \left|\frac{-\delta^2 + 3J_2L^2 + 2\delta L^3}{2\delta L^6}\right|\right\}. \end{aligned}$$

We stress that when β reaches one of the two extremes above, the equilibria become circular equatorial. An outstanding conclusion is that there is not an isolated equilibrium point related with circular non-equatorial motions on $S^2 \times S^2$.

We apply the change (14) to get the normal form of the Hamiltonian around the equilibrium, then we compute the 2-jet in terms of x_1, x_2, y_1 and y_2. Next, the linear terms vanish provided that $N_0 = 2\delta L^4/(\delta^2 - 3J_2L^2 + 2\delta\beta L^6)$, compatible with the value obtained in the previous paragraph as $N_0 = (a_3 + b_3)/2 \equiv a_3$. Dropping constant terms one gets

$$\bar{\mathcal{H}} = c_1(\delta, \beta, J_2, L)x_1^2 + c_2(\delta, \beta, J_2, L)y_1^2 + c_3(\delta, \beta, J_2, L)y_2^2 + \cdots,$$

for concrete (and big) values c_1, c_2 and c_3. The coordinates x_i's, y_i's are defined in (13). Thus, circular non-equatorial trajectories are equilibrium points when $N \equiv N_0$. The inclination of the circular orbits with respect to the equatorial plane is given through the identity $\cos I = N_0/L$. When $c_3 \neq 0$, the normal form has a non-null nilpotent term, thence, the relative equilibrium would be unstable. Even when $c_3 = 0$, if $c_1c_2 < 0$ the equilibrium would be unstable. So, linear stability is possible if and only if $c_3 = 0$ and $c_1c_2 > 0$. A refined analysis of the stability and bifurcations of circular solutions needs to be made in the twice-reduced space, and we have performed it in [24].

Besides, the determinant of the Jacobian matrix associated with the quadratic part of $\bar{\mathcal{H}}$ is identically zero, hence the equilibrium is not isolated.

We have also calculated the second-order averaged Hamiltonian through Lie transformations, that is, the terms of order two of (22), with the aim of clarifying if the equilibria of circular non-equatorial type are isolated. The computations become huge but we could conclude that there are not isolated equilibria of this type.

We cannot obtain more information from the normal form of above. However, it is not a surprise if we compare this result with the one given in [24]. Indeed, studying the set of circular motions in the twice-reduced space, we deduce that the circular non-equatorial trajectories are represented by an isolated equilibrium point of this space. Thus, the continuum of equilibria given in (29) get transformed into an isolated equilibrium point of the twice-reduced space. We should take into account that, in order that the linear part of the 2-jet got vanished, we needed to specify a value for N_0 but ν_0 can get any value, which reflects the fact that there is an infinite number of equilibria.

5.3. Non-circular equatorial solutions

We seek now equilibria of the form $(a_1, a_2, a_3, -a_1, -a_2, a_3)$ with $a_1^2 + a_2^2 + a_3^2 = L^2$. Proceeding as in the previous subsection we arrive at a continuum of equilibria, also a circle

$$a_1^2 + a_2^2 = L^2 - a_3(\delta, J_2, L)^2$$

where a_3 is a specific value of the parameters δ, J_2 and L. In particular

$$a_3 = \frac{C^{2/3} - C^{1/3}(\delta^2 + 2J_2L^2) + (\delta^2 + 2J_2L^2)^2}{8\delta L^2 C^{1/3}}, \tag{30}$$

with

$$C = -\delta^6 - 8J_2^3L^6 - 6\delta^4L^2\left(J_2 - 80L^4\right) - 12\delta^2J_2^2L^4 + 8\sqrt{15}\delta^2L^3\sqrt{-\delta^6 - 12\delta^2J_2^2L^4 - 8J_2^3L^6 - 6\delta^4L^2\left(J_2 - 40L^4\right)},$$

with some conditions among the parameters so that $|a_3| < L$ is ensured.

Applying the changes (17) and (18) to Hamiltonian (22) and calculating the 2-jets, we get that the linear terms vanish if and only if

$$3\delta^2G_0^2 - 15\delta^2L^2 \pm 8\delta L^2G_0^3 + 6J_2L^2G_0^2 = 0. \tag{31}$$

The positive sign corresponds with prograde trajectories whereas the negative one is related with retrograde motions. The positive roots G_0 correspond with the possible values related with non-circular equatorial prograde and retrograde motions. These motions have eccentricity $\sqrt{1 - G_0^2/L^2}$.

After a careful look at the polynomial (31) and taking into account the restrictions on the parameters, it is easy to conclude that there are two possibilities: (i) if $N = G > 0$ and $\delta < 0$; (ii) if $N = -G < 0$ and $\delta > 0$. Then, there are valid roots G_0 such that the motions do not collide with the planet. The eccentricities of these orbits are moderate to high. Notice that for equatorial orbits, $N = (a_3 + b_3)/2 \equiv a_3$, the concrete value of a_3 in (30) is exactly the absolute value of the valid root G_0 for the polynomials (31).

Once the constant terms are removed, the 2-jets acquire the form

$$c_1^{\pm}(\delta, \beta, J_2, L)y_1^2 + c_2^{\pm}(\delta, \beta, J_2, L, k_0)x_2^2 + c_3^{\pm}(\delta, \beta, J_2, L, k_0)y_2^2 + c_4^{\pm}(\delta, \beta, J_2, L, k_0)x_2y_2,$$

where the coordinates x_i's, y_i's are defined in (15) for the prograde motions and in (16) for the retrograde ones. We do not print down the explicit expressions of $c_i^{\pm}$'s as they are too big to be reproduced here (but $c_4^+ \equiv c_4^-$). The value of k_0 is arbitrary. When $c_1^{\pm} \neq 0$, the normal form is nilpotent, thus, the non-circular equatorial orbits become unstable. However, if $c_1^{\pm} = 0$, the stability depends on the relative values of $c_2^{\pm}$, $c_3^{\pm}$ and $c_4^{\pm}$. The specific analysis on the stability and bifurcations of non-circular equatorial solutions is performed in the twice-reduced space, and it appears in the companion paper [24].

The determinant of the Jacobian matrix associated with the quadratic part of $\bar{\mathcal{H}}$ has been calculated and it yields identically zero, this is compatible with the fact that there is a continuum of non-circular equatorial equilibria.

The second-order averaged Hamiltonian does not yield isolated equilibria of non-circular equatorial type. As in the previous subsection, we cannot obtain more information from the normal form of above. Anyway, analysing the set of non-circular equatorial motions in the twice-reduced space [24], we deduce that these trajectories appear as an isolated equilibrium point on this space. So, the continuum of equilibria in (30) get transformed into an isolated equilibrium point of the twice-reduced space. We remark that in the normal form given above, G_0 had a concrete value whereas k_0 can have any value, which is compatible with the feature of an infinite number of equilibria.

5.4. Non-circular non-equatorial solutions

Because of the cumbersome form of the Hamiltonian vector field in terms of a_i's and b_i's, it is not possible to discuss the possible existence of non-circular non-equatorial relative equilibria on $S^2 \times S^2$. Thus we resort to the analysis in Delaunay coordinates as they can be used without trouble. After making the change (20) to (22), we compute the Taylor-expansion of it around $x_1 = x_2 = y_1 = y_2 = 0$ up to degree two.

Examining the four coefficients of the linear part of the 2-jet we get that only one of them is identically zero. In particular, the coefficient factorised by x_1 yields

$$\frac{\delta(G_0^2 - N_0^2)(G_0 - L)(8\beta L^3 G_0^5 + \delta(L + G_0)^2)\sin(2g_0)}{8L^5 G_0^7 (L + G_0)}.$$

Noticing that $|N_0| < G_0 < L$ the only possibilities for this term to vanish are: (i) $8\beta L^3 G_0^5 + \delta(L + G_0)^2 = 0$; (ii) $\sin(2g_0) = 0$. When $\delta > 0$, the polynomial of (i) is strictly positive, thus it has no positive root G_0. If $\delta < 0$, it is possible to get a root $G_0 > 0$ but this happens when the parameters are physically meaningless. Thus, we discard option (i) and focus on the case (ii), and there are two subcases: (a) $\sin(2g_0) = 0$ and $\cos(2g_0) = -1$ or (b) $\sin(2g_0) = 0$ and $\cos(2g_0) = 1$. If we relate N_0 with G_0, in order to make the linear part of the 2-jet zero, we get for the case (a)

$$N_0 = \frac{8\delta L^2 G_0^4 (L + G_0)}{16\delta\beta L^4 G_0^5 - 12 J_2 L^2 G_0^2 (L + G_0) + \delta^2 (L + G_0)(7L^2 - 3G_0^2)},$$

whereas for (b) we get

$$N_0 = \frac{8\delta L^2 G_0^4 (L + G_0)}{16\delta\beta L^4 G_0^5 - 12 J_2 L^2 G_0^2 (L + G_0) + \delta^2 (L + G_0)(5L^2 - G_0^2)}.$$

Some restrictions have to be imposed to the parameters, for (a) and (b), so that $|N_0| < G_0 < L$ is ensured, thus avoiding to mix these equilibria with circular or equatorial motions.

Besides, G_0 has to take a concrete value in terms of the parameters. Indeed this value is given for (a) and (b) through a root of a polynomial of degree 18, respectively a polynomial of degree 16, that we do not print down explicitly. These polynomials have been obtained as the resultants of other polynomials. Then, for (a), G_0 has meaningful values provided that $\delta > 0$ while in case (b) there are realistic solutions for any $\delta \in [-10^{-2}, 10^{-2}]$.

Thus, g_0, N_0 and G_0 take always specific values, however ν_0 may get any value and the linear part of the 2-jet vanishes. It means that there is a continuum of equilibria of non-circular non-equatorial type.

After dropping the constant terms, the 2-jet takes the form

$$\bar{\mathcal{H}} = c_1(\delta, \beta, J_2, L)x_1^2 + c_2(\delta, \beta, J_2, L)y_1^2 + c_3(\delta, \beta, J_2, L)y_2^2 + c_4(\delta, \beta, J_2, L)y_1 y_2 + \cdots$$

where ν_0 is not present in the expressions (as it should be since it is an ignorable angle and N is an exact constant of motion of the problem) and the coordinates x_i's, y_i's are those given in (19). We can arrange $\bar{\mathcal{H}}$ by means of a linear symplectic change to obtain a simpler normal form. This change can be executed generically, excepting for some combinations of the coefficients c_i's. Calling the new coordinates (u_1, u_2, v_1, v_2), we get

$$\bar{\mathcal{H}} = d_1(\delta, \beta, J_2, L)u_1^2 + d_2(\delta, \beta, J_2, L)v_1^2 + d_3(\delta, \beta, J_2, L)v_2^2 + \cdots,$$

where the coefficients d_i's depend on the coefficients c_i's. Clearly, $\bar{\mathcal{H}}$ manifests to be nilpotent, therefore unstable for generic values of the parameters.

The determinant of the Jacobian matrix associated with $\bar{\mathcal{H}}$ vanishes for all d_i's, thus the non-circular non-equatorial equilibria cannot be isolated.

The second-order averaged Hamiltonian has been obtained, but it does not yield isolated equilibria for this situation. As previously, we cannot obtain more information from the normal form of above. The study of non-circular non-equatorial motions has to be completed in the twice-reduced space [24], where we have checked that the non-circular non-equatorial motions are represented as an isolated equilibrium point. This is equivalent to saying that the non-circular non-equatorial equilibria analysed on the space $S^2 \times S^2$ are independent of particular values ν_0.

5.5. Dynamics on $S^2 \times S^2$ coming from the twice-reduced space

The full analysis of the flow related to the twice-reduced space is treated in [24] where a discussion on the relative equilibria, stability, bifurcation lines and the connection with the original system is done with great detail.

We know that the sets of non-isolated relative equilibria analysed in Sections 5.2, 5.3 and 5.4 yield isolated equilibria on the twice-reduced space. However, there are other isolated relative equilibria on the twice-reduced space that do not manifest as continuum of equilibria on $S^2 \times S^2$. Indeed, they correspond with families of periodic solutions on $S^2 \times S^2$ which have not been treated along this paper. The existence and stability of these periodic solutions on $S^2 \times S^2$ reproduces mutatis mutandis that of the relative equilibria in the twice-reduced space. The reason is that the reduction responsible of passing to the twice-reduced space is exact as the third component of the angular momentum is a first integral of the problem, in other words, the argument of the node, when it is defined, is an ignorable coordinate of the vector field.

Thus, the rich dynamics occurring in the twice-reduced space is inherited for $S^2 \times S^2$ as follows: the relative equilibria of the twice-reduced space—where we have discarded the circular equatorial motions—are families of periodic solutions on $S^2 \times S^2$ depending on the parameter N. The stability, linear, non-linear and parametric, of these relative equilibria is inherited by the families of periodic solutions on $S^2 \times S^2$. In the twice-reduced space linear and parametric stability coincide because it is a system of one degree of freedom. The bifurcation lines of relative equilibria are exactly the same bifurcation lines of families of periodic solutions on $S^2 \times S^2$.

6. Reconstruction of the flow

The implications for the Hamiltonian vector field related with (21) can be drawn provided that the relative equilibria are isolated, so first of all we restrict ourselves to extract conclusions of the dynamics of (21) related with the circular equatorial solutions. The implications are indeed of three types: (i) families of periodic solutions connected with the relative equilibria of $S^2 \times S^2$; (ii) KAM tori around some of these equilibria; (iii) other dynamical consequences extracted from the flow on the twice-reduced space.

We outline the three outstanding features of the dynamics of the generalised Størmer problem.

6.1. Families of near circular equatorial periodic solutions

Related to each isolated relative equilibrium point of Section 5.1, there is a family of periodic solutions in $\mathbb{R}^6$ (or, more precisely, in the five-dimensional energy manifold once we fix the energy E) parameterised by L [35,28,1,36]. Besides, the linear stability of a periodic solution is guaranteed if the corresponding equilibrium is parametrically stable [36].

Thus, discarding the cases where $\beta \equiv \beta_c$ and the cases where $J_2 \equiv J_{2c}$ there are families of periodic solutions of (near) circular equatorial type in the original Hamiltonian. These periodic solutions have periods $T = 2\pi L^3 + \mathcal{O}(2)$, and if $c_1^{\pm}$ and $c_2^{\pm}$ are the eigenvalues appearing in (24), the characteristic multipliers of the periodic solutions become

$$\left\{1, 1, 1 + c_1^{\pm} T\imath + \mathcal{O}(2), 1 + c_2^{\pm} T\imath + \mathcal{O}(2), 1 - c_1^{\pm} T\imath + \mathcal{O}(2), 1 - c_2^{\pm} T\imath + \mathcal{O}(2)\right\}.$$

Notice that the small parameters are included in the coefficients $c_i^{\pm}$'s. These periodic solutions are elliptic—i.e., linearly-stable periodic orbits—provided that the relative equilibria are parametrically stable, see [36]. This is indeed the generic situation when the equilibria are isolated and they are not in resonance 1:−1.

So, we have concluded the following:

The Hamiltonian vector field of the generalised Størmer problem (21) *has near prograde and retrograde circular equatorial periodic solutions provided that* $\beta \neq \beta_c$ (*with* β_c *given in* (25) *or in* (28)) *and that* $J_2 \neq J_{2c}$ (*with* J_{2c} *given in* (26) *or in* (27)). *For meaningful values of the parameters, the prograde periodic solutions are elliptic whereas the retrograde periodic solutions are elliptic when the normal form is not in resonance* 1:−1, *i.e., if* $\beta \neq (5\delta + 2L^3)/(2L^6)$.

6.2. Families of KAM 3-tori related to the circular equatorial motions

We prove now the existence of invariant tori of KAM type related to the isolated equilibria of Section 5.1. We apply standard KAM theory [3] combined with reduction theory to get families of KAM tori around a relative equilibria in a reduced space. Concretely, we apply Theorem 2.5 of [36].

We need to take the 4-jet of the Hamiltonian (24), and apply to it the transformation to action-angle coordinates, introducing also a small "artificial" parameter ε through

$$\begin{aligned} x_1 &= \sqrt{\varepsilon}\sqrt{2I_1}\sin\varphi_1, & x_2 &= \sqrt{\varepsilon}\sqrt{2I_2}\sin\varphi_2,\\ y_1 &= \sqrt{\varepsilon}\sqrt{2I_1}\cos\varphi_1, & y_2 &= \sqrt{\varepsilon}\sqrt{2I_2}\cos\varphi_2, \end{aligned}$$

such that ε is supposed to be of the size of the small parameter of the problem. The reason to introduce ε in the scaling is that we need to perform an averaging process with respect to the angular coordinates φ_1 and φ_2 but without altering the size of the higher-order terms, that is, the size of the terms in $\mathcal{O}(2)$, thus we scale by $\sqrt{\varepsilon}$ instead of scaling by ε. The above transformation is symplectic with multiplier ε, thus we need to divide the Hamiltonian by ε, including the Kepler term $-1/(2L^2)$ and the terms inside $\mathcal{O}(2)$.

Thus, the quadratic terms of $\bar{\mathcal{H}}$ get transformed into

$$\bar{\mathcal{H}}^{(2)} = \frac{6\delta^2 - 3J_2L^2 \mp 4\delta L^3}{4L^9} I_1 + \frac{-\delta^2 + 3J_2L^2 + 2\delta L^3(\pm 1 - \beta L^3)}{4L^9} I_2. \tag{32}$$

The cubic terms of the 4-jet are identically zero and we do not print down the explicit expressions of the quartic terms as functions of the action-angle coordinates, as they are too long. Next, in order to put the quartic terms in normal form we need to average them with respect to φ_1 and φ_2, arriving at

$$\begin{aligned} \bar{\mathcal{H}}^{(4)} = \frac{\varepsilon}{2L^{10}}\big(&(39\delta^2 - 12J_2L^2 \mp 12\delta L^3)I_1^2 \\ &+ \left(-14\delta^2 + 24J_2L^2 \pm 12\delta L^3 - 4\delta\beta L^6\right)I_1I_2 \\ &+ \left(\delta^2 - 3J_2L^2 + 2\delta\beta L^6\right)I_2^2\big). \end{aligned} \tag{33}$$

The generating function responsible to bring the quartic terms to normal form is

$$\mathcal{W}^{(4)} = \frac{\varepsilon(\delta + 2\beta L^6)}{4(\mp L^4 + \beta L^7) - 10\delta L} I_1 I_2 \sin\bigl(2(\varphi_1 + \varphi_2)\bigr),$$

or

$$\mathcal{W}^{(4)} = \frac{\varepsilon\delta(\delta + 2\beta L^6)}{2(-7\delta^2 L + 6J_2 L^3 + 2\delta L^4(\pm 3 - \beta L^3))} I_1 I_2 \sin\bigl(2(\varphi_1 - \varphi_2)\bigr).$$

There are two different ways to express $\mathcal{W}^{(4)}$, that depend on the quadrants where the angles g and ν are located when they are defined, accordingly with the expressions given in (11) and (12). Thus, out of the eight possible expressions, the quartic terms previous to the normal form transformation get reduced to two different ones whereas the expressions of the quadratic terms get reduced to one. Notice, however, that there is only one expression of the quartic terms in normal form.

The generating function $\mathcal{W}^{(4)}$ is well defined and is periodic in φ_1 and φ_2 excepting when the denominators of the above expressions become zero and this happens for $\beta = (5\delta \mp 2L^3)/(2L^6)$ and $\beta = (-7\delta^2 + 6J_2L^2 \pm 6\delta L^3)/(2\delta L^6)$, which correspond respectively to the resonances 1:−1 and 1:1 (the upper signs are related to prograde orbits whereas the lower signs to retrograde orbits). At this point we have to recall that for prograde motions the resonance 1:1 is allowed while the resonance 1:−1 is not, while for retrograde motions the resonance 1:−1 is always allowed but the resonance 1:1 may occur only when $\delta > 0$. When we face with a resonant situation and with physically meaningful parameters, it is not possible to average the two angles φ_1 and φ_2 and the normal form would be different, but then we cannot apply KAM theory.

After recovering the Keplerian part, undoing the scaling by $I_1 \to \varepsilon^{-1} I_1$, $I_2 \to \varepsilon^{-1} I_2$, dividing the whole Hamiltonian by the multiplier ε^{-1}, the resulting Hamiltonian function valid in the neighbourhoods of the relative equilibria on $S^2 \times S^2$ reads as

$$\mathcal{H} = -\frac{1}{2L^2} + \bar{\mathcal{H}}^{(2)}(I_1, I_2) + \frac{1}{2}\bar{\mathcal{H}}^{(4*)}(I_1, I_2) + \mathcal{O}(2) + \cdots,$$

where $\bar{\mathcal{H}}^{(4*)} = \varepsilon^{-1}\bar{\mathcal{H}}^{(4)}$ and $\bar{\mathcal{H}}^{(2)}$ and $\bar{\mathcal{H}}^{(4)}$ are respectively given in (32) and in (33) explicitly. We stress that the terms of $\mathcal{O}(2)$ are transformed by the change of coordinates related with the averaging process that can be computed by means of $\mathcal{W}^{(4)}$, however the resulting terms are of the same relative size as the ones previous to the transformations, so they are kept in $\mathcal{O}(2)$.

Now we compute the determinant of the Hessian

$$\begin{pmatrix} \frac{\partial^2 \bar{\mathcal{H}}^{(4*)}}{\partial I_1^2} & \frac{\partial^2 \bar{\mathcal{H}}^{(4*)}}{\partial I_1 \partial I_2} \\ \frac{\partial^2 \bar{\mathcal{H}}^{(4*)}}{\partial I_2 \partial I_1} & \frac{\partial^2 \bar{\mathcal{H}}^{(4*)}}{\partial I_2^2} \end{pmatrix},$$

arriving at the expression

$$D = -\frac{1}{L^{20}}\bigl(10\delta^4 - 39\delta^2 J_2 L^2 \mp 72\delta^3 L^3 + 108 J_2^2 L^4 \pm 108\delta J_2 L^5 + 36\delta^2 L^6 - 50\delta^3 \beta L^6 - 24\delta\beta J_2 L^8 + 4\delta^2\beta^2 L^{12}\bigr).$$

The determinant D does not vanish for $N \geqslant 0$ (i.e., for the sign "+") for physically meaningful values of the parameters whereas it can vanish for $N < 0$ (with the sign "−") for some specific values. We conclude that, generically, $D \neq 0$.

Thus, we have proved that:

Excepting the possible cases of resonance 1:1 *and* 1:−1*, following the discussions of Section* 5.1 *and the above paragraphs, the Hamiltonian vector field of the generalised Størmer problem* (21) *have families of invariant KAM* 3*-tori around the families of periodic solutions of circular equatorial type established in Section* 6.1*. These families of tori exist excepting for a few combinations of the parameters of the problem.*

6.3. Other invariant tori and bifurcations of the generalised Størmer problem

The dynamics on $S^2 \times S^2$ related with the families of periodic solutions in Section 5.5 has to be interpreted in terms of the original Hamiltonian vector field (21). We arrive at the following description.

Families of periodic solutions of $S^2 \times S^2$ are reconstructed as families of invariant 2*-tori corresponding with Hamiltonian* (21)*, that depend on the actions L and N. The parametrically stable periodic solutions are transformed into linearly stable* 2*-tori in the original system while the unstable periodic orbits of $S^2 \times S^2$ are converted into unstable invariant* 3*-tori of* (21)*. The bifurcation lines of periodic solutions get transformed into bifurcation lines of invariant* 2*-tori in the original system, maybe a bit distorted. A complete description on this appears in* [24].

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