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The Application of an Inverse-Free Jarratt-Type Approximation to Nonlinear Integral Equations of Hammerstein-Type

J. A. EZQUERRO, J. M. GUTIÉRREZ
M. A. HERNÁNDEZ AND M. A. SALANOVA
Department of Mathematics and Computation, University of La Rioja
C/ Luis de Ulloa s/n, 26004 Logroño, Spain
jezquer@dmc.unirioja.es

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Abstract—We consider an inverse-free Jarratt-type approximation, whose order of convergence is four, for solving nonlinear equations. The convergence of this method is analysed under two different types of conditions. We use a new technique based on constructing a system of real sequences. Finally, this method is applied to the study of Hammerstein's integral equations. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Many scientific and engineering problems can be brought in the form of a nonlinear equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on a convex subset Ω of a Banach space X with values in another Banach space Y. The resolution of equation (1) by iterative methods has been studied by many authors for a long time, see [1–5]. The study of this methods is habitually based on the well-known Kantorovich-type conditions [6].

The multipoint methods are defined as iterations which use new information at a number of points. In [7], it is imposed that the restriction on one-point iteration of order N that they must depend explicitly on the first $N-1$ derivatives of F. This implies that their informational efficiency is less than or equal to unity. Those restrictions are relieved in only small measure by turning to one-point iterations with memory.

Neither of these restrictions need hold for multipoint methods, that is, for iterations which sample F and its derivatives at a number of values of the independent variable.

A natural generalization of the Newton method is to apply a multipoint scheme [7]. Suppose that we already have the expressions of $F(x_k)$, $F'(x_k)$, and $F'(x_k)^{-1} = \Gamma_k$ at a current step x_k.

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In order to increase the order of convergence without evaluating the second Fréchet derivative, we can add one more evaluation of $F'(c_1x_k + c_2y_k)^{-1}$, where c_1 and c_2 are real constants that are independent of x_k and y_k is generated by a Newton step. Ostrowski [8] and Traub [7] developed two-point schemes of this type for functions of one variable.

According to this idea, Argyros, Chen and Qian (see [9]) give the following new multipoint iteration of order four in Banach spaces:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ G(x_n, y_n) &= F'(x_n)^{-1}\left[F'\left(x_n + \frac{2}{3}(y_n - x_n)\right) - F'(x_n)\right], \\ x_{n+1} &= y_n - \frac{3}{4}G(x_n, y_n)\left[I - \frac{3}{2}G(x_n, y_n)\right](y_n - x_n), \qquad n \geq 0. \end{aligned} \tag{2}$$

First of all, we analyse, under certain assumptions of the pair (F, x_0), the convergence of (2) to a unique zero x^* of (1) by using a new technique consisting of a system of real sequences which satisfy some recurrence relations.

Second, as a consequence of the previous study, we obtain a result on the existence-uniqueness of solutions for the Hammerstein nonlinear equations [10]. The convergence result mentioned above provides a result of this type, but the operator F must satisfy several conditions as a consequence of iteration (2) has R-order of convergence four. We then study the convergence of (2) by a Newton-Kantorovich theorem under milder conditions for the operator F. After that, a new existence-uniqueness result of solutions of equation (2) is given.

Finally, we study a particular Hammerstein equation, which has been already studied by Werner [11]. We compare the results obtained applying iteration (2) with the ones given by Werner for other iterative methods with a similar operational cost. We shall see that better approximations to the discretized solution of this Hammerstein equation are obtained.

We denote $\overline{B(x,r)} = \{y \in X;\ \|y - x\| \leq r\}$ and $B(x,r) = \{y \in X;\ \|y - x\| < r\}$.

2. CONVERGENCE ANALYSIS

We analyse the convergence of (2) to a solution x^* of (1). Let $x_0 \in \Omega$ and suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X.

We assume the following assumptions:

(A_1) $\|\Gamma_0\| \leq \beta$,
(A_2) $\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta$,
(A_3) $\|F''(x)\| \leq M$, $x \in \Omega$,
(A_4) $\|F'''(x)\| \leq N$, $x \in \Omega$,
(A_5) $\|F'''(x) - F'''(y)\| \leq L\|x - y\|$, $x, y \in \Omega$, $L \geq 0$.

Let us denote $a_0 = M\beta\eta$, $b_0 = N\beta\eta^2$, and $c_0 = L\beta\eta^3$. We define the sequences

$$a_{n+1} = a_n f(a_n)^2 g(a_n, b_n, c_n), \tag{3}$$

$$b_{n+1} = b_n f(a_n)^3 g(a_n, b_n, c_n)^2, \tag{4}$$

and

$$c_{n+1} = c_n f(a_n)^4 g(a_n, b_n, c_n)^3, \tag{5}$$

where

$$f(x) = \frac{2}{2 - 2x - x^2 - x^3} \tag{6}$$

and

$$g(x, y, z) = \frac{1}{216}\left(27x^3\left(x^2 + 2x + 5\right) + 18xy + 17z\right). \tag{7}$$

First, we observe that taking into account initial Hypotheses (A_1)–(A_5) and assuming that $y_0 \in \Omega$, we have

$$\|G(x_0, y_0)\| \le \|\Gamma_0\| \|F'\left(x_0 + \frac{2}{3}(y_0 - x_0)\right) - F'(x_0)\| \le \frac{2}{3}\beta M\eta = \frac{2}{3}a_0.$$

Then the iterate x_1 is well defined and

$$\begin{aligned}\|x_1 - x_0\| &\le \left\| I - \frac{3}{4}G(x_0, y_0)\left[I - \frac{3}{2}G(x_0, y_0)\right]\right\| \|y_0 - x_0\| \\ &\le \left(1 + \frac{a_0}{2}(1 + a_0)\right)\|y_0 - x_0\|.\end{aligned}$$

Next we prove the following items are true for all $n \ge 1$, by mathematical induction:

(I_n) $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \le f(a_{n-1})\|\Gamma_{n-1}\|$,
(II_n) $\|y_n - x_n\| = \|\Gamma_n F(x_n)\| \le f(a_{n-1})g(a_{n-1}, b_{n-1}, c_{n-1})\|y_{n-1} - x_{n-1}\|$,
(III_n) $\|G(x_n, y_n)\| \le (2/3)M\|\Gamma_n\|\|y_n - x_n\| \le (2/3)a_n$,
(IV_n) $N\|\Gamma_n\|\|y_n - x_n\|^2 \le b_n$,
(V_n) $L\|\Gamma_n\|\|y_n - x_n\|^3 \le c_n$,
(VI_n) $\|x_{n+1} - x_n\| \le (1 + (a_n/2)(1 + a_n))\|y_n - x_n\|$.

We now assume $x_n, y_n \in \Omega$, for all $n \ge 0$. The proof of that is given in Theorem 2.4. If we suppose

$$a_n\left(1 + \frac{a_n}{2}(1 + a_n)\right) < 1, \qquad n \ge 0, \tag{8}$$

then we have

$$\begin{aligned}\|I - \Gamma_0 F'(x_1)\| &\le \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \le M\|\Gamma_0\|\|x_1 - x_0\| \\ &\le a_0\left(1 + \frac{a_0}{2}(1 + a_0)\right) < 1.\end{aligned}$$

Hence, Γ_1 is defined and $\|\Gamma_1\| \le f(a_0)\|\Gamma_0\|$.

Using Taylor's formula and (2), we get (see [9])

$$\begin{aligned}F(x_1) &= F(y_0) + F'(y_0)(x_1 - y_0) + \int_{y_0}^{x_1} F''(x)(x_1 - x)\,dx \\ &= \int_0^1 F''(y_0 + t(x_1 - y_0))(1 - t)\,dt(x_1 - y_0)^2 \\ &\quad + \int_0^1 \left[F''(x_0 + t(y_0 - x_0))(1 - t) - \frac{1}{2}F''\left(x_0 + \frac{2}{3}t(y_0 - x_0)\right)\right] dt(y_0 - x_0)^2 \\ &\quad - \frac{3}{4}\int_0^1 \left[F''(x_0 + t(y_0 - x_0)) - F''\left(x_0 + \frac{2}{3}t(y_0 - x_0)\right)\right] dt(y_0 - x_0)G(x_0, y_0)(y_0 - x_0) \\ &\quad + \frac{9}{8}\int_0^1 F''(x_0 + t(y_0 - x_0))\,dt(y_0 - x_0)G(x_0, y_0)G(x_0, y_0)(y_0 - x_0).\end{aligned}$$

So

$$\begin{aligned}\|y_1 - x_1\| = \|\Gamma_1 F(x_1)\| &\le \|\Gamma_1\|\,\|F(x_1)\| \le f(a_0)\|\Gamma_0\|\,\|F(x_1)\| \\ &\le f(a_0)g(a_0, b_0, c_0)\|y_0 - x_0\|,\end{aligned}$$

and (II_1) is therefore true. To show (III_1)–(V_1) we, respectively, note that

$$\begin{aligned}\|G(x_1, y_1)\| \le \frac{2}{3}M\|\Gamma_1\|\,\|y_1 - x_1\| &\le \frac{2}{3}M\|\Gamma_0\|\,\|y_0 - x_0\|f(a_0)^2 g(a_0, b_0, c_0) \le \frac{2}{3}a_1, \\ N\|\Gamma_1\|\,\|y_1 - x_1\|^2 &\le N\|\Gamma_0\|\,\|y_0 - x_0\|^2 f(a_0)^3 g(a_0, b_0, c_0)^2 \le b_1,\end{aligned}$$

and

$$L\|\Gamma_1\|\,\|y_1 - x_1\|^3 \le L\|\Gamma_0\|\,\|y_0 - x_0\|^3 f(a_0)^4 g(a_0, b_0, c_0)^3 \le c_1.$$

Finally, we easily deduce

$$\|x_2 - x_1\| \le \left(1 + \frac{a_1}{2}(1 + a_1)\right) \|y_1 - x_1\|.$$

Now if we suppose that (I_n)–(VI_n) are true for a fixed $n \ge 1$, we can prove (I_{n+1})–(VI_{n+1}) by induction.

Our next goal is to analyse real sequences (3)–(5) to obtain the convergence of sequence (2) defined in Banach spaces. So as to obtain the convergence of (2), we only have to prove (2) is a Cauchy sequence and the above assumption (8). First, we provide a technical lemma whose proof is trivial.

LEMMA 2.1. *Let f and g be two real functions given in (6) and (7), respectively. Then*

(i) *f is increasing and $f(x) > 1$ in $(0, 1/2)$,*
(ii) *g is increasing in its three arguments for $x \in (0, 1/2)$, $y > 0$, and $z > 0$,*
(iii) *$f(\gamma x) < f(x)$ and $g(\gamma x, \gamma^2 y, \gamma^3 z) < \gamma^3 g(x, y, z)$, for $\gamma \in (0, 1)$.*

Some properties for the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ given, respectively, by (3)–(5) are now shown.

LEMMA 2.2. *Let f and g be two real functions given by (6) and (7), respectively. Let*

$$p(x) = 27(1 - x)(1 - 2x)\left(x^2 + x + 2\right)\left(x^2 + 2x + 4\right). \tag{9}$$

If $a_0 \in (0, 1/2)$ and $17c_0 + 18a_0b_0 < p(a_0)$, then

(i) *$f(a_0)^2 g(a_0, b_0, c_0) < 1$,*
(ii) *the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are decreasing,*
(iii) *$a_n\left(1 + (a_n/2)(1 + a_n)\right) < 1$, for all $n \ge 0$.*

PROOF. From the hypotheses, item (i) follows immediately. We show item (ii) by mathematical induction on n. The facts that $0 < a_1 < a_0$, $0 < b_1 < b_0$, and $0 < c_1 < c_0$ follow by previous item (i) and Lemma 2.1(i). Next, it is supposed that $a_j < a_{j-1}$, $b_j < b_{j-1}$, and $c_j < c_{j-1}$ for $j = 1, 2, \ldots, n$. Then,

$$a_{n+1} = a_n f(a_n)^2 g(a_n, b_n, c_n) < a_n f(a_0)^2 g(a_0, b_0, c_0) < a_n,$$

since f is increasing and g is also increasing in its three arguments. We have

$$b_{n+1} = b_n f(a_n)^3 g(a_n, b_n, c_n)^2 = b_n \frac{f(a_n)^4 g(a_n, b_n, c_n)^2}{f(a_n)} < b_n$$

and

$$c_{n+1} = c_n f(a_n)^4 g(a_n, b_n, c_n)^3 = c_n \frac{f(a_n)^6 g(a_n, b_n, c_n)^3}{f(a_n)^2} < c_n,$$

by the same reasoning as before and the fact that $f(x) > 1$ in $(0, 1/2)$.

Finally, we have

$$a_n\left(1 + \frac{a_n}{2}(1 + a_n)\right) < a_0\left(1 + \frac{a_0}{2}(1 + a_0)\right) < 1,$$

for all $n \ge 0$, since $\{a_n\}$ is a decreasing sequence and $a_0 \in (0, 1/2)$. ∎

LEMMA 2.3. *Let us suppose the hypotheses of Lemma 2.2 and define* $\gamma = a_1/a_0$. *Then*

(i$_n$) $a_n < \gamma^{4^{n-1}} a_{n-1} < \gamma^{(4^n-1)/3} a_0$, $b_n < (\gamma^{4^{n-1}})^2 b_{n-1} < \gamma^{2/3(4^n-1)} b_0$, and $c_n < (\gamma^{4^{n-1}})^3 c_{n-1} < \gamma^{4^n-1} c_0$, *for all* $n \geq 2$,

(ii$_n$) $f(a_n)g(a_n, b_n, c_n) < \gamma^{4^n-1} f(a_0)g(a_0, b_0, c_0) = (\gamma^{4^n}/f(a_0))$, *for all* $n \geq 1$.

PROOF. We prove (i$_n$) following an inductive procedure. As $a_1 = \gamma a_0$, we have $b_1 = b_0 f(a_0)^3 g(a_0, b_0, c_0)^2 < \gamma^2 b_0$ and $c_1 = c_0 f(a_0)^4 g(a_0, b_0, c_0)^3 < \gamma^3 c_0$, if and only if $f(a_0) > 1$, and by Lemma 2.1 the result holds. If we suppose that (i$_n$) is true, then

$$
\begin{aligned}
a_{n+1} &= a_n f(a_n)^2 g(a_n, b_n, c_n) \\
&< \gamma^{4^{n-1}} a_{n-1} f\left(\gamma^{4^{n-1}} a_{n-1}\right)^2 g\left(\gamma^{4^{n-1}} a_{n-1}, \left(\gamma^{4^{n-1}}\right)^2 b_{n-1}, \left(\gamma^{4^{n-1}}\right)^3 c_{n-1}\right) \\
&< \gamma^{4^{n-1}} a_{n-1} f(a_{n-1})^2 \left(\gamma^{4^{n-1}}\right)^3 g(a_{n-1}, b_{n-1}, c_{n-1}) = \gamma^{4^n} a_n.
\end{aligned}
$$

We also have

$$
\begin{aligned}
b_{n+1} &= b_n f(a_n)^3 g(a_n, b_n, c_n)^2 < \left(\frac{a_{n+1}}{a_n}\right)^2 b_n, \\
c_{n+1} &= c_n f(a_n)^4 g(a_n, b_n, c_n)^3 < \left(\frac{a_{n+1}}{a_n}\right)^3 c_n,
\end{aligned}
$$

if and only if

$$
\begin{aligned}
a_n^2 f(a_n)^3 g(a_n, b_n, c_n)^2 &< a_{n+1}^2 = a_n^2 f(a_n)^4 g(a_n, b_n, c_n)^2, \\
a_n^3 f(a_n)^4 g(a_n, b_n, c_n)^3 &< a_{n+1}^3 = a_n^3 f(a_n)^6 g(a_n, b_n, c_n)^3.
\end{aligned}
$$

These conditions are true since $f(a_n) > 1$. Now $b_{n+1} < (\gamma^{3^n})^2 b_n$ and $c_{n+1} < (\gamma^{4^n})^3 c_n$, since $a_{n+1}/a_n < \gamma^{3^n}$. Moreover,

$$
\begin{aligned}
a_n &< \gamma^{4^{n-1}} a_{n-1} < \gamma^{4^{n-1}} \gamma^{4^{n-2}} a_{n-2} < \cdots < \gamma^{(4^n-1)/3} a_0, \\
b_n &< \left(\gamma^{4^{n-1}}\right)^2 b_{n-1} < \left(\gamma^{4^{n-1}}\right)^2 \left(\gamma^{4^{n-2}}\right)^2 b_{n-2} < \cdots < \gamma^{(2/3)(4^n-1)} b_0,
\end{aligned}
$$

and

$$
c_n < \left(\gamma^{4^{n-1}}\right)^3 c_{n-1} < \left(\gamma^{4^{n-1}}\right)^3 \left(\gamma^{4^{n-2}}\right)^3 c_{n-2} < \cdots < \gamma^{4^n-1} c_0.
$$

On the other hand, we observe that

$$
\begin{aligned}
f(a_n)g(a_n, b_n, c_n) &< f\left(\gamma^{(4^n-1)/3} a_0\right) g\left(\gamma^{(4^n-1)/3} a_0, \gamma^{(2/3)(4^n-1)} b_0, \gamma^{4^n-1} c_0\right) \\
&< \gamma^{4^{n-1}} f(a_0) g(a_0, b_0, c_0) = \frac{\gamma^{4^n}}{f(a_0)}, \qquad n \geq 1.
\end{aligned}
$$

The proof is complete. ∎

After that, we show the following result on the convergence of sequence (2).

THEOREM 2.4. *Let* X, Y *be Banach spaces and* $F : \Omega \subseteq X \to Y$ *be a nonlinear three times Fréchet differentiable operator in an open convex domain* Ω. *Let us assume that* $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ *exists at some* $x_0 \in \Omega$ *and* (A_1)*–*(A_5) *are satisfied. Let us denote* $a_0 = M\beta\eta$, $b_0 = N\beta\eta^2$, *and* $c_0 = L\beta\eta^3$. *Under the conditions of Lemma 2.2, if* $\overline{B(x_0, R\eta)} \subseteq \Omega$, *where* $R = (1 + (a_0/2)(1 + a_0))/(1 - \gamma\Delta)$, $\gamma = a_1/a_0$, *and* $\Delta = 1/f(a_0)$, *the sequence* $\{x_n\}$ *defined in (2) has at least R-order of convergence four, and starting at* x_0 *converges to a solution* x^* *of (1). The* *solution x^* and the iterates x_n and y_n belong to $\overline{B(x_0, R\eta)}$. Moreover, the solution x^* is unique in $B(x_0, 2/(M\beta) - R\eta) \cap \Omega$. Furthermore, the following error bounds are obtained:*

$$\|x^* - x_n\| \le \left(1 + \frac{a_0}{2}\gamma^{(4^n-1)/3}\left(1 + a_0\gamma^{(4^n-1)/3}\right)\right)\left(\gamma^{(4^n-1)/3}\right)\frac{\Delta^n}{1-\gamma^{4^n}\Delta}\eta, \qquad n \ge 0. \tag{10}$$

PROOF. First, we prove that $\{x_n\}$ is a Cauchy sequence. From (II_n), we observe that

$$\begin{aligned}\left(1 + \frac{a_n}{2}(1+a_n)\right)\|y_n - x_n\| &\le \left(1 + \frac{a_0}{2}(1+a_0)\right) f(a_{n-1}) g(a_{n-1}, b_{n-1}, c_{n-1}) \|y_{n-1} - x_{n-1}\| \\ &\le \cdots \le \left(1 + \frac{a_0}{2}(1+a_0)\right)\|y_0 - x_0\| \prod_{j=0}^{n-1} f(a_j) g(a_j, b_j, c_j).\end{aligned}$$

We now have, from Lemma 2.3,

$$\prod_{j=0}^{n-1} f(a_j) g(a_j, b_j, c_j) \le \prod_{j=0}^{n-1}\left(\gamma^{4^j}\Delta\right) = \gamma^{(4^n-1)/3}\Delta^n,$$

where $\gamma = a_1/a_0 < 1$ and $\Delta = 1/f(a_0) < 1$. So, for $m \ge 1$ and $n \ge 1$,

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\le \left(1 + \frac{a_{n+m-1}}{2}(1 + a_{n+m-1})\right)\eta \prod_{j=0}^{n+m-2} f(a_j) g(a_j, b_j, c_j) \\ &\quad + \cdots + \left(1 + \frac{a_n}{2}(1+a_n)\right)\eta \prod_{j=0}^{n-1} f(a_j) g(a_j, b_j, c_j) \\ &< \left(1 + \frac{a_n}{2}(1+a_n)\right)\left(\gamma^{(4^{n+m-1}-1)/3}\Delta^{n+m-1} + \cdots + \gamma^{(4^n-1)/3}\Delta^n\right)\eta \\ &\le \left(1 + \frac{a_0}{2}\gamma^{(4^n-1)/3}\left(1 + a_0\gamma^{(4^n-1)/3}\right)\right)\gamma^{(4^n-1)/3}\Delta^n \frac{1 - \gamma^{(4^n(4^{m-1}+2))/3}\Delta^m}{1 - \gamma^{4^n}\Delta}\eta,\end{aligned} \tag{11}$$

since $\gamma^{(4^i+4^n 3)/3} \ge \gamma^{(4^{i+1})/3}$ for $i = n, n+1, \dots, n+m-1$. For $m \ge 1$ and $n = 0$, we obtain

$$\|x_m - x_0\| < \left(1 + \frac{a_0}{2}(1+a_0)\right)\frac{1 - \gamma^{(4(4^{m-1}-1))/3}\Delta^{m-1}}{1 - \gamma\Delta}\eta < R\eta. \tag{12}$$

Thus, $\{x_n\}$ is a Cauchy sequence.

On the other hand, from (12), it follows that $x_m \in B(x_0, R\eta)$, for all $m \ge 0$. We similarly have that $y_n \in B(x_0, R\eta)$, for all $n \ge 0$.

To see that x^* is a solution of (1), we have $\|\Gamma_n F(x_n)\| \to 0$ as $n \to \infty$. Taking into account that $\|F(x_n)\| \le \|F'(x_n)\|\,\|\Gamma_n F(x_n)\|$ and the sequence $\{\|F'(x_n)\|\}$ is bounded, we infer that $\|F(x_n)\| \to 0$ as $n \to \infty$. Consequently, we obtain $F(x^*) = 0$ by the continuity of F.

To prove the uniqueness, let us assume some other solution z^* of (1) in $B(x_0, 2/(M\beta) - R\eta) \cap \Omega$. From the approximation

$$0 = F(z^*) - F(x^*) = \int_0^1 F'(x^* + t(z^* - x^*))\,dt\,(z^* - x^*),$$

we have to prove that the operator $\int_0^1 F'(x^* + t(z^* - x^*))\,dt$ is invertible and then $z^* = x^*$. Indeed, from

$$\begin{aligned}\|\Gamma_0\| \int_0^1 \|F'(x^* + t(z^* - x^*)) - F'(x_0)\|\,dt &\le M\beta \int_0^1 \|x^* + t(z^* - x^*) - x_0\|\,dt \\ &\le M\beta \int_0^1 \left((1-t)\|x^* - x_0\| + t\|z^* - x_0\|\right) dt < 1,\end{aligned}$$

it follows that $[\int_0^1 F'(x^* + t(z^* - x^*))\,dt]^{-1}$ exists.

Finally, by letting $m \to \infty$ in (11) and (12), we obtain (10), for all $n \geq 0$. In addition, from (10), it follows that the R-order of convergence [12] of (2) is at least four, since

$$\|x^* - x_n\| \leq \left(1 + \frac{a_0}{2}(1 + a_0)\right) \frac{\eta}{\gamma^{1/3}(1 - \gamma\Delta)} \left(\gamma^{1/3}\right)^{4^n}, \qquad n \geq 0.$$

The proof is complete. ∎

We apply our new technique of convergence analysis to the following integral equation.

EXAMPLE. Let

$$[F(x)](s) = 1 - x(s) + \frac{x(s)}{4} \int_0^1 \frac{s}{s+t} x(t)\, dt$$

be the H-equation called integral equation of Chandrasekhar and studied by Argyros, Chen and Qian in [9]. We consider the space $X = C[0,1]$ of all continuous functions on the interval $[0,1]$ with the norm

$$\|x\| = \max_{s \in [0,1]} |x(s)|.$$

Let $x_0 = x_0(s) = 1$. Then the parameters appearing in Theorem 2.4 are (see [9])

$$M = 0.3465, \quad N = 0 = L, \quad \beta = 1.5304, \quad \text{and} \quad \eta = 0.2652.$$

In addition, $a_0 = 0.1406312$ and $b_0 = 0 = c_0$. Consequently, the hypotheses of Theorem 2.4 are satisfied, and therefore, the integral equation of Chandrasekhar has a solution $x^*(s)$ in $\{\psi \in C([0,1])/\|\psi - 1\| \leq 0.287094\}$. Moreover, this solution is unique in $\{\psi \in C([0,1])/\|\psi - 1\| < 3.48447\}$. Furthermore, the next error bounds are obtained:

$$\|x^* - x_1\| \leq 5.67403\,10^{-4} \quad \text{and} \quad \|x^* - x_2\| \leq 2.10679\,10^{-14}.$$

They are a little better than the ones obtained by Argyros, Chen and Qian in [9] for the same method:

$$\|x^* - x_1\| \leq 7.3679\,10^{-4} \quad \text{and} \quad \|x^* - x_2\| \leq 4.5111\,10^{-14}.$$

3. APPLICATION TO NONLINEAR HAMMERSTEIN EQUATIONS

An interesting aspect of the study of iteration convergence to solve equations is to get existence-uniqueness results of solutions. So, in this section, we obtain results of this type for the known integral equation of Hammerstein [13]

$$x(s) = f(s) + \int_a^b \mathcal{K}(s,t)\ell(t, x(t))\, dt, \qquad x \in C([0,1]), \tag{13}$$

where $\mathcal{K}(s,t)$ is the kernel of a lineal integral operator in $C[0,1]$ and $\ell(t,u)$ is a continuous function for $0 \leq t \leq 1$, $-\infty < u < +\infty$.

Note that solving (13) is equivalent to solving (1), where

$$[F(x)](s) = x(s) - f(s) - \int_a^b \mathcal{K}(s,t)\ell(t, x(t))\, dt$$

is an operator defined in the space $C[0,1]$. It is clear that we can get an existence-uniqueness domain of solutions of (13) by Theorem 2.4. But in this theorem, it is proved that the R-order of convergence is four, so that the operator F must satisfy some strong conditions. In the following result, we obtain domains of this type under milder assumptions than the ones that appeared in Theorem 2.4. In particular, under the same conditions that for the Newton method (see [14]), we suppose that (A_1) and (A_2) are satisfied along with a Lipschitz-type condition for F'.

THEOREM 3.1. *Let F be a continuously Fréchet differentiable operator in an open convex domain $\Omega \subseteq X$. Let $x_0 \in \Omega$ be a point where the operator $\Gamma_0 = F'(x_0)^{-1}$ exists and*

$$\|\Gamma_0\| \le \beta, \quad \|y_0 - x_0\| \le \eta, \quad \|F'(x) - F'(y)\| \le K\|x - y\|, \qquad x, y \in \Omega.$$

We define $d_n = d_{n-1} v(d_{n-1})^2 w(d_{n-1})$ with $d_0 = K\beta\eta$, $v(x) = 2/(2 - 2x - x^2 - x^3)$, and $w(x) = (x/8)(8 + 8x + 5x^2 + 2x^3 + x^4)$. If

$$d_0 < \tau = 0.300637\ldots, \tag{14}$$

where τ is the smallest positive root of the polynomial

$$2x^6 + 3x^5 + 8x^4 - 5x^3 - 8x^2 - 24x + 8$$

and $\overline{B(x_0, R\eta)} \subseteq \Omega$, where $R = (1 + (d_0/2)(1 + d_0))/(1 - \gamma\Delta)$, $\gamma = d_1/d_0$, and $\Delta = 1/v(d_0)$, then equation (13) has at least a solution x^* in $\overline{B(x_0, R\eta)}$. Moreover, x^* is unique in $B(x_0, 2/(K\beta) - R\eta) \cap \Omega$.

PROOF. Notice that the hypotheses guarantee the existence of x_1 in (2) and

$$\|x_1 - x_0\| \le \left(1 + \frac{d_0}{2}(1 + d_0)\right) \|y_0 - x_0\|.$$

Besides it can be shown without difficulty that $\Gamma_1 F'(x_0)$ exists and

$$\|\Gamma_1\| \le v(d_0)\|\Gamma_0\|.$$

Then x_2 is defined, and taking into account (2), we have

$$\Gamma_0 F(x_1) = \frac{3}{4} G(x_0, y_0) \left[I - \frac{3}{2} G(x_0, y_0)\right] (x_0 - y_0) + \int_{x_0}^{x_1} \Gamma_0 \left[F'(x) - F'(x_0)\right] dx$$

by Taylor's formula. So

$$\|y_1 - x_1\| \le v(d_0) w(d_0) \|y_0 - x_0\|.$$

On the other hand,

$$K\|\Gamma_1\|\|y_1 - x_1\| \le d_1 \quad \text{and} \quad \|x_2 - x_1\| \le \left(1 + \frac{d_1}{2}(1 + d_1)\right) \|y_1 - x_1\|.$$

Finally, from (14) and $v(d_0)^2 w(d_0) < 1$, we have

$$d_1 = d_0 v(d_0)^2 w(d_0) < d_0.$$

Following an inductive procedure, we can replace x_1 by x_2, x_2 by x_3 and, in general, x_{n-1} by x_n to obtain that there exists Γ_n and the following recurrence relations:

$$\begin{aligned}
\|\Gamma_n\| = \|F'(x_n)^{-1}\| &\le v(d_{n-1})\|\Gamma_{n-1}\|, \\
\|y_n - x_n\| &\le v(d_{n-1}) w(d_{n-1}) \|y_{n-1} - x_{n-1}\|, \\
K\|\Gamma_n\| \, \|y_n - x_n\| &\le d_n, \\
\|x_{n+1} - x_n\| &\le \left(1 + \frac{d_n}{2}(1 + d_n)\right) \|y_n - x_n\|, \\
d_n &< d_{n-1}.
\end{aligned}$$

Consequently, taking into account that $w(d_0) < 1$ and following, a similar way as in Theorem 2.4, we infer that $\{x_n\}$ is a Cauchy sequence and therefore converges to a solution $x^* \in B(x_0, 2/(K\beta) - R\eta) \cap \Omega$ of (1). The uniqueness follows as in Theorem 2.4. ∎

Observe that we need evaluate K, β, and η from the initial point x_0. Then we do certain considerations for equation (13). We assume that ℓ has partial derivative respect to the second argument and satisfies a Lipschitz condition.

It is easy to prove that

$$[F'(x)y]\,(s) = y(s) - \int_a^b \mathcal{K}(s,t)\ell_2'\,(t, x(t))\, y(t)\, dt,$$

where $\ell_2'(t,\mu) = \frac{\partial}{\partial\mu}\ell(t,\mu)$.

On the other hand, fixed $x_0(s)$, we consider $[F'(x_0)y](s) = z(s)$. Then $y(s) = [\Gamma_0 z](s)$, and consequently,

$$\begin{aligned} y(s) &= z(s) + \int_a^b \mathcal{K}(s,t)\ell_2'(t, x_0(t))y(t)\, dt \\ &= z(s) + \int_a^b \mathcal{K}(s,t)\ell_2'(t, x_0(t)) \left(z(t) + \int_a^b \mathcal{K}(t,u)\ell_2'(u, x_0(u))y(u)\, du \right) dt. \end{aligned}$$

By recurrence and taking the max-norm, we deduce

$$\|y(s)\| = \|[\Gamma_0 z]\,(s)\| \le \left(\sum_{n\ge 0} \|h\|^n \, \|\ell_2'\|^n \right) \|z\|,$$

where $h(\xi) = \int_a^b \mathcal{K}(\xi,\sigma)\, d\sigma$. Now if $\|h\|\,\|\ell_2'\| < 1$, then

$$\|\Gamma_0\| \le \frac{1}{1 - \|h\|\,\|\ell_2'\|}.$$

From the definition of the operator F, we deduce that $\|F(x_0)\| = \|x_0 - f\| + \|h\|\|\ell\|$, and therefore,

$$\|\Gamma_0 F(x_0)\| \le \frac{\|x_0 - f\| + \|h\|\,\|\ell\|}{1 - \|h\|\,\|\ell_2'\|}.$$

Besides, taking into account that

$$[(F'(x) - F'(y))\, z]\,(s) = \int_a^b \mathcal{K}(s,t)\,(\ell_2'(t, y(t)) - \ell_2'(t, x(t)))\, z(t)\, dt,$$

it follows

$$\|F'(x) - F'(y)\| \le \|h\| P \|x - y\|,$$

where P is the Lipschitz constant for ℓ_2'.

Therefore, we can give a more explicit result on the existence-uniqueness of solutions of equation (13).

THEOREM 3.2. *Following the same notation, let* $F : \Omega \subseteq C[0,1] \to C[0,1]$, *where* Ω *is an open convex domain and*

$$[F(x)](s) = x(s) - f(s) - \int_a^b \mathcal{K}(s,t)\ell(t, x(t))\, dt.$$

Let $x_0 \in \Omega$ *be a point where the operator* Γ_0 *exists and*

$$\beta = \frac{1}{1 - \|h\|\,\|\ell_2'\|}, \qquad \eta = \frac{\|x_0 - f\| + \|h\|\,\|\ell\|}{1 - \|h\|\,\|\ell_2'\|}, \qquad K = P\|h\|,$$

where $h(\xi) = \int_a^b \mathcal{K}(\xi,\sigma)\,d\sigma$ and P is the Lipschitz constant for ℓ'_2. If $d_0 = K\beta\eta < 0.300637\dots$ and $\overline{B(x_0, R\eta)} \subseteq \Omega$, where $R = (1+(d_0/2)(1+d_0))/(1-\gamma\Delta)$, $\gamma = d_1/d_0$, and $\Delta = 1/v(d_0)$, then a solution of (13) exists at least in $\overline{B(x_0, R\eta)}$. Moreover, this is the only one in $B(x_0, 2/(K\beta) - R\eta) \cap \Omega$.

Notice when the kernel $\mathcal{K}$ and the functions f and ℓ are fixed, then $F(x_0)$ is improved.

After, we study a particular equation of Hammerstein-type. Let us consider the same equation as Werner in [11] and given by

$$x(s) = 1 - \frac{\lambda}{2}\int_0^1 \frac{s}{t+s}\frac{1}{x(t)}\,dt, \qquad s \in [0,1], \quad \lambda \in \left[0, \frac{1}{2}\right] \text{ fixed}, \tag{15}$$

in the space $C[0,1]$ of all continuous functions with the max-norm. Note that (15) is also an H-equation that arises on the radiative transfer theory (see [13] and the references appearing there). From now on

$$F : \Omega \subseteq X \to X, \quad X = C[0,1], \qquad \Omega = \{x \in C[0,1];\ x \text{ is positive}\},$$
$$[F(x)](s) = x(s) - 1 + \frac{\lambda}{2}\int_0^1 \frac{s}{t+s}\frac{1}{x(t)}\,dt, \qquad \lambda \in \left[0, \frac{1}{2}\right].$$

First, we obtain an existence-uniqueness result of solutions depending on the parameter λ. Second, we discretized the above equation to get, by (2), approximations to the solutions. Finally, we compare the speed of convergence of (2) with the classical Newton method and other Newton-type iterations, considered in [11] with a similar operational cost. The speed of convergence is improved by (2).

Before that, we provide some important features of the solution of (15). If $x(s)$ is a solution of (15), then $x(0) = 1$, $x(s)$ is a decreasing function and $x(1) \geq (1 + \sqrt{1 - 2\lambda\ln 2})/2 = T$. Consequently, we consider

$$\Omega_0 = \{x \in C[0,1];\ x \text{ is positive with } T \leq x(s) \leq 1,\ s \in [0,1]\} \subsetneq \Omega, \tag{16}$$

and take $F : \Omega_0 \to X$. According to the general case, as $f(s) = 1$, $\mathcal{K}(s,t) = -(\lambda/2)s/(s+t)$, and $\ell(t, x(t)) = 1/x(t)$, then

$$\|\Gamma_0\| \leq \frac{2\|x_0\|^2}{2\|x_0\|^2 - \lambda\ln 2},$$

if and only if

$$\lambda < \frac{2}{\ln 2}\|x_0\|^2. \tag{17}$$

Moreover, by the Banach lemma, Γ_0 exists and

$$\|[(I - F'(x_0))\,y]\,(s)\| \leq \frac{\lambda}{2}\left|\int_0^1 \frac{s}{s+t}\,dt\right| \frac{\|y\|}{\|x_0\|^2} \leq \frac{\lambda\ln 2}{2\|x_0\|^2}\|y\|.$$

Then $\|I - F'(x_0)\| < 1$, if (17) is satisfied. Furthermore,

$$\|F(x_0)\| = \|x_0 - 1\| + \frac{\lambda}{2}\frac{\ln 2}{\|x_0\|}$$

and

$$\|F'(x) - F'(y)\| \leq \frac{\lambda\ln 2}{T^4}\|x - y\|, \qquad x, y \in \Omega_0.$$

COROLLARY 3.3. *With the notation of Theorem 3.1, let* $F : \Omega_0 \subseteq C[0,1] \to C[0,1]$, *where* Ω_0 *is given by (16) and*

$$[F(x)](s) = x(s) - 1 + \frac{\lambda}{2}\int_0^1 \frac{s}{t+s}\frac{1}{x(t)}\,dt.$$

Let $x_0 \in \Omega_0$, $0 \le \lambda < (2/\ln 2)\|x_0\|^2$,

$$\beta = \frac{2\|x_0\|^2}{2\|x_0\|^2 - \lambda\ln 2}, \quad \eta = \frac{2\|x_0\|^2}{2\|x_0\|^2 - \lambda\ln 2}\left(\|x_0 - 1\| + \frac{\lambda}{2}\frac{\ln 2}{\|x_0\|}\right), \quad \text{and} \quad K = \frac{\lambda\ln 2}{T^4}.$$

If $d_0 = K\beta\eta < 0.300637\ldots$ *and* $\overline{B(x_0,R\eta)} \subseteq \Omega_0$, *where* $R = (1 + (d_0/2)(1+d_0))/(1-\gamma\Delta)$, $\gamma = d_1/d_0$, *and* $\Delta = 1/v(d_0)$, *then equation (15) has at least a solution in* $\overline{B(x_0,R\eta)}$. *Besides this solution is the only one in* $B(x_0, 2/(K\beta) - R\eta) \cap \Omega_0$.

Observe that x_0 must be chosen such that $(2/\ln 2)\|x_0\|^2 > 1/2$, since (15) is defined for $\lambda \in [0,1/2]$.

If we take the same initial function $x_0(s) = 1$, as Werner in [11], we see that the previous corollary is satisfied for all $\lambda \in [0,1/2]$, as a consequence of

$$d_0 = \frac{2(\ln 2)^2\lambda^2}{(2-\lambda\ln 2)^2 T^4} < 0.300637\ldots.$$

Finally, equation (15) is discretized to replace it by a finite dimension problem. The integral appearing in (15) is approximated by a numerical integration formula. We choose $\lambda = 1/2$ and the composite trapezoidal rule with mesh size $1/m$.

So, for $j = 0,1,\ldots,m$, we have

$$0 = x(t_j) - 1 + \frac{1}{4m}\left[\frac{1}{2}\frac{t_j}{t_j+t_0}\frac{1}{x(t_0)} + \sum_{k=1}^{m-1}\frac{t_j}{t_j+t_k}\frac{1}{x(t_k)} + \frac{1}{2}\frac{t_j}{t_j+t_m}\frac{1}{x(t_m)}\right],$$

where $t_j = j/m$.

In this way, we can compare the results obtained with the ones given by Werner in [11] for some different iterations with a similar operational cost.

If we denote $x_n^{(i)} = x_n(t_i)$, $i = 0,1,\ldots,m$, then $(x_n^{(0)}, x_n^{(1)},\ldots,x_n^{(m)})^t$ is the iteration n of (2). The solution is denoted by $(\bar{x}^{(0)},\bar{x}^{(1)},\ldots,\bar{x}^{(m)})^t$.

Taking (2) with $m = 20$ and $x_0^{(i)} = 1$, for $i = 0,1,\ldots,20$, the results of Table 1 are obtained.

Table 1.

i	$\bar{x}^{(i)}$	i	$\bar{x}^{(i)}$
1	0.956652783895941	11	0.830471666177964
2	0.931973961726981	12	0.824546396027649
3	0.912972463361788	13	0.819121327252762
4	0.897743855274231	14	0.814138891021997
5	0.884895332371650	15	0.809558477036342
6	0.873888360619056	16	0.812107706037000
7	0.864307534740268	17	0.808390913203738
8	0.855512839646057	18	0.804976549647592
9	0.848351533450320	19	0.801790182723719
10	0.841610323653732	20	0.798809056636777

Tables 2 and 3 contain the errors

$$\max_{0\le i\le 20}\left|x_n^{(i)} - \bar{x}^{(i)}\right|,$$

for the iterates $(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(20)})^t$ generated by the iterative methods:

I. Stirling's method,
II. Newton's method,
III. the variable γ method appearing in [11],
IV. method (2).

Table 2. $x_0^{(i)} = 1$.

n	I	II	III	IV
1	4.18×10^{-3}	4.80×10^{-3}	9.23×10^{-4}	5.69×10^{-4}
2	1.98×10^{-6}	3.69×10^{-6}	1.99×10^{-8}	2.22×10^{-16}
3	4.33×10^{-13}	2.17×10^{-12}	9.11×10^{-18}	0.0

Table 3. $x_0^{(i)} = 1.2$.

n	I	II	III	IV
1	1.85×10^{-2}	1.75×10^{-2}	4.11×10^{-3}	6.52×10^{-4}
2	4.11×10^{-5}	5.03×10^{-5}	4.11×10^{-7}	1.24×10^{-14}
3	1.88×10^{-10}	4.05×10^{-10}	3.87×10^{-15}	2.22×10^{-16}

The numerical results, using 20 significant decimal figures indicate that iteration (2) converges faster to a solution $(\bar{x}^{(0)}, \bar{x}^{(1)}, \dots, \bar{x}^{(20)})^t$ than the Methods I–III.

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