NORTH-HOLLAND

A Construction Procedure of Iterative Methods with Cubical Convergence

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ABSTRACT

We study the Kantorovich convergence for parameter-based methods for solving nonlinear operator equations in Banach spaces. We also derive a closed form of error bounds in terms of a real parameter $\alpha \in [0, 1]$.

1. INTRODUCTION

In this paper, we study the basic idea of continuation methods [1, 2]. A given problem is included in a family of problems which depends on a parameter α that lies in the interval [0, 1]. In particular, a homotopy can be any continuous connection between two functions f and g. Formally, a homotopy between two functions $f, g: X \to Y$ is a continuous map, $h: [0, 1] \times X \to Y$ where

$$h(\alpha, x) = \alpha f(x) + (1 - \alpha) g(x), h(0, x) = g(x), h(1, x) = f(x).$$

If this map exists, then f is homotopic to g. This is an equivalence relation between the continuous maps of X in Y, where X and Y are any topological spaces.

We extend this idea to the problem of finding the roots of a equation

$$F(x) = 0 \tag{1}$$

in Banach spaces by means of iterations methods.

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Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain $\Omega_0 \subseteq \Omega$. Let us assume that $F'(x_0)^{-1} \in \mathscr{L}(Y, X)$ exists at some $x_0 \in \Omega_0$, where $\mathscr{L}(Y, X)$ is the set of bounded linear operators from Y into X.

Two well known third-order methods for solving (1) are the Chebyshev method

$$x_{n+1} = J_0(x_n) = x_n - \left[I + \frac{1}{2} L_F(x_n)\right] F'(x_n)^{-1} F(x_n), \qquad n \geqslant 0, \quad (2)$$

and the Convex Acceleration of Newton's method

$$x_{n+1} = J_1(x_n) = x_n - \left[I + \frac{1}{2} L_F(x_n)(I - L_F(x_n))^{-1}\right] F'(x_n)^{-1} F(x_n), \qquad n \geqslant 0. \quad (3)$$

We have denoted I the identity operator on X and $L_F(x)$ the linear operator defined by

$$L_F(x) = F'(x)^{-1} F''(x) F'(x)^{-1} F(x), \qquad x \in X,$$

provided that $F'(x)^{-1}$ exists. See Gutiérrez et al. [3] to connect this operator with Newton's method.

In recent years, the convergence of the Chebyshev method and the Convex Acceleration of Newton's method in Banach space setting have been studied by many authors [4–10].

We define a homotopy between the two operators J_0 and J_1 defined, respectively, in (2) and (3). Then we design new parameter-based iterations in Banach spaces as follows

$$x_{\alpha, n+1} = \alpha J_1(x_{\alpha, n}) + (1 - \alpha) J_0(x_{\alpha, n}), \qquad n \geqslant 0,$$

i.e.,

$$x_{\alpha, n+1} = x_{\alpha, n} - \left[I + \frac{1}{2} L_F(x_{\alpha, n}) G_\alpha(x_{\alpha, n})\right] F'(x_{\alpha, n})^{-1} F(x_{\alpha, n}), \qquad n \geqslant 0. \quad (4)$$

where $G_\alpha(x_{\alpha, n}) = I + \alpha L_F(x_{\alpha, n}) H(x_{\alpha, n})$ and $H(x_{\alpha, n}) = (I - L_F(x_{\alpha, n}))^{-1}$.

Observe that the family of iterations (4) includes the Chebyshev method ($\alpha = 0$) and the Convex Acceleration of Newton's method ($\alpha = 1$) as a specific choice of the parameter α. We will provide a Kantorovich-type convergence analysis for this one-parameter family and give error estimates. Some examples to illustrate the previous results are also given.

Denote $\overline{B(x, r)} = \{y \in X; \|y - x\| \leqslant r\}$ and $B(x, r) = \{y \in X; \|y - x\| < r\}$.

2. PRELIMINARIES

We will now apply the majorant principle (see [11, 12]) to study the family (4). For this purpose, let us consider the real equation

$$p(t) = 0, \tag{5}$$

where p is a real function of the real variable t and three times continuously differentiable in $[a, b]$.

Let us consider the scalar sequence $\{t_{\alpha, n}\}$

$$t_{\alpha, n+1} = P_\alpha(t_{\alpha, n}) = t_{\alpha, n} - \left[1 + \frac{L_p(t_{\alpha, n})}{2}\left(1 + \frac{\alpha L_p(t_{\alpha, n})}{1 - L_p(t_{\alpha, n})}\right)\right]\frac{p(t_{\alpha, n})}{p'(t_{\alpha, n})}, \quad n \geqslant 0. \tag{6}$$

We show that the sequences $\{t_{\alpha, n}\}$ and $\{x_{\alpha, n}\}$ are well defined, converge, respectively, to a solution t^* of (5) and a solution x^* of (1), and

$$\|x_{\alpha, n+1} - x_{\alpha, n}\| \leqslant t_{\alpha, n+1} - t_{\alpha, n}, \qquad n \geqslant 0,$$

$$\|x^* - x_{\alpha, n}\| \leqslant t^* - t_{\alpha, n}, \qquad n \geqslant 0.$$

That is, $\{t_{\alpha, n}\}$ is a majorizing sequence of $\{x_{\alpha, n}\}$.

The following simple lemma will be applied.

LEMMA 2.1. *Let p be a nonincreasing convex real function in $[a, b]$ where $p(a) > 0 > p(b)$. Let $t_{\alpha,0} \in [a, b]$ where $p(t_{\alpha,0}) > 0$. Assume $L_{p'}(t) \leqslant L_p(t) < 1$ for $t \in [a, b]$. Then the process defined by (6) is increasing and cubically convergent to t^* for all $0 \leqslant \alpha \leqslant 1$, where t^* is the only root of (5) in $[a, b]$.*

PROOF. From $p(t_{\alpha,0}) > 0$, it follows that $t_{\alpha,0} - t^* \leqslant 0$. We have by the Mean Value theorem that

$$t_{\alpha,1} - t^* = P'_\alpha(s_0)(t_{\alpha,0} - t^*),$$

for some $s_0 \in (t_{\alpha,0}, t^*)$. Moreover, we have $P'_\alpha(t) \geqslant 0$ in $[a, t^*]$ as a consequence of hypotheses and

$$P'_\alpha(t) = \frac{L_{p'}(t)^2}{2(1 - L_{p'}(t))^2}\Big[(1-\alpha)(1 - L_p(t))^2(3 - L_{p'}(t)) + \alpha\big(L_p(t) - L_{p'}(t)\big)\Big].$$

Therefore, $t_{\alpha,1} \leqslant t^*$.

On the other hand,

$$t_{\alpha,1} - t_{\alpha,0} = -\frac{p(t_{\alpha,0})}{p'(t_{\alpha,0})}\left[1 + \frac{1}{2}L_p(t_{\alpha,0})\left(1 + \frac{\alpha L_p(t_{\alpha,0})}{1 - L_p(t_{\alpha,0})}\right)\right] \geqslant 0.$$

Then we obtain $t_{\alpha,n} \leqslant t^*$ and $t_{\alpha,n-1} \geqslant t_{\alpha,n}$ for all $n \geqslant 1$ by mathematical induction, since $(t_{\alpha,n-1}, t^*) \subset (t_{\alpha,0}, t^*)$.

So the sequence (6) is convergent to $s \in [a, b]$. Now from the fact that

$$1 + \frac{1}{2}L_p(s)\left(1 + \frac{\alpha L_p(s)}{1 - L_p(s)}\right) > 0,$$

it is easy to deduce that $s = t^*$. ∎

The next theorem contains sufficient conditions for the convergence of the family defined by (2). These conditions also assure the existence of a solution of equation (1) (see [13]). We denote $\Gamma_{\alpha,n} = F'(x_{\alpha,n})^{-1}$, $n \geqslant 0$, when it exists.

THEOREM 2.2. *Let us assume that $\Gamma_{\alpha,0}$ exists at some $x_{\alpha,0} \in \Omega_0$ and the assumptions of the previous lemma hold. Suppose that the following condi-*

tions are satisfied

$$(c_1)\ \|\Gamma_{\alpha,0}F(x_{\alpha,0})\| \leqslant -\frac{p(t_{\alpha,0})}{p'(t_{\alpha,0})},$$

$$(c_2)\ \|\Gamma_{\alpha,0}F''(x)\| \leqslant -\frac{p''(t)}{p'(t_{\alpha,0})}$$

when $\|x - x_{\alpha,0}\| \leqslant t - t_{\alpha,0} \leqslant t^* - t_{\alpha,0}$,

$$(c_3)\ \|\Gamma_{\alpha,0}F'''(x)\| \leqslant -\frac{p'''(t)}{p'(t_{\alpha,0})}$$

when $\|x - x_{\alpha,0}\| \leqslant t - t_{\alpha,0} \leqslant t^* - t_{\alpha,0}$.

Then if $\overline{B(x_{\alpha,0}, t^*)} \subset \Omega_0$, (1) *has a solution* x^* *and the sequence* (4) *of the approximate solutions* $x_{\alpha,n}$ *is well defined and converges to* x^*. *The error estimate is given by* $\|x^* - x_{\alpha,n}\| \leqslant t^* - t_{\alpha,n}$, $n \geqslant 0$.

PROOF. By Lemma 2.1, the sequence (6) converges to t^*. Then we will prove that we can replace $x_{\alpha,0}$ by $x_{\alpha,1}$ in (c_1)–(c_3).

Firstly, we show the existence of $\Gamma_{\alpha,1}$. Let us estimate the expression

$$\|I - \Gamma_{\alpha,0}F'(x_{\alpha,1})\| \leqslant \int_{x_{\alpha,0}}^{x_{\alpha,1}} \|\Gamma_{\alpha,0}F''(x)\|\,dx \leqslant -\frac{1}{p'(t_{\alpha,0})}\int_{t_{\alpha,0}}^{t_{\alpha,1}} p''(t)\,dt$$

$$= 1 - \frac{p'(t_{\alpha,1})}{p'(t_{\alpha,0})} = q.$$

We have $\Gamma_{\alpha,0}F'(x_{\alpha,1}) = I - [I - \Gamma_{\alpha,0}F'(x_{\alpha,1})]$. Thus, by the Banach Lemma [11], it follows the existence of the inverse of $F'(x_{\alpha,1})$, since $q < 1$. We have

$$\Gamma_{\alpha,1} = F'(x_{\alpha,0})\left[I - \Gamma_{\alpha,0}(F'(x_{\alpha,0}) - F'(x_{\alpha,1}))\right]^{-1}.$$

Hence, we get

$$\|[\Gamma_{\alpha,0}F'(x_{\alpha,1})]^{-1}\| \leqslant \frac{1}{1 - \|I - \Gamma_{\alpha,0}F'(x_{\alpha,1})\|} \leqslant \frac{p'(t_{\alpha,0})}{p'(t_{\alpha,1})}$$

and, consequently, (c_2) and (c_3) follow immediately.

It remains to prove that the condition (c_1) is fulfilled for $x_{\alpha,1}$. Using the analogue of Taylor's formula in the integral form we obtain, by (4),

$$F(x_{\alpha,1}) = \frac{1}{2}F''(x_{\alpha,0})\Gamma_{\alpha,0}F(x_{\alpha,0})(-G_\alpha(x_{\alpha,0}) + I)\Gamma_{\alpha,0}F(x_{\alpha,0})$$
$$+\frac{1}{2}F''(x_{\alpha,0})\Gamma_{\alpha,0}F(x_{\alpha,0})\,L_F(x_{\alpha,0})\,G_\alpha(x_{\alpha,0})\Gamma_{\alpha,0}F(x_{\alpha,0})$$
$$+\frac{1}{8}F''(x_{\alpha,0})\left[L_F(x_{\alpha,0})\,G_\alpha(x_{\alpha,0})\Gamma_{\alpha,0}F(x_{\alpha,0})\right]^2$$
$$+\frac{1}{2}\int_{x_{\alpha,0}}^{x_{\alpha,1}} F'''(x)(x_{\alpha,1} - x)^2\,dx.$$

Taking into account that

$$-G_\alpha(x_{\alpha,0}) + I + L_F(x_{\alpha,0})\,G_\alpha(x_{\alpha,0}) = (1-\alpha)\,L_F(x_{\alpha,0}),$$

we obtain

$$F(x_{\alpha,1}) = \frac{1-\alpha}{2}F''(x_{\alpha,0})\Gamma_{\alpha,0}F(x_{\alpha,0})\,L_F(x_{\alpha,0})\Gamma_{\alpha,0}F(x_{\alpha,0})$$
$$+\frac{1}{8}F''(x_{\alpha,0})\left[L_F(x_{\alpha,0})\,G_\alpha(x_{\alpha,0})\Gamma_{\alpha,0}F(x_{\alpha,0})\right]^2$$
$$+\frac{1}{2}\int_{x_{\alpha,0}}^{x_{\alpha,1}} F'''(x)(x_{\alpha,1} - x)^2\,dx. \tag{7}$$

On the other hand, we have

$$p(t_{\alpha,1}) = p(t_{\alpha,0}) - p(t_{\alpha,0})\left[1 + \frac{w_{\alpha,0}}{2}\left(1 + \frac{\alpha\,w_{\alpha,0}}{1 - w_{\alpha,0}}\right)\right]$$
$$+\frac{1}{2}p(t_{\alpha,0})\,w_{\alpha,0}\left[1 + \frac{w_{\alpha,0}}{2}\left(1 + \frac{\alpha\,w_{\alpha,0}}{1 - w_{\alpha,0}}\right)\right]^2$$
$$+\frac{1}{2}\int_{t_{\alpha,0}}^{t_{\alpha,1}} p'''(v)(t_{\alpha,1} - v)^2\,dv,$$

where $w_{\alpha,0} = L_p(t_{\alpha,0})$. Hence, we obtain

$$p(t_{\alpha,1}) = \frac{1}{2}p(t_{\alpha,0})\,w_{\alpha,0}\left[1 - \alpha + \frac{w_{\alpha,0}}{4}\left(1 + \frac{\alpha\, w_{\alpha,0}}{1 - w_{\alpha,0}}\right)^2\right] + \frac{1}{2}\int_{t_{\alpha,0}}^{t_{\alpha,1}} p'''(v)(t_{\alpha,1} - v)^2\, dv. \tag{8}$$

It follows from (7) and (8) that

$$\|\Gamma_{\alpha,1}F(x_{\alpha,1})\| \leqslant -\frac{p(t_{\alpha,1})}{p'(t_{\alpha,1})}.$$

Therefrom we obtain that $\|x_{\alpha,2} - x_{\alpha,1}\| \leqslant t_{\alpha,2} - t_{\alpha,1}$.

Thus, using induction, it is shown that conditions (c_1)–(c_3) are satisfied for $x_{\alpha,n}$ and $n \geqslant 1$. As a consequence, we get

$$\|x_{\alpha,n+p} - x_{\alpha,n}\| \leqslant t_{\alpha,n+p} - t_{\alpha,n}$$

for any positive integers p and n. Hence, it follows that $\|x^* - x_{\alpha,n}\| \leqslant t^* - t_{\alpha,n}$, $n \geqslant 0$, is satisfied, and the sequence (4) converges to $x^* \in X$. Since $\|F(x_{\alpha,n})\| \leqslant p(t_{\alpha,n})$, we infer that $F(x_{\alpha,n}) \to 0$ as $n \to \infty$, and by the continuity of F we conclude that x^* is a solution of (1). ■

3. MAIN CONVERGENCE RESULT

Frequently, it is difficult to use the above theorem directly, since p is an unknown function. But if the nonlinear operator F satisfies some Kantorovich-type condition, the function p is determinate, and we can give convergence results and error estimates of the family (4). Here the basic assumption made is that the second Fréchet-derivative F'' of F is Lipschitz continuous [14] in some ball.

Following Yamamoto [15], it is assumed that

(i) There exists a continuous linear operator $\Gamma_0 = F'(x_0)^{-1}$, $x_0 \in \Omega_0$.
(ii) $\|\Gamma_0(F''(x) - F''(y))\| \leqslant k\|x - y\|$, $x, y \in \Omega_0$, $k \geqslant 0$.
(iii) $\|\Gamma_0 F(x_0)\| \leqslant a$, $\|\Gamma_0 F''(x_0)\| \leqslant b$.
(iv) The equation

$$p(t) \equiv \frac{k}{6}t^3 + \frac{b}{2}t^2 - t + a = 0 \tag{9}$$

has one negative root and two positive roots r_1 and r_2 $(r_1 \leqslant r_2)$ if $k > 0$, or has two positive roots r_1 and r_2 $(r_1 \leqslant r_2)$ if $k = 0$. Equivalently,

$$a \leqslant \frac{b^2 + 4k - b\sqrt{b^2 + 2k}}{3k\left(b + \sqrt{b^2 + 2k}\right)} \text{ if } k > 0,$$

or $ab \leqslant 1/2$ if $k = 0$.

For each $\alpha \in [0, 1]$, let us define the scalar sequence $\{t_{\alpha, n}\}$ by

$$t_0 = t_{\alpha,0} = 0,\ t_{\alpha, n+1} = P_\alpha(t_{\alpha, n}), \qquad n \geqslant 0, \tag{10}$$

where P_α is defined in (6) and p is the polynomial given by (9).

Notice that the family $\{t_{\alpha, n}\}$ defined by (10) is increasing and converges to r_1 as a consequence of Lemma 2.1, since $L_{p'}(t) < 0$ and $L_p(t) \leqslant 1/2$ (see [13]) in $[0, r_1]$.

Next, we obtain error expressions for the sequence (10). It is difficult to obtain error bounds following the Ostrowski method [16]. In the next two lemmas, we establish estimates for the error in this situation by using a new procedure (discussed in an upcoming article by Gutiérrez and Hernández).

LEMMA 3.1. *Let p be the polynomial given by* (9) *with $k = 0$, that is,*

$$p(t) = \frac{b}{2}t^2 - t + a.$$

We assume that p has two positive roots r_1 and r_2 $(r_1 \leqslant r_2)$. Let $\{t_{\alpha, n}\}$ be the sequence defined in (10).

(a) *When $r_1 < r_2$, let $\vartheta = \dfrac{r_1}{r_2}$, $\lambda_\alpha = 2(1 - \alpha)$ and $\theta_\alpha = \vartheta\sqrt{\lambda_\alpha}$. Then we have*

(a_1) *if $\alpha \in [0, \frac{1}{2})$ and* $ab < \dfrac{2\sqrt{\lambda_\alpha}}{\left(1 + \sqrt{\lambda_\alpha}\right)^2} < 0.5$,

$$r_1 - t_{\alpha, n} \sim \frac{(r_2 - r_1)\theta_\alpha^{3^n}}{\sqrt{\lambda_\alpha} - \theta_\alpha^{3^n}}, \qquad n \geqslant 0,$$

where $\theta_\alpha < 1$.

(a_2) *if* $\alpha \in [\frac{1}{2}, 1)$,

$$r_1 - t_{\alpha,n} \sim \frac{(r_2 - r_1)\theta_\alpha^{3^n}}{\sqrt{\lambda_\alpha} - \theta_\alpha^{3^n}}, \qquad n \geqslant 0,$$

where $\lambda_\alpha \leqslant 1$ *and* $\theta_\alpha < 1$,

(a_3) *if* $\alpha = 1$,

$$r_1 - t_{\alpha,n} = \frac{(r_2 - r_1)\vartheta^{4^n}}{1 - \vartheta^{4^n}}, \qquad n \geqslant 0.$$

(b) *When* $r_1 = r_2$,

$$r_1 - t_{\alpha,n} = r_1\left(\frac{3 - \alpha}{8}\right)^n, \qquad n \geqslant 0.$$

PROOF. Let us write $a_{\alpha,n} = r_1 - t_{\alpha,n}$ and $b_{\alpha,n} = r_2 - t_{\alpha,n}$. Then

$$p(t_{\alpha,n}) = \frac{b}{2} a_{\alpha,n} b_{\alpha,n} \quad \text{and} \quad p'(t_{\alpha,n}) = -\frac{b}{2}(a_{\alpha,n} + b_{\alpha,n}).$$

Define now

$$Q_\alpha(t_{\alpha,n}) = \left(\frac{b_{\alpha,n}^3}{a_{\alpha,n}^3}\right)\left(\frac{a_{\alpha,n+1}}{b_{\alpha,n+1}}\right) = \frac{(r_1 - P_\alpha(t_{\alpha,n}))(r_2 - t_{\alpha,n})^3}{(r_2 - P_\alpha(t_{\alpha,n}))(r_1 - t_{\alpha,n})^3},$$

with P_α defined in (6). As $P_\alpha(r_1) = r_1$, $P_\alpha'(r_1) = 0 = P_\alpha''(r_1)$ and $P_\alpha'''(r_1) = (12(1 - \alpha)/(r_2 - r_1)^2)$, we have for t closely r_1

$$Q_\alpha(t) \sim (r_2 - r_1)^2 \lim_{t \to r_1} \frac{r_1 - P_\alpha(t)}{(r_1 - t)^3} = 2(1 - \alpha) = \lambda_\alpha.$$

When $n \to \infty$, we have $t_{\alpha,n} \to r_1$ and

$$\frac{a_{\alpha,n}}{b_{\alpha,n}} \sim \lambda_\alpha \left(\frac{a_{\alpha,n-1}}{b_{\alpha,n-1}} \right)^3 \sim \cdots \sim \frac{1}{\sqrt{\lambda_\alpha}} \left(\sqrt{\lambda_\alpha} \frac{r_1}{r_2} \right)^{3^n},$$

and (a_1) holds. To see (a_2), it is enough to take into account that $\theta_\alpha < 1$ if $ab < (2\sqrt{\lambda_\alpha}/(1 + \sqrt{\lambda_\alpha})^2)$. See [8] to get ($a_3$).

If $r_1 = r_2$, then $a_{\alpha,n} = b_{\alpha,n}$. Therefore,

$$a_{\alpha,n} = r_1 - t_{\alpha,n} = a_{\alpha,n-1} \frac{3-\alpha}{8}$$

and by recurrence, (b) also holds. ∎

LEMMA 3.2. *Let p be the polynomial given by* (9) *with $k > 0$, that is,*

$$p(t) = \frac{k}{6} t^3 + \frac{b}{2} t^2 - t + a.$$

Let us assume that p has two positive roots r_1 and r_2 ($r_1 \leqslant r_2$), and a negative root, $-r_0$. Let $\{t_{\alpha,n}\}$ the sequence of the family (10).

(a) *If*

$$r_1 < r_2, \lambda_\alpha = \frac{(r_2 - r_1)(r_0 + r_1) + 2(1-\alpha)(r_0 + 2r_1 - r_2)^2}{(r_0 + r_1)^2} < 1$$

and $\theta_\alpha = \frac{r_1}{r_2}\sqrt{\lambda_\alpha}$, then

$$r_1 - t_{\alpha,n} \sim \frac{(r_2 - r_1)\theta_\alpha^{3^n}}{\sqrt{\lambda_\alpha} - \theta_\alpha^{3^n}}, \qquad n \geqslant 0.$$

(b) *If $r_1 = r_2$, we have*

$$r_1 - t_{\alpha,n} \sim r_1 \left(\frac{3(11 - 5\alpha)}{80} \right)^n, \qquad n \geqslant 0.$$

PROOF. The first part is analogous to Lemma 3.1 in the case (a). To prove the second part, let

$$\tilde{Q}_\alpha(t_{\alpha,n}) = \frac{a_{\alpha,n+1}}{a_{\alpha,n}} = \frac{r_1 - P_\alpha(t_{\alpha,n})}{r_1 - t_{\alpha,n}}.$$

Notice that for t closely r_1

$$\tilde{Q}_\alpha(t) \sim \tilde{Q}_\alpha(r_1) = \frac{3(11 - 5\alpha)}{80}$$

and, by recurrence, (b) follows. ■

The point is now to study the sequences $\{x_{\alpha,n}\}$, $\alpha \in [0,1]$, defined in Banach spaces.

LEMMA 3.3. *The iterations* (4) *are well defined for* $\alpha \in [0,1]$ *and* $n \geqslant 0$, *converge to* x^*, *a solution of* (1), *and*

$$\|x_{\alpha,n+1} - x_{\alpha,n}\| \leqslant t_{\alpha,n+1} - t_{\alpha,n}, \tag{11}$$

$$\|x^* - x_{\alpha,n}\| \leqslant r_1 - t_{\alpha,n}. \tag{12}$$

PROOF. For each $\alpha \in [0,1]$ we prove for $n \geqslant 0$

$[\mathrm{I}_n]$ There exists $\Gamma_{\alpha,n} = F'(x_{\alpha,n})^{-1}$.

$[\mathrm{II}_n]$ $\|\Gamma_0 F''(x_{\alpha,n})\| \leqslant -\dfrac{p''(t_{\alpha,n})}{p'(t_0)}$.

$[\mathrm{III}_n]$ $\|\Gamma_{\alpha,n} F'(x_0)\| \leqslant \dfrac{p'(t_0)}{p'(t_{\alpha,n})}$.

$[\mathrm{IV}_n]$ $\|\Gamma_0 F(x_{\alpha,n})\| \leqslant -\dfrac{p(t_{\alpha,n})}{p'(t_0)}$.

$[\mathrm{V}_n]$ There exists $H(x_{\alpha,n}) = [I - L_F(x_{\alpha,n})]^{-1}$ and $\|H(x_{\alpha,n})\| \leqslant \dfrac{1}{1 - L_p(t_{\alpha,n})}$.

[VI_n] There exists $G_\alpha(x_{\alpha,n}) = I + \alpha L_F(x_{\alpha,n}) H(x_{\alpha,n})$ and

$$\|G_\alpha(x_{\alpha,n})\| \leqslant 1 + \frac{\alpha L_p(t_{\alpha,n})}{1 - L_p(t_{\alpha,n})}.$$

Notice that [I_0]–[VI_0] are a consequence of the hypothesis (i)–(iv). Taking into account p defined in (9) satisfies $L_p(t) \leqslant 1/2$, then [V_{n+1}] and [VI_{n+1}] follow as a consequence of [II_{n+1}], [III_{n+1}] and [IV_{n+1}]. Thus, we prove [I_{n+1}]–[IV_{n+1}] by using induction. Applying Altman's technique (see [13, 15]), the conditions [I_{n+1}], [II_{n+1}] and [III_{n+1}] are satisfied immediately.

To prove [IV_{n+1}], let us write $w_{\alpha,n} = L_p(t_{\alpha,n})$ and

$$z_{\alpha,n} = L_F(x_{\alpha,n}) G_\alpha(x_{\alpha,n}) \Gamma_{\alpha,n} F(x_{\alpha,n}).$$

Then

$$\|z_{\alpha,n}\| \leqslant \|L_F(x_{\alpha,n})\| \, \|G_\alpha(x_{\alpha,n})\| \, \|\Gamma_{\alpha,n} F(x_{\alpha,n})\|$$
$$\leqslant -w_{\alpha,n}\left(1 + \frac{\alpha w_{\alpha,n}}{1 - w_{\alpha,n}}\right) \frac{p(t_{\alpha,n})}{p'(t_{\alpha,n})}.$$

We deduce by Taylor's formula and taking into account (4) that

$$\begin{aligned}
F(x_{\alpha,n+1}) &= F(x_{\alpha,n}) + F'(x_{\alpha,n})(x_{\alpha,n+1} - x_{\alpha,n}) \\
&\quad + \frac{1}{2} F''(x_{\alpha,n})(x_{\alpha,n+1} - x_{\alpha,n})^2 \\
&\quad + \int_{x_{\alpha,n}}^{x_{\alpha,n+1}} [F''(x) - F''(x_{\alpha,n})](x_{\alpha,n+1} - x)\, dx \\
&= \frac{1-\alpha}{2} F''(x_{\alpha,n}) \Gamma_{\alpha,n} F(x_{\alpha,n}) L_F(x_{\alpha,n}) \Gamma_{\alpha,n} F(x_{\alpha,n}) \\
&\quad + \frac{1}{8} F''(x_{\alpha,n}) \left(L_F(x_{\alpha,n}) G_\alpha(x_{\alpha,n}) \Gamma_{\alpha,n} F(x_{\alpha,n})\right)^2 \\
&\quad + \int_{x_{\alpha,n}}^{x_{\alpha,n+1}} [F''(x) - F''(x_{\alpha,n})](x_{\alpha,n+1} - x)\, dx.
\end{aligned}$$

Therefore,

$$\|\Gamma_{\alpha,0} F(x_{\alpha,n+1})\| \leqslant \frac{1}{2} p(t_{\alpha,n}) w_{\alpha,n}^2 \left[1 - \alpha + \frac{w_{\alpha,n}}{4}\left(1 + \frac{\alpha w_{\alpha,n}}{1 - w_{\alpha,n}}\right)^2\right]$$

$$+ \frac{k}{6}(t_{\alpha,n+1} - t_{\alpha,n})^3.$$

Repeating the same process for the polynomial p, we obtain

$$p(t_{\alpha,n+1}) = \frac{1}{2} p(t_{\alpha,n}) w_{\alpha,n}^2 \left[1 - \alpha + \frac{w_{\alpha,n}}{4}\left(1 + \frac{\alpha w_{\alpha,n}}{1 - w_{\alpha,n}}\right)^2\right]$$

$$+ \frac{k}{6}(t_{\alpha,n+1} - t_{\alpha,n})^3.$$

Consequently,

$$\|\Gamma_{\alpha,0} F(x_{\alpha,n+1})\| \leqslant p(t_{\alpha,n+1}) = -\frac{p(t_{\alpha,n+1})}{p'(t_{\alpha,0})}. \tag{13}$$

So we conclude the induction.

Furthermore,

$$\|x_{\alpha,n+1} - x_{\alpha,n}\| = \left\|\left(I + \frac{1}{2} L_F(x_{\alpha,n}) G_\alpha(x_{\alpha,n})\right)\Gamma_{\alpha,n} F(x_{\alpha,n})\right\|$$

$$\leqslant -\left[1 + \frac{L_p(t_{\alpha,n})}{2}\left(1 + \frac{\alpha L_p(t_{\alpha,n})}{1 - L_p(t_{\alpha,n})}\right)\right]\frac{p(t_{\alpha,n})}{p'(t_{\alpha,n})}$$

$$= t_{\alpha,n+1} - t_{\alpha,n}.$$

Hence (11) holds and $\{t_{\alpha,n}\}$ majorizes $\{x_{\alpha,n}\}$. The convergence of $\{t_{\alpha,n}\}$ implies the convergence of $\{x_{\alpha,n}\}$ to a limit x^*. When $n \to \infty$ in (13), we deduce $F(x^*) = 0$.

Finally, for $q \geqslant 0$,

$$\| x_{\alpha, n+q} - x_{\alpha, n} \| \leqslant t_{\alpha, n+q} - t_{\alpha, n},$$

and letting $q \to \infty$, we obtain (12). ■

We are now proving the following Kantorovich-type theorem.

THEOREM 3.4. *Let us assume that conditions (i)–(iv) hold and* $\overline{B(x_{\alpha,0}, r_1)} \subset \Omega_0$. *Then, the sequences* (4) *are well defined for* $\alpha \in [0,1]$ *and* $n \geqslant 0$, *lie in* $\overline{B(x_0, r_1)}$ *and converge to a solution* x^* *of* (1). *The solution* x^* *is the unique solution of* (1) *in*

(i) *the ball* $B(x_0, r_2)$ *if* $k = 0$,
(ii) *the ball* $\overline{B(x_0, r_2)}$ *if* $k > 0$.

Furthermore, the following error estimates are true for all $n \geqslant 0$

$$\| x^* - x_{\alpha, n} \| \leqslant r_1 - t_{\alpha, n}$$

where $r_1 - t_{\alpha, n}$ *are estimated in Lemmas* 3.1 *and* 3.2.

PROOF. The convergence to the solution x^* of (1) follows as a consequence of Lemmas 2.1 and 3.3.

To show the uniqueness, we assume that $k > 0$, and there exists another solution y^* of (1) in $\overline{B(x_0, r_2)}$. Following Argyros and Chen [4], we have

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\, dt(y^* - x^*).$$

We prove that $\int_0^1 F'(x^* + t(y^* - x^*))\, dt$ is invertible. Notice that

$$\begin{aligned} I - \Gamma_{\alpha,0} \int_0^1 F'(x^* + t(y^* - x^*))\, dt \\ = -\Gamma_{\alpha,0} \int_0^1 \int_{x_{\alpha,0}}^{x^* + t(y^* - x^*)} F''(z)\, dz\, dt \\ = -\Gamma_{\alpha,0} \int_0^1 \int_{x_{\alpha,0}}^{x^* + t(y^* - x^*)} [F''(x_{\alpha,0}) \\ + (F''(z) - F''(x_{\alpha,0}))]\, dz\, dt \end{aligned}$$

and then

$$\|I - \Gamma_{\alpha,0} \int_0^1 F'(x^* + t(y^* - x^*))\, dt\| \leqslant \frac{k}{6} r_2^2 + \left(\frac{k}{6} r_1 + \frac{b}{2}\right)(r_2 + r_1).$$

Let us define the polynomial

$$q(r) = \frac{k}{6} r^2 + \left(\frac{k}{6} r_1 + \frac{b}{2}\right) r + \left(\frac{k}{6} r_1^2 + \frac{b}{2} r_1 - 1\right). \tag{14}$$

Observe that $q(r_1) = -p'(r_1) < 0$ and $q(0) < 0$. By using Cardano's formulas, we have $r_1 + r_2 = r_0 - 3b/k$ and $r_1 r_2 = 6a/kr_0$ where $-r_0$, r_1 and r_2 are the roots of (9). Then $q(r_2) = -(2a + br_0^2)/r_0 < 0$, and the result holds. In the case $k = 0$, we have $q(r_2) = 0$. So the proof is completed. ■

REMARK. Notice that it is possible to extend the domain of uniqueness of solutions in the case $k > 0$ until $B(x_0, r_3)$ where r_3 is the positive root of polynomial (14).

4. EXAMPLES

EXAMPLE 1. First we consider the system of equations $F(x, y) = 0$, where

$$F(x, y) = (xy - 1,\ xy + x - 2y).$$

Let us consider the max-norm in $x = \mathbb{R}^2$. For a bilinear operator B on X defined by the following calculation scheme

$$B(x, y) = (x_1, x_2) \begin{pmatrix} b_1^{11} & b_1^{12} \\ b_1^{21} & b_1^{22} \\ \hline b_2^{11} & b_2^{12} \\ b_2^{21} & b_2^{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} b_1^{11} x_1 + b_1^{21} x_2 & b_1^{12} x_1 + b_1^{22} x_2 \\ b_2^{11} x_1 + b_1^{21} x_2 & b_2^{12} x_1 + b_2^{22} x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} b_1^{11} x_1 y_1 + b_1^{21} x_2 y_1 + b_1^{12} x_1 y_2 + b_1^{22} x_2 y_2 \\ b_2^{11} x_1 y_1 + b_2^{21} x_2 y_1 + b_2^{12} x_1 y_2 + b_2^{22} x_2 y_2 \end{pmatrix},$$

$$x = (x_1, x_2) \in X, \qquad y = (y_1, y_2) \in X,$$

we consider the norm [17]

$$\|B\| = \sup_{\|x\|=1} \max_{i} \sum_{j=1}^{2} \left| \sum_{k=1}^{2} b_i^{jk} x_k \right|.$$

Use the definition of the first and second derivatives of operator F to obtain

$$F'(x, y) = \begin{pmatrix} y & x \\ y+1 & x-2 \end{pmatrix}$$

and

$$F''(x, y) = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ \hline 0 & 1 \\ 1 & 0 \end{array} \right).$$

We choose $x_0 = (x_0, y_0) = (1.75, 1.75)$ for Theorem 3.4 and study the solution of equation $F(x, y) = 0$ in Ω_0, the ball of center (x_0, y_0) and radius 1.75 (in order to assure that $F'(x, y)^{-1}$ exists). So we get the following results

$$a = \|\Gamma_0 F(x_0, y_0)\| = \frac{216}{336}, \qquad b = \|\Gamma_0 F''(x, y)\| = \frac{16}{21} \quad \text{and} \quad k = 0.$$

Therefore, from the definition (9),

$$p(t) = \frac{8}{21} t^2 - t + \frac{216}{336}.$$

This polynomial has two positive real roots $r_1 = 1.125$ and $r_2 = 1.5$. Hence, by Theorem 3.4, the sequence of iterates is well defined in $B(x_0, r_1)$ and converges to the solution $(x^*, y^*) = (1, 1)$ of $F(x, y) = 0$ for all $0 \leqslant \alpha \leqslant 1$. Moreover, the solution $(x^*, y^*) = (1, 1)$ is unique in the ball $B(x_0, r_2)$.

Starting at $(x_{0,0}, y_{0,0}) = (1.75, 1.75) = (x_{1,0}, y_{1,0})$, we obtain for $\alpha = 0$ and $\alpha = 1$ the sequences given in Tables 1 and 2, respectively.

The following example suggests new approaches to the solution of an integral equation.

TABLE 1
CHEBYSHEV METHOD

n	$x_{0,n}$	$y_{0,n}$
0	1.750000000000000	1.750000000000000
1	1.083090379008746	1.041545189504373
2	1.000112946940708	1.000056473470354
3	1.000000000000320	1.000000000000160
4	1.000000000000000	1.000000000000000

EXAMPLE 2. Let us consider the integral equation

$$F(x)(s) = x(s) - s + \frac{1}{2}\int_0^1 s\cos(x(t))\,dt$$

in the space $X = C[0,1]$ of all continuous functions on the interval [0, 1] with the norm

$$\|x\| = \max_{s\in[0,1]} |x(s)|.$$

Set $x_0 = x_0(s) = s$ for Theorem 3.4 and use the definition of the first and second Fréchet derivatives of the operator F

$$F'(z)\,x(s) = x(s) - \frac{s}{2}\int_0^1 x(t)\sin(z(t))\,dt$$

and

$$F''(z)\,xy(s) = -\frac{s}{2}\int_0^1 x(t)\,y(t)\cos(z(t))\,dt$$

TABLE 2
CONVEX ACCELERATION OF NEWTON'S METHOD

n	$x_{1,n}$	$y_{1,n}$
0	1.750000000000000	1.750000000000000
1	1.037573630035104	1.018786815017552
2	1.000000337688556	1.000000168844278
3	1.000000000000000	1.000000000000000

to obtain

$$a = \frac{\sin 1}{2 - \sin 1 + \cos 1} = b \quad \text{and} \quad k = \frac{1}{2 - \sin 1 + \cos 1} \neq 0.$$

So the polynomial (9) is

$$p(t) = \frac{1}{6(2 - \sin 1 + \cos 1)} \times \left[t^3 + 3(\sin 1) t^2 - 6(2 - \sin 1 + \cos 1) t + 6 \sin 1 \right].$$

This polynomial has two positive real roots $r_1 = 0.60957$ and $r_2 = 1.70991$, and a negative root $-r_0 = -4.84389$. Then, we have that $F(x) = 0$ has a root x^* in $\{v \in C[0,1]; \|v - s\| \leqslant 0.60957\}$ and x^* is unique in $\{v \in C[0,1]; \|v - s\| < r_3\}$ where $r_3 = 3.00806$. Moreover, the best error estimates are attained when $\alpha = 1$ in Lemma 3.2

$$\|x^* - x_{\alpha,n}\| \leqslant r_1 - t_{\alpha,n} \sim \frac{1.10034(0.160128)^{3^n}}{0.449177 - (0.160128)^{3^n}},$$

for all $n \geqslant 0$.

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