Pergamon

Appl. Math. Lett. Vol. 10, No. 2, pp. 63–65, 1997

0893-9659/97 $17.00 + 0.00

PII: S0893-9659(97)00012-8

New Recurrence Relations for Chebyshev Method

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(Received May 1996; accepted August 1996)

Communicated by W. C. Rheinboldt

Abstract—New conditions on the convergence of the Chebyshev method in Banach spaces are stated by using recurrence relations. The results are compared with the ones obtained by other authors.

Keywords—Nonlinear equations in Banach spaces, Third-order method, Chebyshev method, Convergence theorem.

The Chebyshev method for solving a nonlinear equation in Banach spaces is a well-known third-order method. Third-order methods are not considered by many authors because of their high computational cost, mainly for the evaluation of the second Fréchet derivative. However, in some cases, the rise in the velocity of convergence can justify their use. For instance, these methods have been successfully used in the solution of nonlinear integral equations [1,2].

Among the classical third-order methods (Chebyshev, Halley, super-Halley, ...), maybe the first one is the most useful, because it needs one less inversion of a linear operator than the others.

The Chebyshev iteration for an operator F defined between two Banach spaces X and Y is defined by

$$x_{n+1} = x_n - \left[I + \frac{1}{2} L_F(x_n)\right] \Gamma_n F(x_n), \tag{1}$$

where I is the identity operator on X, $\Gamma_n = F'(x_n)^{-1}$ and $L_F(x_n)$ is the linear operator on X,

$$L_F(x_n) = \Gamma_n F''(x_n) \Gamma_n F(x_n).$$

Until now, necessary conditions for the convergence of (1) have been established assuming the second Fréchet derivative of F satisfies a Lipschitz condition

$$\|F''(x) - F''(y)\| \le k \|x - y\|, \tag{2}$$

for x and y in a suitable region of X. See [1,3] for more information. The technique developed by these authors is an extension of the technique followed by Kantorovich and other authors [4,5] for studying Newton's method.

Supported in part by a grant from the University of La Rioja.

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Recently, Smale [6] obtained the convergence of Newton's method for analytic maps from data at one point, instead of the region conditions (2) in the Newton-Kantorovich theorem.

Smale-like theorems for the convergence of iterative processes assume that the following inequalities are satisfied at a point x_0:

$$\frac{1}{k!}\left\|\Gamma_0 F^{(k)}(x_0)\right\| \|\Gamma_0 F(x_0)\|^{k-1} \le h^{k-1}, \qquad k \ge 2. \tag{3}$$

The constant h is different for different processes [7,8].

Our goal in this paper is to prove the convergence of (1), just assuming F'' is bounded and a punctual condition. We also show an example where our conditions are fulfilled and the previous ones fail.

THEOREM 1. *Let $F(x) = 0$ be an equation, where F is a twice continuously Fréchet differentiable operator in an open convex domain $\Omega \subseteq X$. Let $x_0 \in \Omega$ be a point where the operator $\Gamma_0 = F'(x_0)^{-1}$ exists. Define*

$$M_0 = \sup_{x\in\Omega} \|\Gamma_0 F''(x)\|, \qquad \alpha_0 = \|\Gamma_0 F(x_0)\| M_0,$$

when this supreme is finite, and

$$r_0 = \frac{(1+\alpha_0/2)\|\Gamma_0 F(x_0)\|}{1-\alpha_0\left(1+(1+\alpha_0/2)^2\right)}.$$

Then, if

$$\alpha_0 M_0 \le r = 0.326664\ldots \tag{4}$$

(r is the smallest positive root of the polynomial $p(x) = 2x^4+7x^3-4x^2-24x+8$) and $\overline{B(x_0,r_0)} = \{x \in X;\ \|x-x_0\| \le r_0\}$ is contained in Ω, the Chebyshev sequence (1) is well defined, is contained in $\overline{B(x_0,r_0)}$, and converges to x^*, a solution of $F(x) = 0$. Furthermore, x^* is the unique solution in $\overline{B(x_0, 2/M_0 - r_0)}$.

PROOF. For $n \ge 0$, we define $M_n = \sup_{x\in\Omega}\|\Gamma_n F''(x)\|$, $\alpha_n = \|\Gamma_n F(x_n)\| M_n$, $\beta_n = (1+\alpha_n/2)\,\alpha_n$.

Notice that the hypothesis guarantees the existence of x_1 in (1) and

$$\|x_1 - x_0\| \le \left(1+\frac{\alpha_0}{2}\right)\|\Gamma_0 F(x_0)\|.$$

Besides, it can be shown without difficulty that $\Gamma_1 F'(x_0)$ exists and

$$\|\Gamma_1 F'(x_0)\| \le \frac{1}{1-\beta_0} < 2.$$

Then x_2 is defined, and taking into account (1) and

$$\|\Gamma_0 F(x_1)\| \le \frac{\alpha_0}{2}\left(1+\left(1+\frac{\alpha_0}{2}\right)^2\right)\|\Gamma_0 F(x_0)\|,$$

we have

$$\|x_2 - x_1\| \le \left(1+\frac{\alpha_0}{2}\right)\|\Gamma_0 F(x_0)\|.$$

Finally, $\alpha_1 \le \alpha_0$. Indeed, by (4) and the above recurrence relations,

$$\alpha_1 = \|\Gamma_1 F(x_1)\| M_1 \le \frac{\alpha_0^2+\beta_0^2}{2(1-\beta_0)^2} \le \alpha_0.$$

Following an inductive reasoning, we can replace x_1 by x_2, x_2 by x_3, and in general, x_{n-1} by x_n to obtain that there exists $\Gamma_n F'(x_{n-1})$ and the following recurrence relations:

$$\begin{aligned}
\|\Gamma_n F'(x_{n-1})\| &\le \frac{1}{1-\beta_{n-1}} \le \cdots \le \frac{1}{1-\beta_0} < 2, \\
\|\Gamma_{n-1} F(x_n)\| &\le \frac{\alpha_{n-1}}{2}\left(1+\left(1+\frac{\alpha_{n-1}}{2}\right)^2\right)\|\Gamma_{n-1}F(x_{n-1})\|, \\
\|x_{n+1}-x_n\| &\le \left(1+\frac{\alpha_n}{2}\right)\|\Gamma_n F(x_n)\|, \\
\alpha_n &\le \alpha_{n-1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le \left(1+\frac{\alpha_n}{2}\right)\|\Gamma_n F(x_n)\| \\
&\le \left(1+\frac{\alpha_0}{2}\right)\alpha_{n-1}\left(1+\left(1+\frac{\alpha_{n-1}}{2}\right)^2\right)\|\Gamma_{n-1}F(x_{n-1})\| \\
&\le \cdots \le \left(1+\frac{\alpha_0}{2}\right)\|\Gamma_0 F(x_0)\|\left[\alpha_0\left(1+\left(1+\frac{\alpha_0}{2}\right)^2\right)\right]^n.
\end{aligned}$$

As $\alpha_0(1+(1+\alpha_0/2)^2) < 1$, $\{x_n\}$ is a Cauchy sequence, and therefore, converges to $x^* \in \overline{B(x_0, r_0)}$.

To prove that $F(x^*) = 0$, notice that when $n \to \infty$, $\|\Gamma_{n-1}F(x_n)\| \to 0$. As $\|F(x_n)\| \le \|F'(x_{n-1})\|\|\Gamma_{n-1}F(x_n)\|$ and $\{\|F'(x_n)\|\}$ is a bounded sequence, we deduce $\|F(x_n)\| \to 0$ and by continuity, $F(x^*) = 0$.

Now, to show the unicity, suppose that $y^* \in \overline{B(x_0, 2/M_0 - r_0)}$ is another solution of $F(x) = 0$. Then,

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\,dt\,(y^* - x^*).$$

We have to prove that the operator $\int_0^1 F'(x^* + t(y^* - x^*))\,dt$ has an inverse and then $y^* = x^*$. The same proof as in [3] works in this case and the unicity can be deduced. ∎

Next, we give an example where the conditions of Theorem 1 are satisfied, however conditions (2) and (3) fail. Consider the function

$$f(x) = x^3 \ln x^2 + x^2 - 10x + 2.5, \qquad f(0) = 2.5$$

defined in $X = [-1, 1]$ and let $x_0 = 0$. Notice that the derivatives $f^{(k)}(x)$ are not defined in x_0 for $k \ge 3$. Then, the Smale-like conditions (4) do not work in this case. Moreover, neither does f'' satisfy the Lipschitz condition (3). Therefore, the Kantorovich theorem also fails. However,

$$\left|\frac{f(0)}{f'(0)}\right| = 0.25, \qquad \sup_{x\in X}\left|\frac{f''(x)}{f'(0)}\right| = 1.2,$$

$\alpha_0 = 0.3 < r = 0.326664\ldots$, and Theorem 1 can be applied.

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