BULL. AUSTRAL. MATH. SOC. 47H10, 65J15
VOL. 55 (1997) [113-130]

A FAMILY OF CHEBYSHEV-HALLEY TYPE METHODS IN BANACH SPACES

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A family of third-order iterative processes (that includes Chebyshev and Halley's methods) is studied in Banach spaces. Results on convergence and uniqueness of solution are given, as well as error estimates. This study allows us to compare the most famous third-order iterative processes.

INTRODUCTION

Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain $\Omega_0 \subseteq \Omega$. Let us assume that $F'(x_0)^{-1} \in \mathcal{L}(Y,X)$ exists at some $x_0 \in \Omega_0$, where $\mathcal{L}(Y,X)$ is the set of bounded linear operators from Y into X.

Among the third-order methods for solving the equation

$$F(x) = 0 \tag{1}$$

we have:

- Chebyshev's method [**3, 5, 15**],

$$x_{n+1} = x_n - \left[I + \frac{1}{2}L_F(x_n)\right] F'(x_n)^{-1}F(x_n), \quad n \geqslant 0,$$

- Halley's method (or method of tangent hyperbolas) [**1, 2, 4, 6, 7, 10, 13, 15, 20**],

$$x_{n+1} = x_n - \left[I + \frac{1}{2}L_F(x_n)\left[I - \frac{1}{2}L_F(x_n)\right]^{-1}\right] F'(x_n)^{-1}F(x_n), \quad n \geqslant 0,$$

and

- convex acceleration of Newton's method (or super-Halley's method) [**8, 12, 14**],

$$x_{n+1} = x_n - \left[I + \frac{1}{2}L_F(x_n)\left[I - L_F(x_n)\right]^{-1}\right] F'(x_n)^{-1}F(x_n), \quad n \geqslant 0.$$

Received 26th February, 1996
Supported in part by a grant of the University of La Rioja.

We have denoted by I the identity operator on X and by $L_F(x)$ the linear operator defined by

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x), \quad x \in X,$$

provided than $F'(x)^{-1}$ exists. This operator and its connection with Newton's method were studied in [**11**]. For a real function f, the expression

$$L_f(t) = \frac{f(t)f''(t)}{f'(t)^2}$$

is a measure of the convexity of f at t, the degree of logarithmic convexity [**13**].

In view of these methods, we define for $\alpha \in [0,1]$, $x_{\alpha,0} = x_0 \in \Omega_0$ and $n \geqslant 0$ the following one-parameter family of iterative processes

$$(2) \qquad x_{\alpha,n+1} = x_{\alpha,n} - \left[I + \frac{1}{2}L_F(x_{\alpha,n})\left[I - \alpha L_F(x_{\alpha,n})\right]^{-1}\right] F'(x_{\alpha,n})^{-1}F(x_{\alpha,n}).$$

This family extends the family of scalar iterative processes considered by Hernández and Salanova in [**15**], and includes, as particular cases, Chebyshev's method ($\alpha = 0$), Halley's method ($\alpha = 1/2$) and convex acceleration of Newton's method ($\alpha = 1$).

In this paper, we obtain results on existence and uniqueness of solution of (1), convergence of the sequences (2) to this solution under Kantorovich-type assumptions (see [**16, 17, 18**]) and error estimates. Finally, we give some examples to illustrate the previous results, analysing the velocity of convergence of different methods and comparing our error bounds with those that have been given by other authors.

Preliminaries

Following Yamamoto [**20**], we assume throughout this paper that

(i) There exists a continous linear operator $\Gamma_0 = F'(x_0)^{-1}$, $x_0 \in \Omega_0$.

(ii) $\|\Gamma_0(F''(x) - F''(y))\| \leqslant k\,\|x - y\|$, $x, y \in \Omega_0$, $k \geqslant 0$.

(iii) $\|\Gamma_0 F(x_0)\| \leqslant a$, $\qquad \|\Gamma_0 F''(x_0)\| \leqslant b$.

(iv) The equation

$$(3) \qquad p(t) \equiv \frac{k}{6}t^3 + \frac{b}{2}t^2 - t + a = 0$$

has one negative root and two positive roots r_1 and r_2 ($r_1 \leqslant r_2$) if $k > 0$, or has two positive roots r_1 and r_2 ($r_1 \leqslant r_2$) if $k = 0$. Equivalently,

$$a \leqslant \frac{b^2 + 4k - b\sqrt{b^2 + 2k}}{3k\left(b + \sqrt{b^2 + 2k}\right)} \quad \text{if} \quad k > 0,$$

or $ab \leqslant 1/2$ if $k = 0$.

For each $\alpha \in [0,1]$, let us define the scalar sequence $\{t_{\alpha,n}\}$ by

$$(4) \qquad t_{\alpha,0} = t_0 = 0, \; t_{\alpha,n+1} = t_{\alpha,n} - \left[1 + \frac{L_p(t_{\alpha,n})}{2(1-\alpha L_p(t_{\alpha,n}))}\right] \frac{p(t_{\alpha,n})}{p'(t_{\alpha,n})}, \; n \geqslant 0,$$

where p is the polynomial defined by (3).

We write $\Gamma_{\alpha,n} = F'(x_{\alpha,n})^{-1}$ and $H_{\alpha,n} = [I - \alpha L_F(x_{\alpha,n})]^{-1}$, when they exist.

Under the assumptions (i)–(iv) we prove that the sequences $\{t_{\alpha,n}\}$ and $\{x_{\alpha,n}\}$, are well-defined, converge to r_1 and a solution x^* of (1) respectively, and

$$\|x_{\alpha,n+1} - x_{\alpha,n}\| \leqslant t_{\alpha,n+1} - t_{\alpha,n}, \quad n \geqslant 0,$$
$$\|x^* - x_{\alpha,n}\| \leqslant r_1 - t_{\alpha,n}, \quad n \geqslant 0.$$

That is, $\{t_{\alpha,n}\}$ is a majorising sequence of $\{x_{\alpha,n}\}$ (see [**16, 19**]).

First, we give a general result on convergence of scalar sequences that includes, as a particular case, the family $\{t_{\alpha,n}\}$ defined by (4).

LEMMA 2.1. *Let p be the polynomial defined in (3) with two positive roots $r_1 \leqslant r_2$. Then the sequences*

$$s_{\alpha,0} = 0, \; s_{\alpha,n+1} = G_\alpha(s_{\alpha,n}), \; n \geqslant 0,$$

where

$$(5) \qquad G_\alpha(s) = s - \left[1 + \frac{L_p(s)}{2(1-\alpha L_p(s))}\right] \frac{p(s)}{p'(s)}$$

and

$$\alpha \leqslant \min\left\{2, 1 - \frac{L_{p'}(r_1)}{3}\right\},$$

converge to r_1. Moreover these sequences are increasing to r_1.

PROOF: Under the previous assumptions for p, it is well known, [**2**], that

$$0 \leqslant L_p(t) < \frac{1}{2}, \quad \text{for } t \in [0, r_1].$$

Therefore we have $s_{\alpha,n+1} \geqslant s_{\alpha,n}$.

On the other hand, we can write

$$G'_\alpha(s) = \frac{L_p(s)^2}{2(1-\alpha L_p(s))^2} \left[3(1-\alpha) + \alpha(2\alpha - 1)L_p(s) - L_{p'}(s)\right].$$

Taking into account that $L_{p'}$ is negative and increasing for $s \in [0, r_1]$, we have that $G'_\alpha(s) \geqslant 0$, $s \in [0, r_1]$ and the result follows. ∎

Besides, the convergence of the sequences $\{t_{\alpha,n}\}$ is of third order. That follows as a consequence of the following result of Gander [**10**].

Lemma 2.2. *Let r_1 be a simple zero of p and Δ any function satisfying $\Delta(0) = 1$, $\Delta'(0) = 1/2$, $|\Delta''(0)| < \infty$. The iteration*

$$t_{n+1} = t_n - \Delta\left(L_p(t_n)\right)\frac{p(t_n)}{p'(t_n)}$$

is of third order.

Although the convergence is cubic in general, in [**12**] it was established that the sequence $\{t_{1,n}\}$ (called convex acceleration of Newton's method) has convergence of order four when p is a quadratic polynomial. Moreover, in this case, two iterations of Newton's method are equal to one iteration of the sequence $\{t_{1,n}\}$.

Now, we center our study on obtaining error expressions for the sequences (4). When p is a quadratic polynomial, following Ostrowski [**17**], we derive the following error bounds.

Lemma 2.3. *Let p be the polynomial given by (3) with $k = 0$, that is*

$$p(t) = \frac{b}{2}t^2 - t + a.$$

We assume that p has two positive roots $r_1 \leqslant r_2$. Let $\{t_{\alpha,n}\}$ be the sequence defined in (4).

(a) *When $r_1 < r_2$, put $\theta = \dfrac{r_1}{r_2}$ and $R_\alpha = \dfrac{\theta + 2(1-\alpha)}{1 + 2(1-\alpha)\theta}$. Then we have:*

(a$_1$) *If $\alpha \in [0, 1/2)$ and $ab < \dfrac{\sqrt{8(1-\alpha)}}{\left(1 + \sqrt{2(1-\alpha)}\right)^2}$,*

$$(r_2 - r_1)\frac{\theta^{3^n}}{1 - \theta^{3^n}} \leqslant (r_2 - r_1)\frac{\left[\sqrt{R_\alpha}\theta\right]^{3^n}}{\sqrt{R_\alpha} - \left[\sqrt{R_\alpha}\theta\right]^{3^n}} \leqslant r_1 - t_{\alpha,n}$$
$$\leqslant (r_2 - r_1)\frac{\left[\sqrt{2(1-\alpha)}\theta\right]^{3^n}}{\sqrt{2(1-\alpha)} - \left[\sqrt{2(1-\alpha)}\theta\right]^{3^n}}.$$

(a$_2$) *If $\alpha = 1/2$,*

$$r_1 - t_{1/2,n} = (r_2 - r_1)\frac{\theta^{3^n}}{1 - \theta^{3^n}}.$$

(a$_3$) *If $\alpha \in (1/2, 1)$,*

$$(r_2 - r_1)\frac{\left[\sqrt{2(1-\alpha)}\theta\right]^{3^n}}{\sqrt{2(1-\alpha)} - \left[\sqrt{2(1-\alpha)}\theta\right]^{3^n}} \leqslant r_1 - t_{\alpha,n}$$
$$\leqslant (r_2 - r_1)\frac{\left[\sqrt{R_\alpha}\theta\right]^{3^n}}{\sqrt{R_\alpha} - \left[\sqrt{R_\alpha}\theta\right]^{3^n}} \leqslant (r_2 - r_1)\frac{\theta^{3^n}}{1 - \theta^{3^n}}.$$

(a_4) *Finally, if* $\alpha = 1$,

$$r_1 - t_{1,n} = \frac{(r_2 - r_1)\theta^{4^n}}{1 - \theta^{4^n}}.$$

(b) *When* $r_1 = r_2$, *we have*

$$r_1 - t_{\alpha,n} = r_1 \left(\frac{3 - 2\alpha}{4(2-\alpha)} \right)^n.$$

PROOF: Let us write $a_{\alpha,n} = r_1 - t_{\alpha,n}$, $b_{\alpha,n} = r_2 - t_{\alpha,n}$, $n \geqslant 0$. Thus

$$p(t_{\alpha,n}) = \frac{b}{2} a_{\alpha,n} b_{\alpha,n}, \qquad p'(t_{\alpha,n}) = -\frac{b}{2}(a_{\alpha,n} + b_{\alpha,n}).$$

By (4) we have

$$a_{\alpha,n+1} = r_1 - t_{\alpha,n+1} = a_{\alpha,n}^3 \frac{a_{\alpha,n} + 2(1-\alpha) b_{\alpha,n}}{(a_{\alpha,n} + b_{\alpha,n})(a_{\alpha,n}^2 + b_{\alpha,n}^2 + 2(1-\alpha) a_{\alpha,n} b_{\alpha,n})}, \tag{6}$$

and similarly

$$b_{\alpha,n+1} = r_2 - t_{\alpha,n+1} = b_{\alpha,n}^3 \frac{b_{\alpha,n} + 2(1-\alpha) a_{\alpha,n}}{(a_{\alpha,n} + b_{\alpha,n})(a_{\alpha,n}^2 + b_{\alpha,n}^2 + 2(1-\alpha) a_{\alpha,n} b_{\alpha,n})}.$$

If $r_1 < r_2$, we put $\theta = r_1/r_2 < 1$ and $\mu_{\alpha,n} = \dfrac{a_{\alpha,n}}{b_{\alpha,n}}$ to obtain

$$\mu_{\alpha,n+1} = \mu_{\alpha,n}^3 \frac{\mu_{\alpha,n} + 2(1-\alpha)}{1 + 2(1-\alpha)\mu_{\alpha,n}}.$$

Taking into account that the function

$$\frac{x + 2(1-\alpha)}{1 + 2(1-\alpha)x}$$

is decreasing when $\alpha \in [0, 1/2)$, is constant when $\alpha = 1/2$, is increasing when $\alpha \in (1/2, 1)$ and is the identity when $\alpha = 1$, we obtain the first part.

If $r_1 = r_2$, then $a_{\alpha,n} = b_{\alpha,n}$. Therefore, from (6), we have

$$a_{\alpha,n+1} = a_{\alpha,n} \frac{3 - 2\alpha}{4(2-\alpha)}.$$

By recurrence, the second part holds. ∎

When $k > 0$ the real sequences $\{t_{\alpha,n}\}$ in (4) are obtained from a cubic polynomial. In this case, it is difficult to obtain error bounds following Ostrowski's method. In the next lemma, we establish estimates for the error in this situation by using a new procedure.

Lemma 2.4. *Let p be the polynomial given by (3) with $k > 0$, that is*

$$p(t) = \frac{k}{6}t^3 + \frac{b}{2}t^2 - t + a.$$

Let us assume that p has two positive roots $r_1 \leqslant r_2$ and a negative root, $-r_0$. Let $\{t_{\alpha,n}\}$ be the sequence defined in (4). Then, if $r_1 < r_2$ and $\theta = \sqrt{\lambda_\alpha} r_1/r_2 < 1$,

$$r_1 - t_n \sim \frac{(r_2 - r_1)\theta^{3^n}}{\sqrt{\lambda_\alpha} - \theta^{3^n}}, \quad n \geqslant 0,$$

where

$$\lambda_\alpha = \frac{(r_2 - r_1)(r_0 + r_1) + 2(1-\alpha)(r_0 + 2r_1 - r_2)^2}{(r_0 + r_1)^2} < 1, \qquad \theta = \sqrt{\lambda_\alpha}\frac{r_1}{r_2} < 1.$$

If $r_1 = r_2$, we have

$$r_1 - t_n \sim r_1 \left(\frac{3 - 2\alpha}{4(2-\alpha)}\right)^n.$$

Proof: The polynomial p defined above can be written in the form

$$p(t) = \frac{k}{6}(r_1 - t)(r_2 - t)(r_0 + t).$$

Let us write $a_{\alpha,n} = r_1 - t_{\alpha,n}$, $b_{\alpha,n} = r_2 - t_{\alpha,n}$ and

$$Q(t_{\alpha,n}) = \frac{b^3_{\alpha,n} a_{\alpha,n+1}}{a^3_{\alpha,n} b_{\alpha,n+1}} = \frac{(r_1 - G_\alpha(t_{\alpha,n}))(r_2 - t_{\alpha,n})^3}{(r_2 - G_\alpha(t_{\alpha,n}))(r_1 - t_{\alpha,n})^3},$$

with G_α defined by (5).

As $G_\alpha(r_1) = r_1$, $G'_\alpha(r_1) = G''_\alpha(r_1) = 0$, we have for t close to r_1

$$\begin{aligned} Q(t) &\sim (r_2 - r_1)^2 \lim_{t \to r_1} \frac{r_1 - G_\alpha(t)}{(r_1 - t)^3} = \frac{G'''_\alpha(r_1)}{6}(r_2 - r_1)^2 \\ &= \frac{3(1-\alpha)p''(r_1)^2 - p'''(r_1)p'(r_1)}{6p'(r_1)^2}(r_2 - r_1)^2 \qquad (7) \\ &= \frac{(r_2 - r_1)(r_0 + r_1) + 2(1-\alpha)(r_0 + 2r_1 - r_2)^2}{(r_0 + r_1)^2} = \lambda_\alpha. \end{aligned}$$

Since $t_{\alpha,n} \to r_1$ when $n \to \infty$, we obtain

$$\frac{a_{\alpha,n}}{b_{\alpha,n}} \sim \left(\frac{a_{\alpha,n-1}}{b_{\alpha,n-1}}\right)^3 \lambda_\alpha \sim \cdots \sim \left(\sqrt{\lambda_\alpha}\frac{r_1}{r_2}\right)^{3^n} \frac{1}{\sqrt{\lambda_\alpha}},$$

and the first part holds.

If $r_1 = r_2$, let

$$\widetilde{Q}(t_{\alpha,n}) = \frac{a_{\alpha,n+1}}{a_{\alpha,n}} = \frac{r_1 - G_\alpha(t_{\alpha,n})}{r_1 - t_{\alpha,n}}.$$

Notice that for t close to r_1

$$\widetilde{Q}(t) \sim \widetilde{Q}(r_1) = \frac{3-2\alpha}{4(2-\alpha)}.$$

By recurrence, the second part also follows. ∎

We center now our study on the sequences $\{x_{\alpha,n}\}$, $\alpha \in [0,1]$, defined in Banach spaces.

Lemma 2.5. *With the above notation and assumptions, we can write $F(x_{\alpha,n+1})$ in the following way:*

$$F(x_{\alpha,n+1}) = \frac{1}{8}F''(x_{\alpha,n})y_{\alpha,n}^2 + \frac{1-\alpha}{2}F''(x_{\alpha,n})\Gamma_{\alpha,n}F(x_{\alpha,n})y_{\alpha,n} + \int_{x_{\alpha,n}}^{x_{\alpha,n+1}} [F''(x) - F''(x_{\alpha,n})](x_{\alpha,n+1} - x)\,dx,$$

where

$$y_{\alpha,n} = L_F(x_{\alpha,n})H_{\alpha,n}\Gamma_{\alpha,n}F(x_{\alpha,n}). \tag{8}$$

Proof: By Taylor's formula, and using (2), we deduce

$$\begin{aligned} F(x_{\alpha,n+1}) &= F(x_{\alpha,n}) + F'(x_{\alpha,n})(x_{\alpha,n+1} - x_{\alpha,n}) + \frac{1}{2}F''(x_{\alpha,n})(x_{\alpha,n+1} - x_{\alpha,n})^2 \\ &\quad + \int_{x_{\alpha,n}}^{x_{\alpha,n+1}} [F''(x) - F''(x_{\alpha,n})](x_{\alpha,n+1} - x)\,dx \\ &= -\frac{1}{2}F''(x_{\alpha,n})\Gamma_{\alpha,n}F(x_{\alpha,n})H_{\alpha,n}\Gamma_{\alpha,n}F(x_{\alpha,n}) + \frac{1}{2}F''(x_{\alpha,n})\left(\Gamma_{\alpha,n}F(x_{\alpha,n})\right)^2 \\ &\quad + \frac{1}{8}F''(x_{\alpha,n})y_{\alpha,n}^2 + \frac{1}{2}F''(x_{\alpha,n})\Gamma_{\alpha,n}F(x_{\alpha,n})y_{\alpha,n} \\ &\quad + \int_{x_{\alpha,n}}^{x_{\alpha,n+1}} [F''(x) - F''(x_{\alpha,n})](x_{\alpha,n+1} - x)\,dx. \end{aligned}$$

As $H_{\alpha,n} = I + \alpha L_F(x_{\alpha,n})H_{\alpha,n}$, the result holds. ∎

Lemma 2.6. *The iterates (2) are well defined for $\alpha \in [0,1]$ and $n \geqslant 0$, converge to x^*, a solution of (1), and*

$$\|x_{\alpha,n+1} - x_{\alpha,n}\| \leqslant t_{\alpha,n+1} - t_{\alpha,n}, \tag{9}$$

$$\|x^* - x_{\alpha,n}\| \leqslant r_1 - t_{\alpha,n}. \tag{10}$$

PROOF: For each $\alpha \in [0,1]$ we prove for $n \geqslant 0$:

[I_n] $\Gamma_{\alpha,n} = F'(x_{\alpha,n})^{-1}$ exists ($\Gamma_{\alpha,0} = \Gamma_0 = F'(x_0)^{-1}$).

[II_n] $\|\Gamma_0 F''(x_{\alpha,n})\| \leqslant -\dfrac{p''(t_{\alpha,n})}{p'(t_0)}$.

[III_n] $\|\Gamma_{\alpha,n} F'(x_0)\| \leqslant \dfrac{p'(t_0)}{p'(t_{\alpha,n})}$.

[IV_n] $\|\Gamma_0 F(x_{\alpha,n})\| \leqslant -\dfrac{p(t_{\alpha,n})}{p'(t_0)}$.

[V_n] $H_{\alpha,n} = [I - \alpha L_F(x_{\alpha,n})]^{-1}$ exists and $\|H_{\alpha,n}\| \leqslant \dfrac{1}{1-\alpha L_p(t_{\alpha,n})}$.

Since p defined in (3) satisfies $L_p(t) \leqslant 1/2$, (see [**2**]), then [V_{n+1}] follows as a consequence of [II_{n+1}], [III_{n+1}] and [IV_{n+1}]. Thus, we prove [I_{n+1}]–[IV_{n+1}] using induction. Applying Altman's technique, (see [**2**] or [**20**]), [I_{n+1}], [II_{n+1}] and [III_{n+1}] follow immediately.

To prove [IV_{n+1}], let us write $\nu_{\alpha,n} = L_p(t_{\alpha,n})$. Then

$$\|y_{\alpha,n}\| \leqslant \|L_F(x_{\alpha,n})\|\,\|H_{\alpha,n}\|\,\|\Gamma_{\alpha,n}F(x_{\alpha,n})\| \leqslant -\frac{\nu_{\alpha,n}p(t_{\alpha,n})}{(1-\alpha\nu_{\alpha,n})p'(t_{\alpha,n})},$$

were $y_{\alpha,n}$ is given by (8). Therefore, from Lemma 2.5,

$$\|\Gamma_0 F(x_{\alpha,n+1})\| \leqslant \frac{1}{8}\frac{\nu_{\alpha,n}^3 p(t_{\alpha,n})}{(1-\alpha\nu_{\alpha,n})^2} + \frac{1-\alpha}{2}\frac{\nu_{\alpha,n}^2 p(t_{\alpha,n})}{1-\alpha\nu_{\alpha,n}} + \frac{k}{6}(t_{\alpha,n+1}-t_{\alpha,n})^3.$$

Repeating the same process for the polynomial p, we obtain

$$p(t_{\alpha,n+1}) = \frac{1}{8}\frac{\nu_{\alpha,n}^3 p(t_{\alpha,n})}{(1-\alpha\nu_{\alpha,n})^2} + \frac{1-\alpha}{2}\frac{\nu_{\alpha,n}^2 p(t_{\alpha,n})}{1-\alpha\nu_{\alpha,n}} + \frac{k}{6}(t_{\alpha,n+1}-t_{\alpha,n})^3,$$

and consequently,

$$\|\Gamma_0 F(x_{\alpha,n+1})\| \leqslant p(t_{\alpha,n+1}) = -\frac{p(t_{\alpha,n+1})}{p'(t_0)}. \tag{11}$$

So we conclude the induction.

Next, we have

$$\begin{aligned}\|x_{\alpha,n+1}-x_{\alpha,n}\| &= \left\|\left[I + \frac{1}{2}L_F(x_{\alpha,n})H_{\alpha,n}\right]\Gamma_{\alpha,n}F(x_{\alpha,n})\right\| \\ &\leqslant \left[1 + \frac{L_p(t_{\alpha,n})}{2(1-\alpha L_p(t_{\alpha,n}))}\right]\frac{p(t_{\alpha,n})}{p'(t_{\alpha,n})} = t_{\alpha,n+1}-t_{\alpha,n},\end{aligned}$$

then (9) holds, and $\{t_{\alpha,n}\}$ majorises $\{x_{\alpha,n}\}$. The convergence of $\{t_{\alpha,n}\}$ (see Lemma 2.1 and its note) implies the convergence of $\{x_{\alpha,n}\}$ to a limit x^*. By letting $n \to \infty$ in (11), we deduce $F(x^*) = 0$.

Finally, for $p \geqslant 0$,

$$\|x_{\alpha,n+p} - x_{\alpha,n}\| \leqslant t_{\alpha,n+p} - t_{\alpha,n},$$

and by letting $p \to \infty$ we obtain (10). ∎

Lemma 2.7. *Under the previous assumptions we have, for $0 \leqslant \alpha \leqslant 1/2$*

$$\|x^* - x_{\alpha,n+1}\| \leqslant (r_1 - t_{\alpha,n+1}) \left(\frac{\|x^* - x_{\alpha,n}\|}{r_1 - t_{\alpha,n}} \right)^2,$$

and for $1/2 \leqslant \alpha \leqslant 1$,

$$\|x^* - x_{\alpha,n+1}\| \leqslant (r_1 - t_{\alpha,n+1}) \left(\frac{\|x^* - x_{\alpha,n}\|}{r_1 - t_{\alpha,n}} \right)^3.$$

Proof: The argument of Yamamoto (see [**12**] or [**20**] for details), and using

$$I + \frac{1}{2} L_F(x_{\alpha,n}) H_{\alpha,n} = H_{\alpha,n} \left(I - \left(\alpha - \frac{1}{2} \right) L_F(x_{\alpha,n}) \right),$$

shows that

$$\begin{aligned} x^* - x_{\alpha,n+1} = &-H_{\alpha,n}\Gamma_{\alpha,n} \int_{x_{\alpha,n}}^{x^*} [F''(x) - F''(x_{\alpha,n})](x^* - x)\, dx \\ &+ [I - H_{\alpha,n}](x^* - x_{\alpha,n}) - \frac{1}{2} H_{\alpha,n}\Gamma_{\alpha,n} F''(x_{\alpha,n})(x^* - x_{\alpha,n})^2 \\ &+ \frac{1}{2} H_{\alpha,n} L_F(x_{\alpha,n}) \Gamma_{\alpha,n} F(x_{\alpha,n}). \end{aligned}$$

Since $I - H_{\alpha,n} = -\alpha H_{\alpha,n} L_F(x_{\alpha,n})$, we obtain

$$\begin{aligned} x^* - x_{\alpha,n+1} = &-H_{\alpha,n}\Gamma_{\alpha,n} \int_{x_{\alpha,n}}^{x^*} [F''(x) - F''(x_{\alpha,n})](x^* - x)\, dx \\ &+ [I - H_{\alpha,n}](x^* - x_{\alpha,n}) + \alpha H_{\alpha,n} L_F(x_{\alpha,n}) \Gamma_{\alpha,n} \int_{x_{\alpha,n}}^{x^*} F''(x)(x^* - x)\, dx \\ &+ \frac{1}{2} H_{\alpha,n}\Gamma_{\alpha,n} F''(x_{\alpha,n}) \left[(\Gamma_{\alpha,n} F(x_{\alpha,n}))^2 - (x^* - x_{\alpha,n})^2 \right]. \end{aligned} \tag{12}$$

Then, for $0 \leqslant \alpha < 1/2$ it follows that

$$x^* - x_{n+1} = -H_{\alpha,n}\Gamma_{\alpha,n}\int_{x_{\alpha,n}}^{x^*}[F''(x) - F''(x_{\alpha,n})](x^* - x)\,dx$$
$$+ \left(\alpha - \frac{1}{2}\right) H_{\alpha,n}L_F(x_{\alpha,n})\Gamma_{\alpha,n}\int_{x_{\alpha,n}}^{x^*} F''(x)(x^* - x)\,dx$$
$$+ \frac{1}{2}H_{\alpha,n}\Gamma_{\alpha,n}F''(x_{\alpha,n})\left[\Gamma_{\alpha,n}\int_{x_{\alpha,n}}^{x^*} F''(x)(x^* - x)\,dx\right](x^* - x).$$

For $1/2 \leqslant \alpha \leqslant 1$, writing

$$\frac{1}{2} = \left(\alpha - \frac{1}{2}\right) + (1 - \alpha)$$

in (12), we deduce

$$x^* - x_{n+1} = -H_{\alpha,n}\Gamma_{\alpha,n}\int_{x_{\alpha,n}}^{x^*}[F''(x) - F''(x_{\alpha,n})](x^* - x)\,dx$$
$$- \left(\alpha - \frac{1}{2}\right) H_{\alpha,n}\Gamma_{\alpha,n}F''(x_{\alpha,n})\left[\Gamma_{\alpha,n}\int_{x_{\alpha,n}}^{x^*} F''(x)(x^* - x)\,dx\right]^2$$
$$+ (1 - \alpha)H_{\alpha,n}\Gamma_{\alpha,n}F''(x_{\alpha,n})\left[\Gamma_{\alpha,n}\int_{x_{\alpha,n}}^{x^*} F''(x)(x^* - x)\,dx\right](x^* - x).$$

Consequently, for $0 \leqslant \alpha < 1/2$,

$$\|x^* - x_{\alpha,n+1}\|$$
$$\leqslant \left[-\frac{k(r_1 - t_{\alpha,n})^3}{6(1 - \alpha L_p(t_{\alpha,n}))p'(t_{\alpha,n})}\right.$$
$$\left. + \frac{p''(t_{\alpha,n})(r_1 - t_{\alpha,n})}{2(1 - \alpha L_p(t_{\alpha,n}))p'(t_{\alpha,n})^2}\int_{t_{\alpha,n}}^{r_1} p''(z)(r_1 - z)\,dz\right]\left(\frac{\|x^* - x_{\alpha,n}\|}{r_1 - t_{\alpha,n}}\right)^3$$
$$+ \left(\alpha - \frac{1}{2}\right)\frac{L_p(t_{\alpha,n})}{(1 - \alpha L_p(t_{\alpha,n}))p'(t_{\alpha,n})}\int_{t_{\alpha,n}}^{r_1} p''(z)(r_1 - z)\,dz\left(\frac{\|x^* - x_{\alpha,n}\|}{r_1 - t_{\alpha,n}}\right)^2$$
$$\leqslant \left[-\frac{k(r_1 - t_{\alpha,n})^3}{6(1 - \alpha L_p(t_{\alpha,n}))p'(t_{\alpha,n})} + \frac{p''(t_{\alpha,n})(r_1 - t_{\alpha,n})}{2(1 - \alpha L_p(t_{\alpha,n}))p'(t_{\alpha,n})^2}\int_{t_{\alpha,n}}^{r_1} p''(z)(r_1 - z)\,dz\right.$$
$$\left. + \left(\alpha - \frac{1}{2}\right)\frac{L_p(t_{\alpha,n})}{(1 - \alpha L_p(t_{\alpha,n}))p'(t_{\alpha,n})}\int_{t_{\alpha,n}}^{r_1} p''(z)(r_1 - z)\,dz\right]\left(\frac{\|x^* - x_{\alpha,n}\|}{r_1 - t_{\alpha,n}}\right)^2$$
$$\leqslant (r_1 - t_{\alpha,n+1})\left(\frac{\|x^* - x_{\alpha,n}\|}{r_1 - t_{\alpha,n}}\right)^2.$$

In a similar way, we obtain for $1/2 \leqslant \alpha \leqslant 1$

$$\|x^* - x_{\alpha,n+1}\| \leqslant (r_1 - t_{\alpha,n+1}) \left(\frac{\|x^* - x_{\alpha,n}\|}{r_1 - t_{\alpha,n}} \right)^3 . \qquad \square$$

Main Theorem

We are now ready to prove the following Kantorovich-type theorem.

Theorem 3.1. *Let us assume that conditions (i)-(iv) hold and also*

$$\overline{B_\alpha} = \overline{B(x_{\alpha,1}, r_1 - t_{\alpha,1})} = \{x \in X; \|x - x_{\alpha,1}\| \leqslant r_1 - t_{\alpha,1}\} \subseteq \Omega_0. \tag{13}$$

Then:

(a) The sequences (2) are well-defined for $\alpha \in [0,1]$ *and* $n \geqslant 0$, *lie in* B_α *(the interior of* $\overline{B_\alpha}$*) for* $n \geqslant 1$ *and converge to a solution* x^* *of the equation (1).*

(b) The solution is unique in $B(x_0, r_2) \cap \Omega_0$ *if* $r_1 < r_2$ *or in* $\overline{B(x_0, r_1)} \cap \Omega_0$ *if* $r_1 = r_2$.

(c) The following error estimates hold:

(c_1) *For* $\alpha \in [0, 1/2)$, *let* $\tau_{\alpha,n}$ *and* $\sigma_{\alpha,n}$ *be the unique positive root and the smallest positive root of the polynomials*

$$\psi_{\alpha,n}(t) = k_{\alpha,n} t^2 - t + \delta_{\alpha,n}$$

and

$$\phi_{\alpha,n}(t) = k_{\alpha,n} t^2 + t - \delta_{\alpha,n},$$

respectively, where

$$k_{\alpha,n} = \frac{r_1 - t_{\alpha,n+1}}{(r_1 - t_{\alpha,n})^2}, \quad \delta_{\alpha,n} = \|x_{\alpha,n+1} - x_{\alpha,n}\| > 0.$$

Then we have

$$\tau_{\alpha,n} \leqslant \|x^* - x_{\alpha,n}\| \leqslant \sigma_{\alpha,n} \leqslant r_1 - t_{\alpha,n}.$$

(c_2) *For* $\alpha \in [1/2, 1]$, *let* $\tau^*_{\alpha,n}$ *and* $\sigma^*_{\alpha,n}$ *be the smallest positive root and the unique positive root of the polynomials*

$$\phi^*_{\alpha,n}(t) = k^*_{\alpha,n} t^3 + t - \delta_{\alpha,n},$$

and

$$\psi^*_{\alpha,n}(t) = k^*_{\alpha,n} t^3 - t + \delta_{\alpha,n},$$

respectively, where

$$k^*_{\alpha,n} = \frac{r_1 - t_{\alpha,n+1}}{(r_1 - t_{\alpha,n})^3}, \quad \delta_{\alpha,n} = \|x_{\alpha,n+1} - x_{\alpha,n}\| > 0.$$

Then we have

$$\sigma^*_{\alpha,n} \leqslant \|x^* - x_{\alpha,n}\| \leqslant \tau^*_{\alpha,n} \leqslant r_1 - t_{\alpha,n}.$$

PROOF: (a) follows as a consequence of Lemmas 2.1, 2.6 and the condition (13). To prove (b), let x^{**} be a solution of (1) in $B(x_0, r_2) \cap \Omega_0$ or in $\overline{B(x_0, r_2)} \cap \Omega_0$. Then, replacing x^* and r_1 by x^{**} and r_2 in Lemma 2.7, we have for $0 \leqslant \alpha < 1/2$,

$$\frac{\|x^{**} - x_{\alpha,n}\|}{r_2 - t_{\alpha,n}} \leqslant \left(\frac{\|x^{**} - x_{\alpha,n-1}\|}{r_2 - t_{\alpha,n-1}}\right)^2 \leqslant \cdots \leqslant \left(\frac{\|x^{**} - x_0\|}{r_2}\right)^{2^n}.$$

In a similar way, for $1/2 \leqslant \alpha \leqslant 1$,

$$\frac{\|x^{**} - x_{\alpha,n}\|}{r_2 - t_{\alpha,n}} \leqslant \left(\frac{\|x^{**} - x_{\alpha,n-1}\|}{r_2 - t_{\alpha,n-1}}\right)^3 \leqslant \cdots \leqslant \left(\frac{\|x^{**} - x_0\|}{r_2}\right)^{3^n}.$$

So we deduce for $r_1 < r_2$,

$$\|x^{**} - x_{\alpha,n}\| \leqslant (r_2 - t_{\alpha,n})\rho_n,$$

with

$$\rho_n = \left(\frac{\|x^{**} - x_0\|}{r_2}\right)^{2^n}, \quad 0 \leqslant \alpha < 1/2,$$

or

$$\rho_n = \left(\frac{\|x^{**} - x_0\|}{r_2}\right)^{3^n}, \quad 1/2 \leqslant \alpha \leqslant 1.$$

For $r_1 = r_2$

$$\|x^{**} - x_{\alpha,n}\| \leqslant r_1 - t_{\alpha,n}.$$

Since $\rho_n \to 0$ and $\{t_{\alpha,n}\}$ converges to r_1, we obtain in both cases

$$x^{**} = \lim_{n\to\infty} x_{\alpha,n} = x^*.$$

Finally, for $\alpha \in [0, 1/2)$, we deduce from Lemma 2.7

$$\|x^* - x_{\alpha,n}\| - \delta_{\alpha,n} \leqslant \|x^* - x_{\alpha,n+1}\| \leqslant k_{\alpha,n}\|x^* - x_{\alpha,n}\|^2,$$

and then $\phi_{\alpha,n}(\|x^* - x_{\alpha,n}\|) \geqslant 0$.

On the other hand,

$$\phi_{\alpha,n}(r_1 - t_{\alpha,n}) = k_{\alpha,n}(r_1 - t_{\alpha,n})^2 - (r_1 - t_{\alpha,n}) + \delta_{\alpha,n} = \delta_{\alpha,n} - (t_{\alpha,n+1} - t_{\alpha,n}) < 0,$$

and consequently

$$\|x^* - x_{\alpha,n}\| \leqslant \sigma_{\alpha,n} \leqslant r_1 - t_{\alpha,n}.$$

To obtain lower bounds notice that, by using Lemma 2.7 again,

$$\delta_{\alpha,n} - \|x^* - x_{\alpha,n}\| \leqslant \|x^* - x_{\alpha,n+1}\| \leqslant k_{\alpha,n} \|x^* - x_{\alpha,n}\|^2,$$

that is, $\psi_{\alpha,n}(\|x^* - x_{\alpha,n}\|) \geqslant 0$. Consequently,

$$\|x^* - x_{\alpha,n}\| \geqslant \tau_{\alpha,n}.$$

For $\alpha \in [1/2, 1]$, we obtain from Lemma 2.7

$$\|x^* - x_{\alpha,n}\| - \delta_{\alpha,n} \leqslant \|x^* - x_{\alpha,n+1}\| \leqslant k_{\alpha,n} \|x^* - x_{\alpha,n}\|^3.$$

and

$$\delta_{\alpha,n} - \|x^* - x_{\alpha,n}\| \leqslant \|x^* - x_{\alpha,n+1}\| \leqslant k_{\alpha,n} \|x^* - x_{\alpha,n}\|^3,$$

and the result also holds. □

NOTE. The condition (13) can be replaced by $\overline{B(x_0, r_1)} \subseteq \Omega_0$. (Notice that $\overline{B_\alpha} \subseteq \overline{B(x_0, r_1)}$ for $\alpha \in [0, 1]$.)

COROLLARY 3.2. *Under the previous assumptions, we have*

$$0.8\delta_{\alpha,n} < \left(-2 + \sqrt{8}\right)\delta_{\alpha,n} \leqslant \|x^* - x_{\alpha,n}\| \leqslant 2\delta_{\alpha,n}, \quad \alpha \in [0, 1/2),$$

and

$$0.89\delta_{\alpha,n} \leqslant \|x^* - x_{\alpha,n}\| \leqslant 1.5\delta_{\alpha,n}, \quad \alpha \in [1/2, 1].$$

PROOF: Let $\alpha \in [0, 1/2)$. The polynomial $\phi_{\alpha,n}$ has a minimum when $t = 1/(2k_{\alpha,n})$. Also

$$\phi_{\alpha,n}\left(\frac{1}{2k_{\alpha,n}}\right) = \delta_{\alpha,n} - \frac{1}{4k_{\alpha,n}} < 0.$$

Consequently $k_{\alpha,n}\delta_{\alpha,n} < 1/4$. Since $\phi_{\alpha,n}(2\delta_{\alpha,n}) < 0$, we deduce $\sigma_{\alpha,n} < 2\delta_{\alpha,n}$.

On the other hand, let

$$\widetilde{\psi}_{\alpha,n}(t) = \frac{1}{4\delta_{\alpha,n}} t^2 + t - \delta_{\alpha,n}.$$

As $\psi_{\alpha,n}(t) < \widetilde{\psi}_{\alpha,n}(t)$ for $t > 0$, and $\widetilde{\psi}_{\alpha,n}\left((-2 + \sqrt{8})\delta_{\alpha,n}\right) = 0$, we obtain $(-2 + \sqrt{8})\delta_{\alpha,n} < \tau_{\alpha,n}$.

For $\alpha \in [1/2, 1]$ the result follows in a similar way (see [20] for details). □

Table 1

n	$t_{0,n}$	$t_{1/2,n}$	$t_{1,n}$
0	0.000000000000000	0.000000000000000	0.000000000000000
1	8.148148148148148	8.571428571428571	9.333333333333333
2	9.936934743362793	9.980430528375734	9.999847409781033
3	9.999995122930333	9.999999925494193	10.000000000000000
4	10.000000000000000	10.000000000000000	10.000000000000000
5	10.000000000000000	10.000000000000000	10.000000000000000

Examples

Example 1. First we consider the real polynomial equation

$$p(t) \equiv (t-10)(t-20) = t^2 - 30t + 200 = 0.$$

In Table 1 we compare the sequences $\{t_{0,n}\}$ (Chebyshev's method), $\{t_{1/2,n}\}$ (Halley's method) and $\{t_{1,n}\}$ (convex acceleration of Newton's method), starting from the same point $t_{0,0} = t_{1/2,0} = t_{1,0} = 0$.

Example 2. Now we consider the system of equations $F(x,y) = 0$, where

$$F(x,y) = \left(x^2 - y - 2,\, y^3 - x^2 + y + 1\right).$$

Starting at $(x_{0,0}, y_{0,0}) = \left(x_{1/2,0}, y_{1/2,0}\right) = (x_{1,0}, y_{1,0}) = (6,3)$ we obtain the sequences given in Tables 2–4.

In the previous examples we have analysed the velocity of convergence of different sequences of the family (2), attaining the best results for $\alpha = 1$. In Theorem 3.1 we have only studied the sequences $\{x_{\alpha,n}\}$ for $\alpha \in [0,1]$. However, it is possible to obtain convergent sequences for $\alpha > 1$, and the convergence could be even faster, as happens in the real case. It is not difficult to see from (7) that $G'''(r_1) = 0$ for

$$\alpha = 1 - \frac{L_{p'}(r_1)}{3} \geqslant 1.$$

Then, the order of convergence increases for this value. For instance, when $p''(t)$ is constant, we obtain a fourth-order method for $\alpha = 1$, as was remarked in [**12**]. This result was extended to Banach spaces in the same paper.

The following example suggest new approaches to the solution of a integral equation (see also [**4, 5, 9**]).

Table 2. Chebyshev's method

n	$x_{0,n}$	$y_{0,n}$
0	6.000000000000000000	3.000000000000000000
1	2.719439840217446527	1.727937814357567444
2	1.830721648025601499	1.138820258896789643
3	1.733249320860063453	1.003038870277247239
4	1.732050823057711034	1.000000046348271498
5	1.732050807568877294	1.000000000000000000
6	1.732050807568877294	1.000000000000000000

Table 3. Halley's method

n	$x_{1/2,n}$	$y_{1/2,n}$
0	6.000000000000000000	3.000000000000000000
1	2.527771600300525920	1.581818181818181818
2	1.774166519412615274	1.057032561548124306
3	1.732099394264064235	1.000113501450570644
4	1.732050807569220348	1.000000000000974628
5	1.732050807568877294	1.000000000000000000
6	1.732050807568877294	1.000000000000000000

Table 4. Convex acceleration of Newton's method

n	$x_{1,n}$	$y_{1,n}$
0	6.000000000000000000	3.000000000000000000
1	2.195930445526441461	1.173690932311621967
2	1.726757444904059338	0.998981852656109923
3	1.732050203990691682	1.000000000352888045
4	1.732050807568877294	1.000000000000000000
5	1.732050807568877294	1.000000000000000000

EXAMPLE 3. In the space $X = C[0,1]$ of all continous functions on the interval $[0,1]$ with the norm

$$\|x\| = \max_{s\in[0,1]} |x(s)|,$$

we consider the equation $F(x) = 0$, where

$$F(x)(s) = x(s) - s + \frac{1}{2}\int_0^1 s\cos(x(t))\,dt, \quad x \in C[0,1],\ s \in [0,1].$$

With the notation of Theorem 3.1 and for $x_0 = x_0(s) = s$, we use the definition of the first and second Fréchet derivatives of the operator F to obtain

$$a = b = \frac{\sin 1}{2 - \sin 1 + \cos 1}, \quad k = \frac{1}{2 - \sin 1 + \cos 1}.$$

So the polynomial (3) is

$$p(t) = \frac{1}{6(2 - \sin 1 + \cos 1)}[t^3 + 3(\sin 1)t^2 - 6(2 - \sin 1 + \cos 1)t + 6\sin 1].$$

The positive roots of p are

$$r_1 = 0.6095694860276291, \quad r_2 = 1.70990829134757.$$

Then, we have that $F(x) = 0$ has a root in $\overline{B(x_0, r_1)}$. Besides, this is the unique root in $B(x_0, r_2)$. Some error bounds

$$\|x^* - x_{\alpha,n}\| \leqslant r_1 - t_{\alpha,n}$$

are shown in Table 5.

Table 5. Error bounds

n	$r_1 - t_{0,n}$	$r_1 - t_{1/2,n}$	$r_1 - t_{1,n}$
0	0.6095694860276291	0.6095694860276291	0.6095694860276291
1	0.0534834955243040	0.0495130055348865	0.0349873303274992
2	0.0001520166774545	0.0000984825547302	0.0000560164474543
3	0.0000000000042804	0.0000000000009129	0.0000000000001218
4	0.0000000000000000	0.0000000000000000	0.0000000000000000

Notice that the best error bounds are attained when $\alpha = 1$ (convex acceleration of Newton's method). For this same equation, and using Halley's method, Döring [9] obtained the bound

$$\|x^* - x_{1/2,2}\| \leqslant 0.000825.$$

Later, Candela and Marquina, [4, 5], gave the bounds

$$\|x^* - x_{0,2}\| \leqslant 0.00037022683427694$$

and

$$\|x^* - x_{1/2,2}\| \leqslant 0.00014987029635502$$

for Chebyshev's and Halley's methods respectively. Observe that the bounds given in Table 5 really improve the previous ones.

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