Journal of Algebra 323 (2010) 2363–2370

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Regularity conditions on skew and symmetric elements in superalgebras with superinvolution ☆

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A R T I C L E   I N F O

Article history:
Received 23 April 2008
Available online 11 March 2010
Communicated by Efim Zelmanov

Keywords:
Associative superalgebras
Semiprime superalgebras
Superinvolutions
Symmetric and skewsymmetric elements
Regularity conditions

A B S T R A C T

We study semiprime superalgebras with superinvolution, under certain additional regularity conditions. More precisely, we assume regularity conditions either on every nonzero homogeneous symmetric element, or on every nonzero homogeneous skewsymmetric element.

1. Introduction

Given an associative algebra A with involution, the algebraic properties of the set of symmetric elements H or the set of skewsymmetric elements K determine, in several cases, the algebraic structure of A as algebra. This is why the relationships between A, H and K have been investigated profusely by many authors.

In [1], I.N. Herstein compiled several of the results obtained until 1976 about this subject. In this paper we are concerned with one of them: Osborn's Theorem.

J.M. Osborn established in [9] that if A is a simple algebra of characteristic not 2 in which every nonzero symmetric element is invertible, then A must either be a division algebra or the set of all 2×2 matrices over a field. Later, in [10], he extended this result to a semisimple algebra A which is 2-torsion free (that is, $2x = 0$ implies $x = 0$), so that, if every symmetric element in A is either nilpotent or invertible, then A is either a division algebra, or a direct sum of a division algebra and its opposite, or the 2×2 matrices over a field.

☆ The authors have been supported by the Spanish Ministerio de Educación y Ciencia (MTM 2007-67884-C04-03).

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.02.022

Here we are interested in studying both of Osborn's theorems (as well as related work regarding the skewsymmetric elements) in associative superalgebras with superinvolution. All these results in the nongraded case can be found in [1].

Specifically we want to describe the semiprime superalgebras in which the nonzero homogeneous symmetric elements are invertible, and also the semiprime superalgebras with noncommutative even part in which every nonzero homogeneous skewsymmetric element is invertible. In both cases the superalgebra is either a division superalgebra or the direct sum of a division superalgebra and its opposite.

We also want to study prime superalgebras in which every nonzero homogeneous symmetric element is not nilpotent or prime superalgebras in which every nonzero homogeneous skewsymmetric element is not nilpotent. We will prove that, in these superalgebras, $xx^* \neq 0$ for every homogeneous element x.

Additional regularity conditions are imposed to skew or symmetric elements in order to analyze the structure of the superalgebra.

For a complete introduction to the basic definitions and examples of superalgebras, superinvolutions and prime and semiprime superalgebras, we refer the reader to [4] and [8].

Throughout the paper, unless otherwise stated, A will denote a nontrivial semiprime associative superalgebra with superinvolution $*$ over a commutative unital ring ϕ of scalars with $\frac{1}{2} \in \phi$. By a nontrivial superalgebra we understand a superalgebra with a nonzero odd part. Z will denote the even part of the center of A, H the Jordan superalgebra of symmetric elements of A, and K the Lie superalgebra of skew elements of A. If P is a subset of A, we will denote by $P_H = P \cap H$ and $P_K = P \cap K$. The following containments are straightforward to check, and they will be used throughout without explicit mention: $[K,K] \subseteq K$, $[K,H] \subseteq H$, $[H,H] \subseteq K$, $H \circ H \subseteq H$, $H \circ K \subseteq K$ and $K \circ K \subseteq H$.

If $Z \neq 0$ one can consider the localization $Z^{-1}A = \{z^{-1}a\colon\ 0 \neq z \in Z,\ a \in A\}$. If A is prime, then $Z^{-1}A$ is a central prime associative superalgebra over the field $Z^{-1}Z$. We call this superalgebra the central closure of A. We also say that A is a central order in $Z^{-1}A$. This terminology is not the standard one, for which the definition involves the extended centroid.

Let A be a prime superalgebra, and let $V = Z_H - \{0\}$ be the subset of regular symmetric elements. Note that if $Z \neq 0$, $Z_H \neq 0$. Also $Z^{-1}A = V^{-1}A$, since for all $0 \neq z \in Z$, $a \in A$ we have $z^{-1}a = (zz^*)^{-1}(z^*a)$. It will be more convenient for us, in order to extend the superinvolution in a natural way, to work with V rather than with Z. We may consider $V^{-1}A$ as a superalgebra over the field $V^{-1}Z_H$. Then the superinvolution on A is extended to a superinvolution of the same kind on $V^{-1}A$ over $V^{-1}Z_H$ via $(v^{-1}a)^* = v^{-1}a^*$. It is then easy to check that $H(V^{-1}A,*) = V^{-1}H$ and $K(V^{-1}A,*) = V^{-1}K$. Moreover, $Z(V^{-1}A)_0 = V^{-1}Z$ and $V^{-1}Z \cap V^{-1}H = V^{-1}Z_H$. We will say that the superalgebra $V^{-1}A$ over the field $V^{-1}Z_H$ is the $*$-central closure of A.

The following results are instrumental for the paper:

Lemma 1.1. *(See [8], Lemma 1.2.) If A is a semiprime superalgebra, then A and A_0 are semiprime.*

Lemma 1.2. *(See [4], Theorem 3.2.) The only two F-superinvolutions in $M_{1,1}(F)$ are the following*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ c & a \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & b \\ -c & a \end{pmatrix}.$$

Lemma 1.3. *(See the proof of Theorem 2.8 in [7].) Let A be a prime associative superalgebra. If A has a nonzero supercommutative ideal, then A is also supercommutative (that this means that for any homogeneous elements $a, b \in A$, $ab = (-1)^{\bar{a}\bar{b}}ba$).*

Lemma 1.4. *(See [1], Theorem 2.3.2.) Let A be a noncommutative algebra with involution, and suppose that the maximal nil ideal of A, N, is not zero, and also that every nonzero skewsymmetric element is invertible. Then A/N is commutative and $N^2 = 0$.*

2. Osborn's Theorem

We begin with some general results about semiprime and simple superalgebras which will be applied very often in the following.

Lemma 2.1. *Let A be a semiprime nontrivial superalgebra, then A satisfies*:

(i) *For every nonzero element $u_1 \in A_1$ we have $u_1 A_1$, $A_1 u_1 \neq 0$.*
(ii) *If $1 \in A$ and A_0 is simple, then A is simple.*
(iii) *If A is also a finite dimensional odd simple superalgebra over a field F, then A does not admit F-superinvolutions.*

Proof. Suppose that $u_1 A_1 = 0$, then $u_1(A_1 + A_1^2) = 0$ and so $0 \neq u_1 \in (A_1 + A_1^2) \cap Ann_l(A_1 + A_1^2)$, that contradicts the semiprimeness of A. So $u_1 A_1 \neq 0$. A similar argument shows that $A_1 u_1 \neq 0$ and we have (i).

To prove (ii), suppose that I is a nonzero ideal of A. Then I_0 is an ideal of A_0 and since A_0 is simple it follows that either $I_0 = A_0$, in which case $1 \in I_0$ and $I = A$, or $I_0 = 0$, and then $I^2 = 0$, a contradiction because A is semiprime.

Finally, under the conditions of (iii), $A = A_0 + uA_0$, where $u \in Z(A)_1$ is such that $u^2 = 1$ (see [12]). But then, if $*$ is a F-superinvolution on A, $u^* = \lambda u$ for some $\lambda \in F$ because $Z(A) = F + F.u$ and $1 = (u^2)^* = -(u^*)^2 = -(\lambda u)^2 = -\lambda^2$ and $u = (u^*)^* = \lambda^2 u$, a contradiction. □

The next result, due to A. Elduque, is used to prove the next theorem. It is included here for completeness.

Lemma 2.2. *Let A be a simple superalgebra with A_0 artinian. Then A is an artinian superalgebra.*

Proof *(A. Elduque).* We claim first that A_0 is either simple or the direct sum of two simple algebras. Suppose that A_0 is not simple and let I be a proper ideal of A_0. Then $(I + A_1 I A_1) + (A_1 I + I A_1)$ is a nonzero graded ideal of A, with A a simple superalgebra, therefore $I + A_1 I A_1 = A_0$. Let $J = I \cap A_1 I A_1$ and suppose that $J \neq 0$, since J is an ideal of A_0, as before we have $A_0 = J + A_1 J A_1 \subseteq I + A_1(A_1 I A_1)A_1 = I + A_0 I A_0 \subseteq I$, a contradiction because I is a proper ideal and therefore $J = 0$ and $A_0 = I \oplus A_1 I A_1$. Moreover, for every J ideal of I, J is an ideal of A_0 and so $I = J$, whence I is simple. Therefore either A_0 is artinian and simple, or A_0 is the direct sum of two proper ideals which are simple. We notice that in both cases A_0 has a unitary element.

Now we will show that A is unitary. Let e be the unitary element of A_0 and consider the Peirce decomposition relative to e:

$$A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}.$$

We notice that $A_0 \subseteq A_{11}$ and $A_{10} \oplus A_{01} \oplus A_{00} \subseteq A_1$. Since $A_{10}A_{00} \subseteq A_{10}$ and $A_{10}A_{00} \subseteq A_1 A_1 = A_0$, it follows that $A_{10}A_{00} = 0$. Moreover, since $A_{01}A_{00} = 0$, we deduce that A_{00} is a left ideal of A. In the same way we can prove that A_{00} is a right ideal of A, and so $A_{00} = 0$ because A is simple. We can repeat this reasoning to prove that A_{10} is a left ideal of A and also that A_{01} is a right ideal of A. Therefore $A_{10}A_{01}$ is an ideal of A, and $A_{10}A_{01} \subseteq A_0$. But A is simple, so $A_{10}A_{01} = 0$. Since $A_{01}A_{10} \subseteq A_{00} = 0$ we deduce that A_{10} is an ideal of A and A_{01} is an ideal of A, which implies that $A_{10} = A_{01} = 0$, because A is simple. Therefore A has a unitary element.

Finally we show that A is artinian. Let $I(1) \supseteq I(2) \supseteq \cdots$ be a chain of left graded ideals of A, then $I(1)_0 \supseteq I(2)_0 \supseteq \cdots$ and $A_1 I(1)_1 \supseteq A_1 I(2)_1 \supseteq \cdots$ are chains of left ideals in A_0, which is an artinian algebra, therefore there exists $N \in \mathbf{Z}^+$ such that $I(N)_0 = I(N+1)_0 = I(N+2)_0 = \cdots$ and $A_1 I(N)_1 = A_1 I(N+1)_1 = A_1 I(N+2)_1 = \cdots$. Since A has a unitary element, $A_1^2 = A_0$ because A is simple, and $A_0 I(N)_1 \subseteq I(N)_1$ because $I(N)$ is a left ideal of A, we have $I(N)_1 = A_0 I(N)_1 = A_1(A_1 I(N)_1) =$

$A_1(A_1I(N+1)_1) = A_0I(N+1)_1 = I(N+1)_1$. Therefore $I(N)_0 = I(N+1)_0 = \cdots$, $I(N)_1 = I(N+1)_1 = \cdots$ and so A is artinian. $\square$

Theorem 2.3. *Let A be a semiprime superalgebra with superinvolution such that every nonzero homogeneous symmetric element is invertible. Then*

(i) *either A is a division superalgebra, or*
(ii) *$A = B \oplus B^*$ where B is a division superalgebra, and the superinvolution in A is then the exchange superinvolution.*

Proof. By Lemma 1.1 A_0 is semiprime, and so we have a semiprime algebra in which the nonzero symmetric elements are invertible. By the nongraded case (see Theorem 2.1.7 in [1]) it follows that either a) A_0 is a division algebra, or b) $A_0 = I_0 \oplus I_0^*$ where I_0 is a division algebra, or c) $A_0 \cong M_2(F)$ with the symplectic involution.

If a) is satisfied and A_0 is a division algebra, then let $0 \neq u_1 \in A_1$. By Lemma 2.1(i) there exists $v_1 \in A_1$ such that $u_1v_1 \neq 0$. But $u_1v_1 \in A_0$ and therefore, since A_0 is a division algebra, u_1 is invertible and A is a division superalgebra. We have obtained (i).

If b) is satisfied, so, $A_0 = I_0 \oplus I_0^*$ with I_0 a division algebra, we claim first that A is $*$-simple, that is, A has not nonzero ideals N such that $N^* = N$. Indeed, let N be a nonzero proper ideal of A such that $N^* = N$, and since A is semiprime it follows that $N_0 \neq 0$. Then either $N_0 = I_0$ or $N_0 = I_0^*$ because 1 is not in N. But $N_0^* = N_0$ and so $N_0 \subseteq I_0 \cap I_0^* = 0$, a contradiction. So A is $*$-simple and from Lemma 11 in [11] either A is simple or $A \cong B \oplus B^*$ with B a simple superalgebra.

If A is simple, since A_0 is artinian, from Lemma 2.2, A is artinian and then by Theorem 3 in [11] we have that either $A \cong M_{p,q}(D)$ with D a division algebra, or $A \cong M_n(\Delta)$ with Δ a division superalgebra. In the latter case A_0 is simple, but now we have $A_0 = I_0 \oplus I_0^*$. So $A \cong M_{p,q}(D)$ with D a division algebra, and since $A_0 \cong I_0 \oplus I_0^*$ with I_0 a division algebra then $p = q = 1$, D is a field, F, and $A = M_{1,1}(F)$. Now by Lemma 1.2 we have a contradiction because it is not true in A that every nonzero homogeneous symmetric element is invertible.

Therefore A is not simple and $A \cong B \oplus B^*$ with B a simple superalgebra. But $A_0 \cong I_0 \oplus I_0^*$ with I_0 a division algebra, so $B_0 \cong I_0$ is a division algebra and as in the case a) B is a division superalgebra, so (ii) follows.

Finally, if c) holds and $A_0 \cong M_2(F)$ by Lemma 2.1(ii) A is simple, and clearly A_0 is artinian, so applying Theorem 3 in [11] $A \cong M_2(F) + uM_2(F)$ with $u \in Z(A)_1$ such that $u^2 = 1$. But from Lemma 2.1(iii) we deduce that this situation is not possible. $\square$

With the same hypothesis about the symmetric elements, we also want to get information when A is not semiprime, that is, when A has nilpotent ideals, and in particular nil ideals. As we can expect, a nil ideal in a superalgebra, A, is an ideal such that every homogeneous element is nilpotent. If A has a nonzero nil ideal, it is easy to prove by the Zorn Lemma that A has a maximal nil ideal, which in fact is the nil radical of A. For a subalgebra S of A, $S^3 = 0$ means that the product of any three homogeneous elements in S is zero. An ideal I of A is said to be nilpotent of index n, for a positive integer n, if $I^n = 0$.

Theorem 2.4. *Let A be a superalgebra with superinvolution such that every nonzero homogeneous symmetric element is invertible. If A has a nonzero nilpotent ideal, then there exists a maximal nilpotent ideal N such that $N^6 = 0$ and A/N is either a field or the direct sum of two copies of a field.*

Proof. If A has a nonzero nilpotent ideal, let N be the maximal nil ideal of A and then we have $0 \neq N$ and $N^* = N$. We will show that N is also nilpotent. Notice that if $0 \neq a \in N_i$ then since $a + a^* \in N$ is symmetric and $N \neq A$ because $1 \notin N$, we have $a^* = -a$ for any $a \in N_i$. Also, if $a \in N_0$, $(a^2)^* = (a^*)^2 = a^2$, and so a^2 is a symmetric element in N, therefore $a^2 = 0$ for any $a \in N_0$, and $N_0^3 = 0$. Now it is not complicated to check that $N^6 = 0$, and hence N is nilpotent.

Let $r \in A_j$ and since for any $a \in N_i$ we have $ra \in N$, it follows that $(ra)^* = -ra$, that is, $(-1)^{\bar{r}\bar{a}}ar^* = ra$. Therefore if $r \in A_j$, $t \in A_k$, then $[r,t]a = (rt - (-1)^{\bar{r}\bar{t}}tr)a = rta - (-1)^{\bar{r}\bar{t}}tra = (-1)^{\bar{r}(\bar{t}+\bar{a})}tar^* - (-1)^{\bar{r}\bar{t}}tra = (-1)^{\bar{r}\bar{t}}tra - (-1)^{\bar{r}\bar{t}}tra = 0$. Thus, since $[r,tu] = [r,t]u + (-1)^{\bar{t}\bar{r}}t[r,u]$ for any homogeneous elements $r,t,u \in A$ we have $CN = 0$, where C is the ideal generated by the elements $[r,t]$ with r,t homogeneous elements in A. Therefore C has no nonzero symmetric homogeneous element because $N \neq 0$. But $C^* = C$ because $[r,t]^* = -[r^*,t^*]$ for all r,t homogeneous elements in A, and for all $a \in C_i$ $a^* + a$ is symmetric, therefore $a^* = -a$, and, as we did for N we can prove that $C^6 = 0$, that is $C \subseteq N$.

So A/N is supercommutative and it is endowed with a superinvolution $(a+N)^* = a^* + N$, because $N^* = N$, and the set of symmetric elements is $(H+N)/N$, where H is the set of symmetric elements of A. Therefore the nonzero homogeneous symmetric elements in A/N are invertible, so by Theorem 2.3 either A/N is either a field or the direct sum of two copies of a field. In both cases A/N is a trivial superalgebra. $\square$

Note that every prime superalgebra with superinvolution must satisfy that $H \neq 0$, for otherwise $K = A$ and so for every $a \in A_0$, a^2 is a symmetric element, whence $a^2 = 0$. Thus $A^3 = 0$ and then $A^6 = 0$, a contradiction because A is prime.

Theorem 2.5. *Let A be a prime superalgebra with superinvolution such that there is no nonzero nilpotent homogeneous symmetric element. Then $xx^* \neq 0$ for every $0 \neq x \in A_i$.*

Proof. First we claim that to prove that $xx^* \neq 0$ for every $0 \neq x \in A_i$ it is enough to show this for the nonzero even elements. Indeed, if there exists $0 \neq x_1 \in A_1$ such that $x_1x_1^* = 0$ then by Lemma 2.1(i) there exists $y_1 \in A_1$ such that $y_1x_1 \neq 0$ and $(y_1x_1)(y_1x_1)^* = -y_1x_1x_1^*y_1^* = 0$, which is a contradiction if $xx^* \neq 0$ for every $x \in A_0$.

So we need to show that $xx^* \neq 0$ for every $x \in A_0$. Suppose that $x \in A_0$ satisfies $xx^* = 0$. We notice that $x^*Hx \subseteq H$, where H is the set of symmetric elements of A, and also that the elements of x^*Hx are nilpotent. From the hypothesis it follows that $x^*Hx = 0$. In the same way we can prove that $xHx^* = 0$. Therefore $x^*(r+r^*)x = 0$ for every $r \in A_j$, and so $x^*rx = -x^*r^*x = -(x^*rx)^*$. Thus x^*rx is skew. Moreover $x^*Ax \neq 0$ because A is prime.

Consider an homogeneous element $0 \neq k = x^*rx$. Since $x^*x = 0$, it follows that $k^2 = 0$. Also, since $xHx^* = 0$, we have $khk = x^*rxhx^*rx = 0$. Hence

$$[k,h]^3 = [k,h]\big(kh - (-1)^{\bar{k}\bar{h}}hk\big)^2 = -[k,h](-1)^{\bar{h}\bar{k}}kh^2k = -(-1)^{\bar{h}\bar{k}}khkh^2k + hk^2h^2k = 0,$$

and so, since $[k,H] \subseteq H$, we have $[k,H] = 0$. Now it follows from [3] that either $H \subseteq Z$, or there exist a nonzero ideal I such that $I \subseteq \bar{H}$ (where $\bar{H}$ denotes the subalgebra of A generated by H), or A is a central order in a quaternion superalgebra. Since $[k,H] = 0$ it follows that $[k,\bar{H}] = 0$ (because $[r,tu] = [r,t]u + (-1)^{\bar{t}\bar{r}}t[r,u]$), and so if $I \subseteq \bar{H}$, then $[k,I] = 0$, that is, $k \in Z(A)$. But this is a contradiction because $k^2 = 0$ and A is prime. If $H \subseteq Z$, localizing A by H we have that $V^{-1}A$ is a prime superalgebra with superinvolution in which the nonzero homogeneous symmetric elements are invertible, and it also holds that $h^{-1}k$ is nilpotent. This is a contradiction with Theorem 2.3. Finally if A is a central order in a quaternion superalgebra, then $Z^{-1}A = V^{-1}A \cong M_{1,1}(F)$ and from Lemma 1.2 there are nonzero homogeneous symmetric elements which are nilpotent, contradicting our hypothesis. $\square$

Notice that, in the above theorem, one can prove in a similar way that $x^*x \neq 0$ for every $0 \neq x \in A_i$. We are interested in proving a skew version of Theorem 2.5.

Theorem 2.6. *Let A be a prime superalgebra with superinvolution such that there is no nonzero homogeneous skewsymmetric element which is nilpotent. Then $xx^* \neq 0$ for every $0 \neq x \in A_i$.*

Proof. As in the proof of Theorem 2.5 it is enough to show that $xx^* \neq 0$ for every $0 \neq x \in A_0$. If $K = 0$, then every element in A is symmetric and so A is supercommutative and prime, therefore $xx^* \neq 0$

for every $0 \neq x \in A_0$. In fact we can suppose that A is not supercommutative, for otherwise A has not homogeneous zero divisors.

Suppose that $x^*x = 0$ for some $0 \neq x \in A_0$. Since $x^*Kx \subseteq K$ and all elements of x^*Kx are nilpotent, $x^*Kx = 0$. Likewise $xKx^* = 0$. Hence $x^*rx = x^*r^*x = (x^*rx)^*$ for all $r \in A_i$. But, since A is prime, there exists a homogeneous element $r \in A_i$ such that $h = x^*rx \neq 0$. We notice that $0 \neq h \in H$ and also that $h^2 = 0$. Thus $hKh = x^*rxKx^*rx = 0$ because $xKx^* = 0$. Now let $k \in K_i$. Writing $h \circ k = hk + (-1)^{\bar{h}\bar{k}}kh$, we have $(h \circ k)^2 = (-1)^{\bar{k}\bar{h}}hk^2h$, and so $(h \circ k)^4 = 0$. Now by the hypothesis $h \circ k = 0$, that is, $[h, K^2] = 0$.

However, by Lemma 4.1 in [4] K^2 is a Lie ideal of A, and so, by Theorem 2.1 in [6], either $K^2 \subseteq Z$, or K^2 is dense in A or A is a central order in a quaternion superalgebra. If K^2 is dense in A, then there exists a nonzero ideal I of A such that $I \subseteq \overline{K^2}$ (where $\overline{K^2}$ denotes the subalgebra of A generated by K^2) and since $[h, K^2] = 0$ it follows that $[h, I] = 0$. Thus $h \in Z(A)$, which contradicts the primeness of A because $h^2 = 0$. If A is a central order in a quaternion superalgebra, then $Z^{-1}A = V^{-1}A = M_{1,1}(F)$ and from Lemma 1.2 we have a contradiction with the hypothesis. So $K^2 \subseteq Z$, and if $0 \neq u \in K_i$ then by the hypothesis $0 \neq u^2 \in Z$ and hence $Z \neq 0$.

We localize A at V and then $V^{-1}A$ is a prime superalgebra such that $(K(V^{-1}A))^2 \subseteq Z(V^{-1}A)_0$, and for every $0 \neq k \in K(V^{-1}A)_i$, $0 \neq k^2 \in Z(V^{-1}A)_0$, that is, k is invertible. Now we will show that $V^{-1}A$ is simple. If $0 \neq I$ is a proper ideal of $V^{-1}A$, then $J = I^*I \subseteq I$ is nonzero because $V^{-1}A$ is prime, and also J is $*$-ideal. But J must consist of symmetric elements, because the nonzero skewsymmetric elements are invertible, and then J is supercommutative. By Lemma 1.3 A is supercommutative. This contradicts our supposition at the beginning of the proof. So $V^{-1}A$ is simple, and for any nonzero homogeneous elements $u, v \in K(V^{-1}A)$, since they are invertible, we have $0 \neq uv \in Z(V^{-1}A)_0$. Therefore, since $Z(V^{-1}A)_0$ is a field, $u = \alpha v$, where $\alpha \in Z(V^{-1}A)_0$. But from Theorem 4.2 in [4] either $\overline{K(V^{-1}A)} = V^{-1}A$ or $V^{-1}A$ is a quaternion superalgebra. In the first case A will be commutative because $V^{-1}A \cong F.1 + F.u$, a contradiction. And if $V^{-1}A$ is a quaternion superalgebra, by Lemma 1.2 we also have a contradiction, because A has no nonzero nilpotent homogeneous skewsymmetric element. $\square$

3. A skew version of Osborn's Theorem

In this section we will prove the skew analogs of Theorems 2.3 and 2.4. In the nongraded case these results are due to I.N. Herstein and S. Montgomery in [2].

Theorem 3.1. *Let A be a semiprime superalgebra with superinvolution in which every nonzero homogeneous skewsymmetric element is invertible, and such that A_0 is noncommutative. Then*

(i) *either A is a division superalgebra, or*
(ii) *$A = B \oplus B^*$ with B a division superalgebra, and the superinvolution in A is then the exchange superinvolution.*

Proof. If A is a semiprime superalgebra with superinvolution in which every nonzero homogeneous skewsymmetric element is invertible, then by Lemma 1.1 A_0 is a noncommutative semiprime algebra with involution in which every nonzero skewsymmetric element is invertible. By Theorem 2 in [2] we know that either A_0 is a division algebra, or is the direct sum of a division algebra and its opposite, relative to the exchange involution, or is the ring of 2×2 matrices over a field F.

As in the proof of Theorem 2.3 we can show that if A_0 is a division algebra, then A is a division superalgebra. Also as in the proof of Theorem 2.3 we can show that if A_0 is the direct sum of a division algebra and its opposite, then either $A \cong M_{1,1}(F)$ or $A \cong B \oplus B^*$ with B a division superalgebra. A is not isomorphic to $M_{1,1}(F)$ because of Lemma 1.2.

Finally, if $A_0 \cong M_2(F)$ by Lemma 2.1(ii) and (iii) we obtain a contradiction. $\square$

Next we are going to prove a similar result to Theorem 2.4, namely, that if A has a nonzero nil ideal, then A/N has a specific structure, where N is the maximal nil ideal of A.

But firstly let us say some words about radicals in superalgebras. The Jacobson radical, $J(A)$, of an associative superalgebra, A, was defined in [5]. It was proved there that it is equal to the Jacobson radical of A as nongraded algebra. Before the statement of Theorem 2.4 we have defined the nil radical of an associative superalgebra, A. It will now be denoted by $Nil(A)$. Clearly,

$$Nil(A) \subseteq J(A).$$

Finally if we consider $rad(A) = \bigcap\{P\colon\ P \text{ is a prime ideal of } A\}$, we can prove as in the nongraded case that

$$rad(A) \subseteq Nil(A) \subseteq J(A).$$

Theorem 3.2. *Let A be a superalgebra with superinvolution such that every nonzero homogeneous skewsymmetric element is invertible. Suppose that the maximal nil ideal of A (the nil radical of A), N, is not zero and also that A_0 is noncommutative. Then*

(i) *A/N is a trivial superalgebra and it is either a field or the direct sum of two copies of a field with the exchange involution.*
(ii) *$N^4 = 0$.*

Proof. Let I be a nonzero proper $*$-ideal. Since I contains no invertible element, then I consist of symmetric elements, that is, $a^* = a$ for all $a \in I$. But then I is supercommutative. Now for every $a_i \in A_i$ and for all $x_j \in I_j$ we have $a_i x_j = (a_i x_j)^* = (-1)^{ij} x_j a_i^*$. Therefore for all a_i, b_j homogeneous elements in A_i and A_j respectively, and for all $x_k \in I_k$ it follows that $[a_i, b_j]x_k = (a_i b_j - (-1)^{ij} b_j a_i)x_k = (-1)^{k(i+j)} x_k (a_i b_j)^* - (-1)^{ij+ik} b_j x_k a_i^* = (-1)^{ki+kj+ij} x_k b_j^* a_i^* - (-1)^{ij+ik+kj} x_k b_j^* a_i^* = 0$. Hence $[A, A]I = 0$, so if C is the ideal generated by $[A, A]$ we have $CI = 0$ for every $*$-ideal I.

Let N be the nil radical of A. Thent $N \neq A$, since A has a unitary element. Moreover $N^* = N$ and so $CN = 0$. But C is $*$-ideal, and $C \neq 0$ because A_0 is noncommutative, and $C \neq A$ since $CN = 0$, so $C^2 = 0$. It then follows that $C \subseteq N$ and A/N is supercommutative.

Now since $N \cap K = 0$ we have $x_i^* = x_i$ for all $x_i \in N_i$, and so if $a_j \in A_j$ then $a_j x_i \in N$ and $a_j x_i = (a_j x_i)^* = (-1)^{ij} x_i a_j^*$. If we consider now $y_k \in N_k$ then $a_j x_i y_k = (-1)^{ij} x_i a_j^* y_k = (-1)^{ji+jk} x_i y_k a_j$, and so $x_i y_k \in Z_s(A) = \{a_j \in A_j\colon\ a_j b_l = (-1)^{lj} b_l a_j \text{ for every } b_l \in A_l\}$. Hence if $x_0, y_0 \in N_0$, then $x_0 y_0 = y_0 x_0$ is a symmetric element of N_0, and $x_0 y_0 K \subseteq N \cap K = 0$. Since the elements of K are invertible we have $N_0^2 = 0$ and so $N^4 = 0$. $\square$

Theorem 3.3. *Let A be a superalgebra with superinvolution and let $J(A)$ be its Jacobson radical. If either*

(i) *every nonzero homogeneous symmetric element is invertible or nilpotent, or*
(ii) *every nonzero homogeneous skewsymmetric element is invertible or nilpotent,*

then $A/J(A)$ is either a division superalgebra, or the direct sum of a division superalgebra and its opposite, or is $M_{1,1}(F)$ with the symplectic superinvolution.

Proof. Let $J(A) = J_0(A) \oplus J_1(A)$ be the Jacobson radical of A. Notice that $J(A)^* = J(A)$. We claim that $J_0(A) = J(A_0)$. Indeed, we have that $J_0(A) \subseteq J(A_0)$. Consider $J(A_0) + (A_1 J(A_0) + J_1(A))$. It is a left ideal of A in which every homogeneous element is a left quasi-invertible element, because from Lemma 2 in [5] we know that $x_1 \in A_1$ is a left quasi-invertible element in A if and only if x_1^2 is a left quasi-invertible element in A_0. So $J(A_0) + (A_1 J(A_0) + J_1(A))$ must be equal to $J(A)$ and therefore $J(A_0) = J_0(A)$. Hence $A_0/J_0(A)$ is a semisimple algebra with involution satisfying our hypothesis about symmetric or skewsymmetric elements. It follows from Theorem 2.3.4 in [1] that $A_0/J_0(A)$ is either i) a division algebra, or ii) the direct sum of a division algebra and its opposite, or the 2×2 matrices over a field, or a commutative algebra with trivial involution.

Suppose that $(A/J(A))_0$ is commutative with trivial involution, and so all the even elements are symmetric. If there exists a nonzero nilpotent element $x \in (A/J(A))_0$, then it follows that $(A/J(A))_0 x$ is a nilpotent ideal of $(A/J(A))_0$ and this is a contradiction, because $(A/J(A))_0$ is semisimple. Therefore there is no nonzero nilpotent even element in $A/J(A)$. Now we notice that for every $x \in (A/J(A))_1$, xx^* is skew and even, so, since all the even elements are symmetric, $xx^* = 0$. But then for every $x, y \in (A/J(A))_1$ we have $xyxy = x(yx)^* y = -xx^* y^* y = 0$ using that yx is even and symmetric. Hence xy is a nilpotent even element, and so $xy = 0$, that is, $(A/J(A))_1^2 = 0$. But then $(A/J(A))_1$ is an ideal in $A/J(A)$ such that $(A/J(A))_1^2 = 0$, a contradiction, because $A/J(A)$ is semisimple.

If $(A/J(A))_0$ is a division algebra, since $A/J(A)$ is semiprime, by Lemma 2.1(i) we have that $A/J(A)$ is a division superalgebra.

If $(A/J(A))_0 \cong M_2(F)$, since $A/J(A)$ is semiprime, by Lemma 2.1(ii) $A/J(A)$ is simple and now applying Lemma 2.1(iii) we obtain a contradiction.

Finally if $(A/J(A))_0$ is the direct sum of a division algebra and its opposite, we can prove that $A/J(A)$ is $*$-simple as in ii) in the proof of Theorem 2.3. So either $A/J(A)$ is simple or is the direct sum of a division superalgebra and its opposite. But if $A/J(A)$ is simple, since $(A/J(A))_0$ is artinian, by Lemma 2.2 $A/J(A)$ is artinian and also $A/J(A)$ is an even simple superalgebra because $(A/J(A))_0$ is the direct sum of a division algebra and its opposite. So $A/J(A) \cong M_{1,1}(F)$ and the superinvolution is the symplectic superinvolution. □

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