Mean convergence of Fourier-Dunkl series

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Abstract

In the context of the Dunkl transform a complete orthogonal system arises in a very natural way. This paper studies the weighted norm convergence of the Fourier series expansion associated to this system. We establish conditions on the weights, in terms of the A_p classes of Muckenhoupt, which ensure the convergence. Necessary conditions are also proved, which for a wide class of weights coincide with the sufficient conditions.

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1. Introduction

For $\alpha > -1$, let J_α denote the Bessel function of order α:

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\,\Gamma(\alpha+n+1)}$$

(a classical reference on Bessel functions is [17]). Throughout this paper, by $\frac{J_\alpha(z)}{z^\alpha}$ we denote the even function

$$\frac{1}{2^\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(\alpha+n+1)}, \quad z \in \mathbb{C}. \tag{1}$$

In this way, for complex values of z, let

$$\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)};$$

the function $\mathcal{I}_\alpha$ is a small variation of the so-called modified Bessel function of the first kind and order α, usually denoted by I_α. Also, let us take

$$E_\alpha(z) = \mathcal{I}_\alpha(z) + \frac{z}{2(\alpha+1)} \mathcal{I}_{\alpha+1}(z), \quad z \in \mathbb{C}.$$

These functions are related with the so-called Dunkl transform on the real line (see [6] and [7] for details), which is a generalization of the Fourier transform. In particular, $E_{-1/2}(x) = e^x$ and the Dunkl transform

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of order $\alpha = -1/2$ becomes the Fourier transform. Very recently, many authors have been investigating the behaviour of the Dunkl transform with respect to several problems already studied for the Fourier transform; for instance, Paley-Wiener theorems [1], multipliers [4], uncertainty [16], Cowling-Price's theorem [11], transplantation [14], Riesz transforms [15], and so on. The aim of this paper is to pose and analyse in this new context the weighted L^p convergence of the associated Fourier series in the spirit of the classical scheme which, for the trigonometric Fourier series, can be seen in Hunt, Muckenhoupt and Wheeden's paper [10].

The function $\mathcal{I}_\alpha$ is even, and $E_\alpha(ix)$ can be expressed as

$$E_\alpha(ix) = 2^\alpha \Gamma(\alpha+1) \left(\frac{J_\alpha(x)}{x^\alpha} + \frac{J_{\alpha+1}(x)}{x^{\alpha+1}} xi \right).$$

Let $\{s_j\}_{j\ge1}$ be the increasing sequence of positive zeros of $J_{\alpha+1}$. The real-valued function $\operatorname{Im} E_\alpha(ix) = \frac{x}{2(\alpha+1)} \mathcal{I}_{\alpha+1}(ix)$ is odd and its zeros are $\{s_j\}_{j\in\mathbb{Z}}$ where $s_{-j} = -s_j$ and $s_0 = 0$. In connection with the Dunkl transform on the real line, two of the authors introduced the functions e_j, $j \in \mathbb{Z}$, as follows:

$$\begin{aligned} e_0(x) &= 2^{(\alpha+1)/2}\Gamma(\alpha+2)^{1/2}, \\ e_j(x) &= \frac{2^{\alpha/2}\Gamma(\alpha+1)^{1/2}}{|\mathcal{I}_\alpha(is_j)|} E_\alpha(is_j x), \quad j \in \mathbb{Z}\setminus\{0\}. \end{aligned}$$

The case $\alpha = -1/2$ corresponds to the classical trigonometric Fourier setting: $\mathcal{I}_{-1/2}(z) = \cos(iz)$, $\mathcal{I}_{1/2}(z) = \frac{\sin(iz)}{iz}$, $s_j = \pi j$, $E_{-1/2}(is_j x) = e^{i\pi j x}$, and $\{e_j\}_{j\in\mathbb{Z}}$ is the trigonometric system with the appropriate multiplicative constant so that it is orthonormal on $(-1,1)$ with respect to the normalized Lebesgue measure $(2\pi)^{-1/2}\,dx$.

For all values of $\alpha > -1$, in [5] the sequence $\{e_j\}_{j\in\mathbb{Z}}$ was proved to be a complete orthonormal system in $L^2((-1,1), d\mu_\alpha)$, $d\mu_\alpha(x) = (2^{\alpha+1}\Gamma(\alpha+1))^{-1}|x|^{2\alpha+1}\,dx$. That is to say

$$\int_{-1}^{1} e_j(x)\overline{e_k(x)}\,d\mu_\alpha(x) = \delta_{jk}$$

and for each $f \in L^2((-1,1), d\mu_\alpha)$ the series

$$\sum_{j=-\infty}^{\infty} \left(\int_{-1}^{1} f(y)\overline{e_j(y)}\,d\mu_\alpha(y) \right) e_j(x),$$

which we will refer to as Fourier-Dunkl series, converges to f in the norm of $L^2((-1,1), d\mu_\alpha)$. The next step is to ask for which $p \in (1,\infty)$, $p \neq 2$, the convergence holds in $L^p((-1,1), d\mu_\alpha)$. The problem is equivalent, by the Banach-Steinhauss theorem, to the uniform boundedness on $L^p((-1,1), d\mu_\alpha)$ of the partial sum operators $S_n f$ given by

$$S_n f(x) = \int_{-1}^{1} f(y) K_n(x,y)\,d\mu_\alpha(y),$$

where $K_n(x,y) = \sum_{j=-n}^{n} e_j(x)\overline{e_j(y)}$. We are interested in weighted norm estimates of the form

$$\|S_n(f)U\|_{L^p((-1,1),d\mu_\alpha)} \le C\|fV\|_{L^p((-1,1),d\mu_\alpha)},$$

where C is a constant independent of n and f, and U, V are nonnegative functions on $(-1,1)$.

Before stating our results, let us fix some notation. The conjugate exponent of $p \in (1,\infty)$ is denoted by p'. That is,

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \text{or} \quad p' = \frac{p}{p-1}.$$

For an interval $(a,b) \subseteq \mathbb{R}$, the Muckenhoupt class $A_p(a,b)$ consists of those pairs of nonnegative functions (u,v) on (a,b) such that

$$\left(\frac{1}{|I|}\int_I u(x)\,dx\right)\left(\frac{1}{|I|}\int_I v(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \le C,$$

for every interval $I \subseteq (a,b)$, with some constant $C > 0$ independent of I. The smallest constant satisfying this property is called the A_p constant of the pair (u,v).

We say that $(u,v) \in A_p^\delta(a,b)$ (where $\delta > 1$) if $(u^\delta, v^\delta) \in A_p(a,b)$. It follows from Hölder's inequality that $A_p^\delta(a,b) \subseteq A_p(a,b)$.

If $u \equiv 0$ or $v \equiv \infty$, it is trivial that $(u,v) \in A_p(a,b)$ for any interval (a,b). Otherwise, for a bounded interval (a,b), if $(u,v) \in A_p(a,b)$ then the functions u and $v^{-\frac{1}{p-1}}$ are integrable on (a,b).

Throughout this paper, C denotes a positive constant which may be different in each occurrence.

2. Main results

We state here some A_p conditions which ensure the weighted L^p boundedness of these Fourier-Dunkl orthogonal expansions. For simplicity, we separate the general result corresponding to arbitrary weights in two theorems, the first one for $\alpha \ge -1/2$ and the second one for $-1 < \alpha < -1/2$.

Theorem 1. *Let $\alpha \ge -1/2$ and $1 < p < \infty$. Let U, V be weights on $(-1,1)$. Assume that*

$$\Big(U(x)^p |x|^{(\alpha+\frac{1}{2})(2-p)}, V(x)^p |x|^{(\alpha+\frac{1}{2})(2-p)}\Big) \in A_p^\delta(-1,1) \tag{2}$$

for some $\delta > 1$ (or $\delta = 1$ if $U = V$). Then there exists a constant C independent of n and f such that

$$\|S_n(f)U\|_{L^p((-1,1),d\mu_\alpha)} \le C\|fV\|_{L^p((-1,1),d\mu_\alpha)}.$$

Theorem 2. *Let $-1 < \alpha < -1/2$ and $1 < p < \infty$. Let U, V be weights on $(-1,1)$. Let us suppose that U, V satisfy the conditions*

$$\Big(U(x)^p |x|^{(2\alpha+1)(1-p)}, V(x)^p |x|^{(2\alpha+1)(1-p)}\Big) \in A_p^\delta(-1,1), \tag{3}$$

$$\big(U(x)^p |x|^{2\alpha+1}, V(x)^p |x|^{2\alpha+1}\big) \in A_p^\delta(-1,1) \tag{4}$$

for some $\delta > 1$ (or $\delta = 1$ if $U = V$). Then there exists a constant C independent of n and f such that

$$\|S_n(f)U\|_{L^p((-1,1),d\mu_\alpha)} \le C\|fV\|_{L^p((-1,1),d\mu_\alpha)}.$$

As we mentioned in the introduction, the case $\alpha = -1/2$ corresponds to the classical trigonometric case. Accordingly, (2) reduces then to $(U^p, V^p) \in A_p^\delta(-1,1)$. It should be noted also that taking real and imaginary parts in these Fourier-Dunkl series we would obtain the so-called Fourier-Bessel series on $(0,1)$ (see [18, 2, 3, 9]), but the known results for Fourier-Bessel series do not give a proof of the above theorems. Also in connection with Fourier-Bessel series on $(0,1)$, Lemma 3 below can be used to improve some results of [9].

Theorems 1 and 2 establish some sufficient conditions for the L^p boundedness. Our next result presents some necessary conditions. To avoid unnecessary subtleties, we exclude the trivial cases $U \equiv 0$ and $V \equiv \infty$.

Theorem 3. *Let $-1 < \alpha$, $1 < p < \infty$, and U, V weights on $(-1,1)$, neither $U \equiv 0$ nor $V \equiv \infty$. If there exists some constant C such that, for every n and every f,*

$$\|S_n(f)U\|_{L^p((-1,1),d\mu_\alpha)} \le C\|fV\|_{L^p((-1,1),d\mu_\alpha)},$$

then $U \le CV$ almost everywhere on $(-1,1)$, and

$$\begin{aligned} U(x)^p |x|^{(\alpha+\frac{1}{2})(2-p)} &\in L^1((-1,1),dx),\\ \Big(V(x)^p |x|^{(\alpha+\frac{1}{2})(2-p)}\Big)^{-\frac{1}{p-1}} = V(x)^{-p'} |x|^{(\alpha+\frac{1}{2})(2-p')} &\in L^1((-1,1),dx),\\ U(x)^p |x|^{2\alpha+1} &\in L^1((-1,1),dx),\\ \Big(V(x)^p |x|^{(2\alpha+1)(1-p)}\Big)^{-\frac{1}{p-1}} = V(x)^{-p'} |x|^{2\alpha+1} &\in L^1((-1,1),dx). \end{aligned}$$

Notice that the first two integrability conditions imply the other two if $\alpha \ge -1/2$, while the last two imply the other if $-1 < \alpha < -1/2$.

When U, V are power-like weights, it is easy to check that the conditions of Theorem 3 are equivalent to the A_p conditions (2), (3), (4). By power-like weights we mean finite products of the form $|x-t|^\gamma$, for some constants t, γ. For these weights, therefore, Theorems 1, 2 and 3 characterize the boundedness of the Fourier-Dunkl expansions. For instance, we have the following particular case:

Corollary. *Let $b, A, B \in \mathbb{R}$, $1 < p < \infty$, and*

$$U(x) = |x|^b (1-x)^A (1+x)^B.$$

Then, there exists some constant C such that

$$\|U S_n f\|_{L^p((-1,1),d\mu_\alpha)} \le C \|U f\|_{L^p((-1,1),d\mu_\alpha)}$$

for every f and n if and only if $-1 < Ap < p-1$, $-1 < Bp < p-1$ and

$$-1 + p\Big(\alpha+\frac{1}{2}\Big)_+ < bp + 2\alpha + 1 < p - 1 + p(2\alpha+1) - p\Big(\alpha+\frac{1}{2}\Big)_+,$$

where $(\alpha+\frac{1}{2})_+ = \max\{\alpha+\frac{1}{2}, 0\}$.

In the unweighted case ($U = V = 1$) the boundedness of the partial sum operators S_n, or in other words the convergence of the Fourier-Dunkl series, holds if and only if

$$\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$$

in the case $\alpha \ge -1/2$, and for the whole range $1 < p < \infty$ in the case $-1 < \alpha < -1/2$.

Remark. These conditions for the unweighted case are exactly the same as in the Fourier-Bessel case when the orthonormal functions are $2^{1/2}|J_{\alpha+1}(s_n)|^{-1} J_\alpha(s_n x) x^{-\alpha}$ and the orthogonality measure is $x^{2\alpha+1}\,dx$ on the interval $(0,1)$.

Other variants of Bessel orthogonal systems exist in the literature, see [2, 3, 18]. For instance, one can take the functions $2^{1/2}|J_{\alpha+1}(s_n)|^{-1} J_\alpha(s_n x)$, which are orthonormal with respect to the measure $x\,dx$ on the interval $(0,1)$. The conditions for the boundedness of these Fourier-Bessel series, as can be seen in [3], correspond to taking $A = B = 0$ and $b = \alpha - \frac{2\alpha+1}{p}$ in our corollary. Another usual case is to take the functions $(2x)^{1/2}|J_{\alpha+1}(s_n)|^{-1} J_\alpha(s_n x)$, which are orthonormal with respect to the measure dx on $(0,1)$. Passing from one orthogonality to another consists basically in changing the weights. Then, from the weighted L^p boundedness of any of these systems we easily deduce a corresponding weighted L^p boundedness for any of the other systems.

In the case of the Fourier-Dunkl series on $(-1,1)$ we feel, however, that the natural setting is to start from $J_\alpha(z) z^{-\alpha}$, since these functions, defined by (1), are holomorphic on $\mathbb{C}$; in particular, they are well defined on the interval $(-1,1)$.

3. Auxiliary results

We will need to control some basic operator in weighted L^p spaces on $(-1,1)$. For a function $g:(0,2)\to\mathbb{R}$, the Calderón operator is defined by

$$Ag(x)=\frac{1}{x}\int_0^x |g(y)|\,dy+\int_x^2\frac{|g(y)|}{y}\,dy,$$

that is, the sum of the Hardy operator and its adjoint. The weighted norm inequality

$$\|Ag\|_{L^p((0,2),u)}\le C\|g\|_{L^p((0,2),v)}$$

holds for every $g\in L^p((0,2),v)$, provided that $(u,v)\in A_p^\delta(0,2)$ for some $\delta>1$, and $\delta=1$ is enough if $u=v$ (see [12, 13]). Let us consider now the operator J defined by

$$Jf(x)=\int_{-1}^1\frac{f(y)}{2-x-y}\,dy$$

for $x\in(-1,1)$ and suitable functions f. With the notation $f_1(t)=f(1-t)$, we have

$$|Jf(x)|=\left|\int_0^2\frac{f(1-t)}{1-x+t}\,dt\right|\le A(f_1)(1-x)$$

and a simple change of variables proves that the weighted norm inequality

$$\|Jf\|_{L^p((-1,1),u)}\le C\|f\|_{L^p((-1,1),v)}$$

holds for every $f\in L^p((-1,1),v)$, provided that $(u,v)\in A_p^\delta(-1,1)$ for some $\delta>1$ (or $\delta=1$ if $u=v$).

The Hilbert transform on the interval $(-1,1)$ is defined as

$$Hg(x)=\int_{-1}^1\frac{g(y)}{x-y}\,dy.$$

The above weighted norm inequality holds also for the Hilbert transform with the same $A_p^\delta(-1,1)$ condition (see [10, 13]). In both cases, the norm inequalities hold with a constant C depending only on the A_p^δ constant of the pair (u,v).

Our first objective is to obtain a suitable estimate for the kernel $K_n(x,y)$. With this aim, we will use some well-known properties of Bessel (and related) functions, that can be found on [17]. For the Bessel functions we have the asymptotics

$$J_\nu(z)=\frac{z^\nu}{2^\nu\Gamma(\nu+1)}+O(z^{\nu+2}),\tag{5}$$

if $|z|<1$, $|\arg(z)|\le\pi$; and

$$J_\nu(z)=\sqrt{\frac{2}{\pi z}}\left[\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+O(e^{\operatorname{Im}(z)}z^{-1})\right],\tag{6}$$

if $|z|\ge1$, $|\arg(z)|\le\pi-\theta$. The Hankel function of the first kind, denoted by $H_\nu^{(1)}$, is defined as

$$H_\nu^{(1)}(z)=J_\nu(z)+iY_\nu(z),$$

where Y_ν denotes the Weber function, given by

$$Y_\nu(z)=\frac{J_\nu(z)\cos\nu\pi-J_{-\nu}(z)}{\sin\nu\pi},\ \text{if }\nu\notin\mathbb{Z},$$
$$Y_n(z)=\lim_{\nu\to n}\frac{J_\nu(z)\cos\nu\pi-J_{-\nu}(z)}{\sin\nu\pi},\ \text{if }n\in\mathbb{Z}.$$

From these definitions, we have

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)}{i \sin \nu\pi}, \text{ if } \nu \notin \mathbb{Z},$$
$$H_n^{(1)}(z) = \lim_{\nu\to n} \frac{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)}{i \sin \nu\pi}, \text{ if } n \in \mathbb{Z}.$$

For the function $H_\nu^{(1)}$, the asymptotic

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z-\nu\pi/2-\pi/4)}[C + O(z^{-1})] \tag{7}$$

holds for $|z| > 1$, $-\pi < \arg(z) < 2\pi$, with some constant C.

As usual for the L^p convergence of orthogonal expansions, the results are consequences of suitable estimates for the kernel $K_n(x, y)$. The next lemma contains an estimate for the difference between the kernel $K_n(x, y)$ and an integral containing the product of two E_α functions. This integral can be evaluated using Lemma 1 in [5]. Next, to obtain the estimate we consider an appropriate function in the complex plane having poles in the points s_j and integrate this function along a suitable path.

Lemma 1. *Let $\alpha > -1$. Then, there exists some constant $C > 0$ such that for each $n \geq 1$ and $x, y \in (-1, 1)$,*

$$\left| K_n(x, y) - \int_{-M_n}^{M_n} E_\alpha(izx)\overline{E_\alpha(izy)}\, d\mu_\alpha(z) \right| \leq C \left(\frac{|xy|^{-(\alpha+1/2)}}{2 - x - y} + 1 \right),$$

where $M_n = (s_n + s_{n+1})/2$.

Proof. Using elementary algebraic manipulations, the kernel $K_n(x, y)$ can be written as

$$K_n(x, y) = 2^{\alpha+1}\Gamma(\alpha + 2) + \frac{2^{\alpha+1}\Gamma(\alpha + 1)}{(xy)^\alpha} \sum_{j=1}^{n} \frac{J_\alpha(s_j x)J_\alpha(s_j y) + J_{\alpha+1}(s_j x)J_{\alpha+1}(s_j y)}{J_\alpha(s_j)^2}. \tag{8}$$

Let us find a function whose residues at the points s_j are the terms in the series, so that this series can be expressed as an integral. The identities

$$-J'_{\alpha+1}(z)H^{(1)}_{\alpha+1}(z) + J_{\alpha+1}(z)(H^{(1)}_{\alpha+1})'(z) = \frac{2i}{\pi z}$$

(see [19, p. 76]), and

$$zJ'_{\alpha+1}(z) + (\alpha + 1)J_{\alpha+1}(z) = -zJ_\alpha(z),$$

give

$$-J'_{\alpha+1}(s_j)H^{(1)}_{\alpha+1}(s_j) = \frac{2i}{\pi s_j}$$

and

$$J'_{\alpha+1}(s_j) = -J_\alpha(s_j)$$

for every $j \in \mathbb{N}$. Then,

$$\begin{aligned}
-\frac{2i}{\pi}&|xy|^{1/2} \frac{J_\alpha(s_j x)J_\alpha(s_j y) + J_{\alpha+1}(s_j x)J_{\alpha+1}(s_j y)}{J_\alpha(s_j)^2} \\
&= -\frac{2i}{\pi}|xy|^{1/2} \frac{J_\alpha(s_j x)J_\alpha(s_j y) + J_{\alpha+1}(s_j x)J_{\alpha+1}(s_j y)}{J'_{\alpha+1}(s_j)^2} \\
&= |xy|^{1/2} s_j H^{(1)}_{\alpha+1}(s_j) \frac{J_\alpha(s_j x)J_\alpha(s_j y) + J_{\alpha+1}(s_j x)J_{\alpha+1}(s_j y)}{J'_{\alpha+1}(s_j)} \\
&= \lim_{z\to s_j} (z - s_j)H_{x,y}(z) = \operatorname{Res}(H_{x,y}, s_j),
\end{aligned}$$

where we define

$$H_{x,y}(z) = |xy|^{1/2}\, zH^{(1)}_{\alpha+1}(z)\frac{J_\alpha(zx)J_\alpha(zy) + J_{\alpha+1}(zx)J_{\alpha+1}(zy)}{J_{\alpha+1}(z)}$$

(the factor $|xy|^{1/2}$ is taken for convenience). The fact that $J_\nu(-z) = e^{\nu\pi i}J_\nu(z)$ gives $\operatorname{Res}(H_{x,y}, s_j) = \operatorname{Res}(H_{x,y}, -s_j)$.

Since the definition of $H^{(1)}_{\alpha+1}(z)$ differs in case $\alpha \in \mathbb{Z}$, for the rest of the proof we will assume that $\alpha \notin \mathbb{Z}$; the other case can be deduced by considering the limit.

The function $H_{x,y}(z)$ is analytic in $\mathbb{C} \setminus ((-\infty, -M_n] \cup [M_n, \infty) \cup \{\pm s_j : j = 1, 2, \dots\})$. Moreover, the points $\pm s_j$ are simple poles. So, we have

$$\int_{\mathbf{S}\cup\mathbf{I}(\varepsilon)} H_{x,y}(z)\, dz = 0, \tag{9}$$

where $\mathbf{I}(\varepsilon)$ is the interval $[-M_n, M_n]$ warped with upper half circles of radius ε centered in $\pm s_j$, with $j = 1, \dots, n$ and $\mathbf{S}$ is the path of integration given by the interval $M_n + i[0, \infty)$ in the direction of increasing imaginary part and the interval $-M_n + i[0, \infty)$ in the opposite direction. The existence of the integral is clear for the path $\mathbf{I}(\varepsilon)$; for $\mathbf{S}$ this fact can be checked by using (5), (6) and (7). Indeed, on $\mathbf{S}$ we obtain that $\left|\frac{H^{(1)}_{\alpha+1}(z)}{J_{\alpha+1}(z)}\right| \le Ce^{-2\operatorname{Im}(z)}$. Similarly, on $\mathbf{S}$ one has

$$\left||xy|^{1/2} zJ_\alpha(zx)J_\alpha(zy)\right| \le Ce^{\operatorname{Im}(z)(x+y)}h^\alpha_{x,y}(|z|)$$

where

$$h^\alpha_{x,y}(|z|) = \max\{|xz|^{\alpha+1/2}, 1\}\max\{|yz|^{\alpha+1/2}, 1\}$$

for $-1 < \alpha < -1/2$, and

$$h^\alpha_{x,y}(|z|) = 1$$

for $\alpha \ge -1/2$. Thus

$$|H_{x,y}(z)| \le C\left(h^\alpha_{x,y}(|z|) + h^{\alpha+1}_{x,y}(|z|)\right)e^{-\operatorname{Im}(z)(2-x-y)}, \tag{10}$$

and the integral on $\mathbf{S}$ is well defined.

From the definition of $H_{x,y}(z)$, we have

$$\begin{aligned}\int_{\mathbf{I}(\varepsilon)} H_{x,y}(z)\, dz = \int_{\mathbf{I}(\varepsilon)} &\frac{|xy|^{1/2}zJ_{-\alpha-1}(z)}{i\sin(\alpha+1)\pi}\cdot\frac{J_\alpha(zx)J_\alpha(zy) + J_{\alpha+1}(zx)J_{\alpha+1}(zy)}{J_{\alpha+1}(z)}\, dz\\ &- |xy|^{1/2}\frac{e^{-(\alpha+1)\pi i}}{i\sin(\alpha+1)\pi}\int_{\mathbf{I}(\varepsilon)} z\left(J_\alpha(zx)J_\alpha(zy) + J_{\alpha+1}(zx)J_{\alpha+1}(zy)\right)\, dz.\end{aligned}$$

The function in the first integral is odd, and the function in the second integral has no poles at the points s_j. Then, the first integral equals the integral over the symmetric path $-\mathbf{I}(\varepsilon) = \{z : -z \in \mathbf{I}(\varepsilon)\}$. Putting $|z - s_j| = \varepsilon$ for the positively oriented circle, this gives

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbf{I}(\varepsilon)} H_{x,y}(z)\, dz &= \lim_{\varepsilon\to 0}\frac{-1}{2}\sum_{|s_j|<M_n}\int_{|z-s_j|=\varepsilon}\frac{|xy|^{1/2}zJ_{-\alpha-1}(z)}{i\sin(\alpha+1)\pi}\cdot\frac{J_\alpha(zx)J_\alpha(zy) + J_{\alpha+1}(zx)J_{\alpha+1}(zy)}{J_{\alpha+1}(z)}\, dz\\ &\quad - |xy|^{1/2}\frac{e^{-(\alpha+1)\pi i}}{i\sin(\alpha+1)\pi}\int_{-M_n}^{M_n} z\left(J_\alpha(zx)J_\alpha(zy) + J_{\alpha+1}(zx)J_{\alpha+1}(zy)\right)\, dz\\ &= -\pi i\sum_{|s_j|<M_n}\operatorname{Res}(H_{x,y}, s_j)\\ &\quad - |xy|^{1/2}\frac{e^{-(\alpha+1)\pi i}}{i\sin(\alpha+1)\pi}(1 - e^{2\pi i\alpha})\int_0^{M_n} z\left(J_\alpha(zx)J_\alpha(zy) + J_{\alpha+1}(zx)J_{\alpha+1}(zy)\right)\, dz\end{aligned}$$

$$= -4|xy|^{1/2} \sum_{j=1}^{n} \frac{J_\alpha(s_j x) J_\alpha(s_j y) + J_{\alpha+1}(s_j x) J_{\alpha+1}(s_j y)}{J_\alpha(s_j)^2}$$
$$+ 2|xy|^{1/2} \int_0^{M_n} z \left(J_\alpha(zx) J_\alpha(zy) + J_{\alpha+1}(zx) J_{\alpha+1}(zy) \right) dz.$$

This, together with (9), gives

$$\sum_{j=1}^{n} \frac{J_\alpha(s_j x) J_\alpha(s_j y) + J_{\alpha+1}(s_j x) J_{\alpha+1}(s_j y)}{J_\alpha(s_j)^2}$$
$$= \frac{1}{4|xy|^{1/2}} \int_{\mathbf{S}} H_{x,y}(z)\, dz + \frac{1}{2} \int_0^{M_n} z \left(J_\alpha(zx) J_\alpha(zy) + J_{\alpha+1}(zx) J_{\alpha+1}(zy) \right) dz.$$

Then, it follows from (8) that

$$K_n(x,y) = 2^{\alpha+1}\Gamma(\alpha+2) + \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(xy)^\alpha |xy|^{1/2}} \int_{\mathbf{S}} H_{x,y}(z)\, dz$$
$$+ \frac{2^\alpha \Gamma(\alpha+1)}{(xy)^\alpha} \int_0^{M_n} z (J_\alpha(zx) J_\alpha(zy) + J_{\alpha+1}(zx) J_{\alpha+1}(zy))\, dz.$$

Now, it is easy to check the identity

$$\frac{2^\alpha \Gamma(\alpha+1)}{(xy)^\alpha} \int_0^{M_n} z (J_\alpha(zx) J_\alpha(zy) + J_{\alpha+1}(zx) J_{\alpha+1}(zy))\, dz = \int_{-M_n}^{M_n} E_\alpha(izx) \overline{E_\alpha(izy)}\, d\mu_\alpha(z),$$

so that

$$\left| K_n(x,y) - \int_{-M_n}^{M_n} E_\alpha(izx) \overline{E_\alpha(izy)}\, d\mu_\alpha(z) \right| \le 2^{\alpha+1}\Gamma(\alpha+2) + \frac{2^{\alpha-1}\Gamma(\alpha+1)}{|xy|^{\alpha+1/2}} \left| \int_{\mathbf{S}} H_{x,y}(z)\, dz \right|.$$

We conclude showing that

$$\left| \int_{\mathbf{S}} H_{x,y}(z)\, dz \right| \le C \left(\frac{1}{2-x-y} + |xy|^{\alpha+1/2} \right), \tag{11}$$

for $-1 < x, y < 1$. For $\alpha \ge -1/2$, the bound (11) follows from (10). Indeed, in this case

$$\left| \int_{\mathbf{S}} H_{x,y}(z)\, dz \right| \le C \int_0^\infty e^{-t(2-x-y)}\, dt = \frac{C}{2-x-y}.$$

For $-1 < \alpha < -1/2$, we have $|H_{x,y}(z)| \le C|xy|^{\alpha+1/2} e^{-\operatorname{Im}(z)(2-x-y)}$ if $z \in \mathbf{S}$. With this inequality we obtain (11) as follows:

$$\left| \int_{\mathbf{S}} H_{x,y}(z)\, dz \right| \le C|xy|^{\alpha+1/2} \int_0^\infty e^{-t(2-x-y)}\, dt = C\frac{|xy|^{\alpha+1/2}}{2-x-y} \le C \left(|xy|^{\alpha+1/2} + \frac{1}{2-x-y} \right). \qquad \square$$

From the previous lemma and the identity (see [5])

$$\int_{-1}^{1} E_\alpha(ixz) \overline{E_\alpha(iyz)}\, d\mu_\alpha(z) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \frac{x\mathcal{I}_{\alpha+1}(ix)\mathcal{I}_\alpha(iy) - y\mathcal{I}_{\alpha+1}(iy)\mathcal{I}_\alpha(ix)}{x-y},$$

which holds for $\alpha > -1$, $x, y \in \mathbb{C}$, and $x \ne y$, we obtain that

$$|K_n(x,y) - B(M_n,x,y) - B(M_n,y,x)| \le C \left(\frac{|xy|^{-(\alpha+1/2)}}{2-x-y} + 1 \right) \tag{12}$$

with

$$B(M_n,x,y) = \frac{M_n^{2(\alpha+1)}}{2^{\alpha+1}\Gamma(\alpha+2)} \frac{x\mathcal{I}_{\alpha+1}(iM_n x)\mathcal{I}_\alpha(iM_n y)}{x-y}$$

or, by the definition of $\mathcal{I}_\alpha$ and the fact that $\frac{J_\alpha(z)}{z^\alpha}$ is even,

$$B(M_n,x,y) = 2^\alpha\Gamma(\alpha+1)\frac{M_n x J_{\alpha+1}(M_n|x|)J_\alpha(M_n|y|)}{|x|^{\alpha+1}|y|^\alpha(x-y)}.$$

4. Proof of Theorem 1

We can split the partial sum operator S_n into three terms suitable to apply (12):

$$\begin{aligned} S_n f(x) &= \int_{-1}^1 f(y)B(M_n,x,y)\,d\mu_\alpha(y) + \int_{-1}^1 f(y)B(M_n,y,x)\,d\mu_\alpha(y) \\ &\quad + \int_{-1}^1 f(y)\Big[K_n(x,y) - B(M_n,x,y) - B(M_n,y,x)\Big]\,d\mu_\alpha(y) \\ &=: T_{1,n}f(x) + T_{2,n}f(x) + T_{3,n}f(x). \end{aligned} \tag{13}$$

With this decomposition, the theorem will be proved if we see that

$$\|UT_{j,n}f\|^p_{L^p((-1,1),d\mu_\alpha)} \leq C\|Vf\|^p_{L^p((-1,1),d\mu_\alpha)}, \qquad j=1,2,3,$$

for a constant C independent of n and f.

4.1. The first term

We have

$$\begin{aligned} T_{1,n}f(x) &= \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{-1}^1 f(y)B(M_n,x,y)|y|^{2\alpha+1}\,dy \\ &= \frac{M_n^{1/2}xJ_{\alpha+1}(M_n|x|)}{2|x|^{\alpha+1}}\int_{-1}^1 \frac{f(y)M_n^{1/2}J_\alpha(M_n|y|)|y|^{\alpha+1}}{x-y}\,dy. \end{aligned}$$

According to (5) and (6) and the assumption that $\alpha \geq -1/2$, we have

$$|J_\alpha(z)| \leq Cz^{-1/2}, \qquad |J_{\alpha+1}(z)| \leq Cz^{-1/2},$$

for every $z > 0$. Using these inequalities and the boundedness of the Hilbert transform under the A_p condition (2) gives

$$\begin{aligned} &\|UT_{1,n}f\|^p_{L^p((-1,1),d\mu_\alpha)} \\ &\quad = C\int_{-1}^1 \left|\int_{-1}^1 \frac{f(y)M_n^{1/2}J_\alpha(M_n|y|)|y|^{\alpha+1}}{x-y}\,dy\right|^p U(x)^p M_n^{p/2}|J_{\alpha+1}(M_n|x|)|^p|x|^{2\alpha+1-\alpha p}\,dx \\ &\quad \leq C\int_{-1}^1 \left|\int_{-1}^1 \frac{f(y)M_n^{1/2}J_\alpha(M_n|y|)|y|^{\alpha+1}}{x-y}\,dy\right|^p U(x)^p|x|^{(\alpha+\frac12)(2-p)}\,dx \\ &\quad \leq C\int_{-1}^1 \left|f(x)M_n^{1/2}J_\alpha(M_n|x|)|x|^{\alpha+1}\right|^p V(x)^p|x|^{(\alpha+\frac12)(2-p)}\,dx \\ &\quad \leq C\int_{-1}^1 |f(x)|^pV(x)^p|x|^{2\alpha+1}\,dx = C\|Vf\|^p_{L^p((-1,1),d\mu_\alpha)}. \end{aligned}$$

4.2. The second term

This term is given by

$$\begin{aligned} T_{2,n}f(x) &= \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{-1}^{1} f(y)B(M_n,y,x)|y|^{2\alpha+1}\,dy \\ &= \frac{M_n^{1/2}J_\alpha(M_n|x|)}{2|x|^\alpha}\int_{-1}^{1}\frac{f(y)yM_n^{1/2}J_{\alpha+1}(M_n|y|)|y|^\alpha}{y-x}\,dy \end{aligned}$$

and everything goes as with the first term.

4.3. The third term

According to (12),

$$|T_{3,n}f(x)| \le C|x|^{-(\alpha+1/2)}\int_{-1}^{1}\frac{f(y)|y|^{\alpha+1/2}}{2-x-y}\,dy + C\int_{-1}^{1}|f(y)|\,|y|^{2\alpha+1}\,dy$$

so it is enough to have both

$$\int_{-1}^{1}\left|\int_{-1}^{1}\frac{f(y)|y|^{\alpha+1/2}}{2-x-y}\,dy\right|^p U(x)^p|x|^{2\alpha+1-p(\alpha+1/2)}\,dx \tag{14}$$

and

$$\left|\int_{-1}^{1}|f(x)|\,|x|^{2\alpha+1}\,dx\right|^p\int_{-1}^{1}U(x)^p|x|^{2\alpha+1}\,dx \tag{15}$$

bounded by

$$C\int_{-1}^{1}|f(x)|^pV(x)^p|x|^{2\alpha+1}\,dx.$$

For the boundedness of (14) it suffices to impose

$$\left(U(x)^p|x|^{2\alpha+1-p(\alpha+1/2)}, V(x)^p|x|^{2\alpha+1-p(\alpha+1/2)}\right) \in A_p^\delta(-1,1),$$

but this is exactly (2). By duality, the boundedness of (15) is equivalent to

$$\left(\int_{-1}^{1}U(x)^p|x|^{2\alpha+1}\,dx\right)\left(\int_{-1}^{1}V(x)^{-p/(p-1)}|x|^{2\alpha+1}\,dx\right)^{p-1} < \infty.$$

Now, it is easy to check that

$$\begin{aligned} &\left(\int_{-1}^{1}U(x)^p|x|^{2\alpha+1}\,dx\right)\left(\int_{-1}^{1}V(x)^{-p/(p-1)}|x|^{2\alpha+1}\,dx\right)^{p-1} \\ &\qquad\le \left(\int_{-1}^{1}U(x)^p|x|^{(\alpha+\frac12)(2-p)}\,dx\right)\left(\int_{-1}^{1}\left(V(x)^p|x|^{(\alpha+\frac12)(2-p)}\right)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \le C, \end{aligned}$$

the last inequality following from the A_p condition (2).

5. Proof of Theorem 2

We begin with a simple lemma on A_p weights.

Lemma 2. *Let $1 < p < \infty$, $(u, v) \in A_p(-1, 1)$, $(u_1, v_1) \in A_p(-1, 1)$. Let w, ζ be weights on $(-1, 1)$ such that either*

$$w \le C(u + u_1) \quad \text{and} \quad \zeta \ge C_1(v + v_1)$$

or

$$w^{-1} \ge C(u^{-1} + u_1^{-1}) \quad \text{and} \quad \zeta^{-1} \le C_1(v^{-1} + v_1^{-1})$$

for some constants C, C_1. Then $(w, \zeta) \in A_p(-1, 1)$ with a constant depending only on C, C_1 and the A_p constants of (u, v) and (u_1, v_1).

Proof. Assume that $w \le C(u + u_1)$ and $\zeta \ge C_1(v + v_1)$. For any interval $I \subseteq (-1, 1)$,

$$\left(\frac{1}{|I|}\int_I \zeta^{-\frac{1}{p-1}}\right)^{p-1} \le \frac{1}{C_1}\min\left\{\left(\frac{1}{|I|}\int_I v^{-\frac{1}{p-1}}\right)^{p-1}, \left(\frac{1}{|I|}\int_I v_1^{-\frac{1}{p-1}}\right)^{p-1}\right\}.$$

Therefore,

$$\left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I \zeta^{-\frac{1}{p-1}}\right)^{p-1} \le \frac{C}{C_1}\left(\frac{1}{|I|}\int_I u\right)\left(\frac{1}{|I|}\int_I v^{-\frac{1}{p-1}}\right)^{p-1} + \frac{C}{C_1}\left(\frac{1}{|I|}\int_I u_1\right)\left(\frac{1}{|I|}\int_I v_1^{-\frac{1}{p-1}}\right)^{p-1}.$$

This proves that $(w, \zeta) \in A_p(-1, 1)$ with a constant depending on C, C_1 and the A_p constants of (u, v) and (u_1, v_1).

Assume now that $w^{-1} \ge C(u^{-1} + u_1^{-1})$ and $\zeta^{-1} \le C_1(v^{-1} + v_1^{-1})$. Then

$$\frac{1}{|I|}\int_I w \le \frac{1}{C}\min\left\{\frac{1}{|I|}\int_I u, \frac{1}{|I|}\int_I u_1\right\} \tag{16}$$

for any interval $I \subseteq (-1, 1)$. On the other hand, the inequality

$$\frac{1}{2}(a^\lambda + b^\lambda) \le (a + b)^\lambda \le 2^\lambda(a^\lambda + b^\lambda), \qquad a, b \ge 0,\ \lambda > 0 \tag{17}$$

gives

$$\zeta^{-\frac{1}{p-1}} \le C_1^{\frac{1}{p-1}}\left(v^{-1} + v_1^{-1}\right)^{\frac{1}{p-1}} \le C_1^{\frac{1}{p-1}} 2^{\frac{1}{p-1}}\left(v^{-\frac{1}{p-1}} + v_1^{-\frac{1}{p-1}}\right),$$

and

$$\left(\frac{1}{|I|}\int_I \zeta^{-\frac{1}{p-1}}\right)^{p-1} \le 2^p C_1\left(\frac{1}{|I|}\int_I v^{-\frac{1}{p-1}}\right)^{p-1} + 2^p C_1\left(\frac{1}{|I|}\int_I v_1^{-\frac{1}{p-1}}\right)^{p-1}.$$

This, together with (16), proves that $(w, \zeta) \in A_p(-1, 1)$ with a constant depending on C, C_1 and the A_p constants of (u, v) and (u_1, v_1). □

Now, we use the following estimate for the Bessel functions, which is a consequence of (5), (6) and $-1 < \alpha < -1/2$:

$$|z^{1/2}J_\alpha(z)| \le C(1 + z^{\alpha+1/2}), \quad z \ge 0,$$

and

$$|z^{1/2}J_{\alpha+1}(z)| \le C(1 + z^{\alpha+1/2})^{-1}, \quad z \ge 0.$$

In particular, there exists a constant C such that, for $x \in (-1, 1)$ and $n \ge 0$, we have

$$M_n^{1/2}|J_\alpha(M_n|x|)| \le C|x|^{-1/2}(1 + |M_n x|^{\alpha+1/2})$$

and

$$M_n^{1/2}|J_{\alpha+1}(M_n|x|)| \le C\,\frac{|x|^{-1/2}}{1+|M_n x|^{\alpha+1/2}}.$$

Moreover, the inequality (17) gives

$$2^{\alpha+1/2}|x|^{\alpha+1/2}(|x|+M_n^{-1})^{-(\alpha+1/2)} \le 1+|M_n x|^{\alpha+1/2} \le 2|x|^{\alpha+1/2}(|x|+M_n^{-1})^{-(\alpha+1/2)}$$

so that we get

$$M_n^{1/2}|J_\alpha(M_n|x|)| \le C|x|^\alpha(|x|+M_n^{-1})^{-(\alpha+1/2)} \tag{18}$$

and

$$M_n^{1/2}|J_{\alpha+1}(M_n|x|)| \le C|x|^{-(\alpha+1)}(|x|+M_n^{-1})^{\alpha+1/2}. \tag{19}$$

To handle these expressions, the following result will be useful:

Lemma 3. *Let $1<p<\infty$, a sequence $\{M_n\}$ of positive numbers that tends to infinity, two nonnegative functions U and V defined on the interval $(-1,1)$, $-1<\alpha<-1/2$ and $\delta>1$ ($\delta=1$ if $U=V$). If (3) and (4) are satisfied, then*

$$\Big(U(x)^p(|x|+M_n^{-1})^{p(\alpha+1/2)}|x|^{(2\alpha+1)(1-p)}, V(x)^p(|x|+M_n^{-1})^{p(\alpha+1/2)}|x|^{(2\alpha+1)(1-p)}\Big) \in A_p^\delta(-1,1), \tag{20}$$

$$\Big(U(x)^p(|x|+M_n^{-1})^{-p(\alpha+1/2)}|x|^{2\alpha+1}, V(x)^p(|x|+M_n^{-1})^{-p(\alpha+1/2)}|x|^{2\alpha+1}\Big) \in A_p^\delta(-1,1), \tag{21}$$

"uniformly", i.e., with A_p^δ constants independent of n.

Proof. As a first step, let us observe that (3) and (4) imply

$$\Big(U(x)^p|x|^{(2\alpha+1)(1-\frac{1}{2}p)}, V(x)^p|x|^{(2\alpha+1)(1-\frac{1}{2}p)}\Big) \in A_p^\delta(-1,1).$$

To prove this, just put

$$U(x)^p|x|^{(2\alpha+1)(1-\frac{1}{2}p)} = \Big[U(x)^p|x|^{(2\alpha+1)(1-p)}\Big]^{1/2}\Big[U(x)^p|x|^{(2\alpha+1)}\Big]^{1/2}$$

(the same with V) and check the A_p^δ condition using the Cauchy-Schwarz inequality and (3), (4).

Now, (17) yields

$$\Big[U(x)^p(|x|+M_n^{-1})^{p(\alpha+\frac{1}{2})}|x|^{(2\alpha+1)(1-p)}\Big]^{-\delta} \ge \frac{1}{2}\Big[U(x)^p|x|^{(2\alpha+1)(1-\frac{1}{2}p)}\Big]^{-\delta} + \frac{1}{2}\Big[U(x)^p M_n^{-p(\alpha+\frac{1}{2})}|x|^{(2\alpha+1)(1-p)}\Big]^{-\delta}$$

and

$$\Big[V(x)^p(|x|+M_n^{-1})^{p(\alpha+\frac{1}{2})}|x|^{(2\alpha+1)(1-p)}\Big]^{-\delta} \le 2^{-p\delta(\alpha+\frac{1}{2})}\Big[V(x)^p|x|^{(2\alpha+1)(1-\frac{1}{2}p)}\Big]^{-\delta} + 2^{-p\delta(\alpha+\frac{1}{2})}\Big[V(x)^p M_n^{-p(\alpha+\frac{1}{2})}|x|^{(2\alpha+1)(1-p)}\Big]^{-\delta}.$$

Thus, Lemma 2 gives (20) with an A_p^δ constant independent of n, since the A_p^δ constant of the pair

$$\Big(U(x)^p M_n^{-p(\alpha+\frac{1}{2})}|x|^{(2\alpha+1)(1-p)}, V(x)^p M_n^{-p(\alpha+\frac{1}{2})}|x|^{(2\alpha+1)(1-p)}\Big)$$

is the same constant of the pair

$$\Big(U(x)^p|x|^{(2\alpha+1)(1-p)}, V(x)^p|x|^{(2\alpha+1)(1-p)}\Big)$$

i.e., it does not depend on n. The proof of (21) follows the same argument, since

$$\Big[U(x)^p(|x|+M_n^{-1})^{-p(\alpha+\frac12)}|x|^{2\alpha+1}\Big]^\delta \\ \le 2^{-p\delta(\alpha+\frac12)}\Big[U(x)^p|x|^{(2\alpha+1)(1-\frac12 p)}\Big]^\delta + 2^{-p\delta(\alpha+\frac12)}\Big[U(x)^pM_n^{p(\alpha+\frac12)}|x|^{2\alpha+1}\Big]^\delta$$

and

$$\Big[V(x)^p(|x|+M_n^{-1})^{-p(\alpha+\frac12)}|x|^{2\alpha+1}\Big]^\delta \ge \frac12\Big[V(x)^p|x|^{(2\alpha+1)(1-\frac12 p)}\Big]^\delta + \frac12\Big[V(x)^pM_n^{p(\alpha+\frac12)}|x|^{2\alpha+1}\Big]^\delta. \qquad \square$$

We already have all the ingredients to start with the proof of Theorem 2. Let us take the same decomposition $S_nf = T_{1,n}f + T_{2,n} + T_{3,n}f$ as in (13) in the previous section and consider each term separately.

5.1. The first term

As in the proof of Theorem 1, by using (19) we have

$$\|UT_{1,n}f\|^p_{L^p((-1,1),d\mu_\alpha)} = \int_{-1}^1\left|\int_{-1}^1\frac{f(y)M_n^{1/2}J_\alpha(M_n|y|)|y|^{\alpha+1}}{x-y}\,dy\right|^p U(x)^pM_n^{p/2}|J_{\alpha+1}(M_n|x|)|^p|x|^{2\alpha+1-\alpha p}\,dx \\ \le C\int_{-1}^1\left|\int_{-1}^1\frac{f(y)M_n^{1/2}J_\alpha(M_n|y|)|y|^{\alpha+1}}{x-y}\,dy\right|^p U(x)^p(|x|+M_n^{-1})^{p(\alpha+1/2)}|x|^{(2\alpha+1)(1-p)}\,dx.$$

Now, by the A_p condition (20), this is bounded by

$$C\int_{-1}^1\left|f(x)M_n^{1/2}J_\alpha(M_n|x|)|x|^{\alpha+1}\right|^p V(x)^p(|x|+M_n^{-1})^{p(\alpha+1/2)}|x|^{(2\alpha+1)(1-p)}\,dx,$$

which, by (18) is in turn bounded by

$$C\int_{-1}^1|f(x)|^pV(x)^p|x|^{2\alpha+1}\,dx = C\|Vf\|^p_{L^p((-1,1),d\mu_\alpha)}.$$

5.2. The second term

The definition of $T_{2,n}$ and (18) yield

$$\|UT_{2,n}f\|^p_{L^p((-1,1),d\mu_\alpha)} = \int_{-1}^1\left|\int_{-1}^1\frac{f(y)yM_n^{1/2}J_{\alpha+1}(M_n|y|)|y|^{\alpha}}{y-x}\,dy\right|^p U(x)^pM_n^{p/2}|J_{\alpha}(M_n|x|)|^p|x|^{2\alpha+1-\alpha p}\,dx \\ \le C\int_{-1}^1\left|\int_{-1}^1\frac{f(y)yM_n^{1/2}J_{\alpha+1}(M_n|y|)|y|^{\alpha}}{y-x}\,dy\right|^p U(x)^p(|x|+M_n^{-1})^{-p(\alpha+1/2)}|x|^{2\alpha+1}\,dx.$$

Now, by the A_p condition (21), this is bounded by

$$C\int_{-1}^1\left|f(x)xM_n^{1/2}J_{\alpha+1}(M_n|x|)|x|^{\alpha}\right|^p V(x)^p(|x|+M_n^{-1})^{-p(\alpha+1/2)}|x|^{2\alpha+1}\,dx,$$

which, by (19) is in turn bounded by

$$C\int_{-1}^1|f(x)|^pV(x)^p|x|^{2\alpha+1}\,dx = C\|Vf\|^p_{L^p((-1,1),d\mu_\alpha)}.$$

5.3. The third term

Taking limits when $n \to \infty$ in (20) we get (2), so the proof of the boundedness of the third summand in Theorem 1 is still valid for Theorem 2.

6. Proof of Theorem 3

The following lemma is a small variant of a result proved in [8]. We give here a proof for the sake of completeness.

Lemma 4. *Let $\nu > -1$. Let h be a Lebesgue measurable nonnegative function on $[0,1]$, $\{\rho_n\}$ a positive sequence such that $\lim_{n\to\infty} \rho_n = +\infty$ and $1 \le p < \infty$. Then*

$$\lim_{n\to\infty} \int_0^1 |\rho_n^{1/2} J_\nu(\rho_n x)|^p h(x)\, dx \ge M \int_0^1 h(x) x^{-p/2}\, dx \tag{22}$$

(in particular, that limit exists), where M is a positive constant independent of h and $\{\rho_n\}$.

Proof. We can assume that $h(x)x^{\nu p}$ is integrable on $(0,\delta)$ for some $\delta \in (0,1)$, since otherwise

$$\int_0^1 |\rho_n^{1/2} J_\nu(\rho_n x)|^p h(x)\, dx = \infty$$

for each n, as follows from (5), and (22) is trivial. Assume also for the moment that $h(x)x^{-p/2}$ is integrable on $(0,1)$. For each $x \in (0,1)$ and n, let us put

$$\varphi(x,n) = (\rho_n x)^{1/2} J_\nu(\rho_n x) - \sqrt{\frac{2}{\pi}} \cos\left(\rho_n x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

The estimate (6) gives

$$\lim_{n\to\infty} \varphi(x,n) = 0$$

for each $x \in (0,1)$. Moreover, in case $\rho_n x \ge 1$ the same estimate gives

$$|\varphi(x,n)| \le \frac{C}{\rho_n x} \le C \tag{23}$$

with a constant C independent of n and x, while for $\rho_n x \le 1$ it follows from (5) that

$$|\varphi(x,n)| \le C\left((\rho_n x)^{\nu+1/2} + 1\right). \tag{24}$$

Without loss of generality we can assume that $\rho_n \ge 1$. Then, (23) and (24) give $|\varphi(x,n)| \le C(x^{\nu+1/2}+1)$ with a constant C independent of x and n, so that, by the dominate convergence theorem,

$$\lim_{n\to\infty} \int_0^1 \left| (\rho_n x)^{1/2} J_\nu(\rho_n x) - \sqrt{\frac{2}{\pi}} \cos\left(\rho_n x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \right|^p h(x) x^{-p/2}\, dx = 0. \tag{25}$$

Therefore,

$$\lim_{n\to\infty} \int_0^1 |\rho_n^{1/2} J_\nu(\rho_n x)|^p h(x)\, dx = \lim_{n\to\infty} \int_0^1 \left| \sqrt{\frac{2}{\pi}} \cos\left(\rho_n x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \right|^p h(x) x^{-p/2}\, dx. \tag{26}$$

Now we use Fejér's lemma: if $f \in L^1(0,2\pi)$, and g is a continuous, 2π-periodic function, then

$$\lim_{\lambda\to\infty} \frac{1}{2\pi} \int_0^{2\pi} g(\lambda t) f(t)\, dt = \widehat{g}(0)\widehat{f}(0) = \frac{1}{2\pi} \int_0^{\pi} g(t)\, dt\, \frac{1}{2\pi} \int_0^{\pi} f(t)\, dt$$

where $\widehat{f}$, $\widehat{g}$ denote the Fourier transforms of f, g. After a change of variables, Fejér's lemma applied to the right hand side of (26) gives

$$\lim_{n\to\infty}\int_0^1 |\rho_n^{1/2} J_\nu(\rho_n x)|^p h(x)\,dx = M\int_0^1 h(x)x^{-p/2}\,dx$$

for some constant M, thus proving (22).

Finally, in case $h(x)x^{-p/2}$ is not integrable on $(0,1)$, let us take the sequence of increasing measurable sets

$$K_j = \{x\in(0,1) : h(x)x^{-p/2}\le j\},\qquad j\in\mathbb{N},$$

and define $h_j = h$ on K_j and $h_j = 0$ on $(0,1)\setminus K_j$. Applying (22) to each h_j and then the monotone convergence theorem proves that

$$\lim_{n\to\infty}\int_0^1 |\rho_n^{1/2} J_\nu(\rho_n x)|^p h(x)\,dx = \infty,$$

which is (22). □

We can now prove Theorem 3.

Proof of Theorem 3. The first partial sum of the Fourier expansion is

$$S_0 f = e_0\int_{-1}^1 f\overline{e_0}\,d\mu_\alpha = (\alpha+1)\int_{-1}^1 f(x)|x|^{2\alpha+1}\,dx,$$

so that the inequality $\|S_0(f)U\|_{L^p((-1,1),d\mu_\alpha)}\le C\|fV\|_{L^p((-1,1),d\mu_\alpha)}$ gives, by duality,

$$U(x)^p|x|^{2\alpha+1}\in L^1((-1,1),dx),\quad V(x)^{-p'}|x|^{2\alpha+1}\in L^1((-1,1),dx).$$

In fact, this is needed just to ensure that the partial sums of the Fourier expansions of all functions in $L^p(V^p\,d\mu_\alpha)$ are well defined and belong to $L^p(U^p\,d\mu_\alpha)$. These are the last two integrability conditions of Theorem 3.

Now, if

$$\|S_n(f)U\|_{L^p((-1,1),d\mu_\alpha)}\le C\|fV\|_{L^p((-1,1),d\mu_\alpha)}$$

then the difference

$$\begin{aligned} S_n f - S_{n-1}f &= e_n\int_{-1}^1 f\overline{e_n}\,d\mu_\alpha + e_{-n}\int_{-1}^1 f\overline{e_{-n}}\,d\mu_\alpha\\ &= e_n\int_{-1}^1 f\overline{e_n}\,d\mu_\alpha + \overline{e_n}\int_{-1}^1 f e_n\,d\mu_\alpha\end{aligned}$$

is bounded in the same way. Taking even and odd functions, and using that $\operatorname{Re} e_n$ is even and $\operatorname{Im} e_n$ is odd, gives

$$\|U\operatorname{Re} e_n\|_{L^p((-1,1),d\mu_\alpha)}\|V^{-1}\operatorname{Re} e_n\|_{L^{p'}((-1,1),d\mu_\alpha)}\le C \tag{27}$$

and the same inequality with $\operatorname{Im} e_n$. Recall that

$$\operatorname{Re} e_n(x) = 2^{\alpha/2}\Gamma(\alpha+1)^{1/2}\,\frac{|s_n|^\alpha}{|J_\alpha(s_n)|}\,\frac{J_\alpha(s_n x)}{(s_n x)^\alpha}.$$

Taking into account that $|J_\nu(x)|$ is an even function (recall that $J_\alpha(z)/z^\alpha$ is taken as an even function) and $|J_\alpha(s_n)|\le C s_n^{-1/2}$ (this follows from (6)), Lemma 4 gives

$$\liminf_{n\to\infty}\int_{-1}^1\left|\frac{1}{J_\alpha(s_n)}J_\nu(s_n x)\right|^p h(x)\,dx\ge C\int_{-1}^1 h(x)|x|^{-p/2}\,dx$$

for every measurable nonnegative function h. Therefore,

$$\liminf_{n\to\infty} \|U \operatorname{Re} e_n\|_{L^p((-1,1),d\mu_\alpha)} \geq C \left(\int_{-1}^{1} U(x)^p |x|^{-p\alpha-\frac{p}{2}+2\alpha+1} \, dx \right)^{\frac{1}{p}}$$

and the corresponding lower bound for $\liminf_n \|V^{-1} \operatorname{Re} e_n\|_{L^{p'}((-1,1),d\mu_\alpha)}$ holds. The same bounds hold for $\operatorname{Im} e_n$. Thus, (27) implies

$$\left(\int_{-1}^{1} U(x)^p |x|^{-p\alpha-\frac{p}{2}+2\alpha+1} \, dx \right)^{\frac{1}{p}} \left(\int_{-1}^{1} V(x)^{-p'} |x|^{-p'\alpha-\frac{p'}{2}+2\alpha+1} \, dx \right)^{\frac{1}{p'}} \leq C$$

or, in other words, the first two integrability conditions of Theorem 3.

Take now $f = U/(1+V+UV)$ and any measurable set $E \subseteq (-1,1)$. Then $f \in L^2(d\mu_\alpha)$ by Hölder's inequality, the obvious inequality $|f| \leq UV^{-1}$ and the integrability conditions $U \in L^p(d\mu_\alpha)$, $V^{-1} \in L^{p'}(d\mu_\alpha)$, already proved. Since $\{e_j\}_{j\in\mathbb{Z}}$ is a complete orthonormal system in $L^2((-1,1),d\mu_\alpha)$, we have $S_n(f\chi_E) \to f\chi_E$ in the $L^2(d\mu_\alpha)$ norm. Therefore, there exists some subsequence $S_{n_j}(f\chi_E)$ converging to $f\chi_E$ almost everywhere. Fatou's lemma then gives

$$\int_{-1}^{1} |f\chi_E|^p U^p \, d\mu_\alpha \leq \liminf_{j\to\infty} \int_{-1}^{1} |S_{n_j}(f\chi_E)|^p U^p \, d\mu_\alpha.$$

Under the hypothesis of Theorem 3, each of the integrals on the right hand side is bounded by

$$C^p \int_{-1}^{1} |f\chi_E|^p V^p \, d\mu_\alpha$$

(observe, by the way, that $fV \in L^p(d\mu_\alpha)$, since $|fV| \leq 1$). Thus,

$$\int_{-1}^{1} |f\chi_E|^p U^p \, d\mu_\alpha \leq C^p \int_{-1}^{1} |f\chi_E|^p V^p \, d\mu_\alpha$$

for every measurable set $E \subseteq (-1,1)$. This gives $fU \leq CfV$ almost everywhere, and $U \leq CV$. □

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