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# On extensions of free nilpotent Lie algebras of type 2

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ABSTRACT

In this paper, we study the structure of Lie algebras which have free $t$-nilpotent Lie algebras $\mathfrak{n}_{2,t}$ of type 2 as nilradical and give a detailed construction for them. We prove that the dimension of any Lie algebra $\mathfrak{g}$ of this class is $\dim \mathfrak{n}_{2,t} + k$. If $\mathfrak{g}$ is solvable, $k \leq 2$; otherwise, the Levi subalgebra of $\mathfrak{g}$ is $\mathfrak{sl}_2(\mathbb{K})$, the split simple 3-dimensional Lie algebra of $2 \times 2$ matrices of trace zero, and then $k \leq 4$. As an application of the main results we get the classification over algebraically closed fields of Lie algebras with nilradical $\mathfrak{n}_{2,1}$, $\mathfrak{n}_{2,2}$ and $\mathfrak{n}_{2,3}$.
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## 1. Introduction

From Levi's Theorem, any finite-dimensional Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$, direct sum of vector spaces, where $\mathfrak{r}$ is the solvable radical (the maximal solvable ideal of $\mathfrak{g}$) and $\mathfrak{s}$ is a semisimple subalgebra called *Levi subalgebra* (or Levi factor). The Lie

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algebra $\mathfrak{g}$ is faithful if the adjoint representation of the subalgebra $\mathfrak{s}$ in $\mathfrak{r}$ is faithful (equivalently, $\mathfrak{g}$ contains no nonzero semisimple ideals). Note that any Lie algebra with radical $\mathfrak{r}$ can be decomposed into the direct sum (as ideals) of a semisimple Lie algebra and a faithful Lie algebra with the same radical. Let now consider the nilradical $\mathfrak{n}$ of $\mathfrak{g}$, i.e. the largest nilpotent ideal inside $\mathfrak{g}$. The nilradical is contained in the solvable radical and:

$$[\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{n}. \tag{1}$$

In other words, the action of $\mathfrak{g}$ on $\frac{\mathfrak{r}}{\mathfrak{n}}$ is trivial.

Concerning the problem of the classification of Lie algebras of a given radical, Malcev [12] (see also [17, Theorem 4.4, Section 4]) proved the following structure result:

**Theorem 1.1.** *Any faithful Lie algebra $\mathfrak{g}$ with solvable radical $\mathfrak{r}$ is isomorphic to a Lie algebra of the form $\mathfrak{s} \oplus_{id} \mathfrak{r}$ where $\mathfrak{s}$ is a semisimple subalgebra of a Levi subalgebra $\mathfrak{s}_0$ of the algebra of derivations of $\mathfrak{r}$, named* $\operatorname{Der} \mathfrak{r}$*. Moreover, given $\mathfrak{s}_1$ and $\mathfrak{s}_2$ semisimple subalgebras of $\mathfrak{s}_0$, the algebras $\mathfrak{s}_1 \oplus_{id} \mathfrak{r}$ and $\mathfrak{s}_2 \oplus_{id} \mathfrak{r}$ are isomorphic if and only if $\mathfrak{s}_2 = A\mathfrak{s}_1 A^{-1}$, where $A$ is an automorphism of $\mathfrak{r}$.* □

The product in $\mathfrak{s} \oplus_{id} \mathfrak{r}$ is given by considering $\mathfrak{s}$ and $\mathfrak{r}$ as subalgebras and $[d, a] := d(a)$ for $d \in \mathfrak{s}$ and $a \in \mathfrak{r}$; since $\mathfrak{s} \subseteq \operatorname{Der} \mathfrak{r}$, $[d_1, d_2] = d_1 d_2 - d_2 d_1$ is the Lie bracket for $d_i \in \mathfrak{s}$.

The previous theorem reduces the problem of classifying Lie algebras with a given solvable radical $\mathfrak{r}$, to the analysis of derivations and automorphisms of the solvable Lie algebra $\mathfrak{r}$. The same argument is valid if we consider the problem of classifying Lie algebras with a given nilradical. According to [13], solvable Lie algebras can be classified through nilpotent ones and, from [19], any nilpotent Lie algebra is a quotient of a free nilpotent Lie algebra of the same type and nilindex. So, as the former problem reduces to the latter one, it seems to be quite natural to start studying the classification of Lie algebras with a given nilradical. This is the starting point in [2] where Malcev's decompositions of Lie algebras are studied, or in [16] where the problem of classifying Lie algebras whose radical is just the nilradical of a parabolic subalgebra of a semisimple Lie algebra is treated. In this paper we classify Lie algebras with nilradical a free nilpotent Lie algebra $\mathfrak{n}_{2,t}$ of type 2 and nilindex $t$ (i.e. $\mathfrak{n}_{2,t}^{t+1} = 0$, $\mathfrak{n}_{2,t}^{t} \neq 0$ and $\dim \mathfrak{n}_{2,t} - \dim \mathfrak{n}_{2,t}^2 = 2$). Following [19], the Levi subalgebra of $\operatorname{Der} \mathfrak{n}_{2,t}$ is $\mathfrak{sl}_2(\mathbb{K})$, the split 3-dimensional simple Lie algebra of $2 \times 2$ matrices of trace zero. So, our main idea in this paper is to apply basic representation theory of $\mathfrak{sl}_2(\mathbb{K})$ to attack the problem. This technique has been used previously in [21] to get the classification of the 9-dimensional nonsolvable indecomposable Lie algebras.

This paper is organized as follows. Section 2 gives some basic definitions and facts on free nilpotent Lie algebras $\mathfrak{n}_{2,t}$ and includes a complete description of $\operatorname{Der} \mathfrak{n}_{2,t}$, the Lie algebra of derivations of $\mathfrak{n}_{2,t}$. Sections 3 and 4 deal with the structure and general construction of Lie algebras with nilradical $\mathfrak{n}_{2,t}$. It turns out that the solvable radical of

Lie algebras of this class is an extension of $\mathfrak{n}_{2,t}$ by a suitable vector subspace of $\operatorname{Der}\mathfrak{n}_{2,t}$ of dimension at most two. The results in Sections 3 and 4 will be used in Section 5 to review and present in a unified way the classifications of Lie algebras with nilradical $\mathfrak{n}_{2,2}$ and $\mathfrak{n}_{2,3}$ given in [18] (where only solvable extensions are considered) and [1]. The Lie algebra $\mathfrak{n}_{2,2}$ is a 3-dimensional Heisenberg algebra and $\mathfrak{n}_{2,3}$ is a 5-dimensional quasi-classical Lie algebra denoted as $\mathcal{L}_{5,3}$ in [1]. Here quasi-classical means endowed with a symmetric non-degenerate invariant form according to [14, Definition 2.1]. In [3], the authors establish that $\mathfrak{n}_{2,3}$ and $\mathfrak{n}_{3,2}$ (free nilpotent of type 3) are the unique quasi-classical free nilpotent Lie algebras. Following [15], quasi-classical Lie algebras are related to consistent Yang–Mills gauge theories, so the results in this paper are interesting for theoretical physics.

Throughout the paper all the vector spaces are finite dimensional over a field $\mathbb{K}$ of characteristic zero. We follow [9] and [10] for basic definitions and results on Lie algebras.

## 2. Lie algebras of type 2

(The results in this section are partially included in [5, Section 3], where free nilpotent Lie algebras of characteristic zero and arbitrary type are treated.)

A nilpotent Lie algebra $\mathfrak{n}$ is said to be *t-nilpotent* in case $\mathfrak{n}^t \neq 0$ and $\mathfrak{n}^{t+1} = 0$ and of *type d* if $d = \dim\mathfrak{n} - \dim\mathfrak{n}^2$. So type 2 implies $\dim\mathfrak{n} = 2 + \dim\mathfrak{n}^2$.

The free nilpotent Lie algebra $\mathfrak{n}_{2,t}$ is defined as the quotient:

$$\mathfrak{n}_{2,t} = \frac{\mathfrak{FL}_2}{\mathfrak{FL}_2^{t+1}} \tag{2}$$

of the free Lie algebra $\mathfrak{FL}_2$ on $\{x_1, x_2\}$ by the ideal

$$\mathfrak{FL}_2^{t+1} = \sum_{j \geq t+1} \operatorname{span}\big\langle \big[\dots[x_{i_1}x_{i_2}]\dots x_{i_j}\big] : i_s \in \{1,2\}\big\rangle.$$

We consider right-normalized iterated commutators denoted by $[x_{i_1}, ..., x_{i_s}]$. From [19, Proposition 4, Proposition 2] and [6, Proposition 1.4] we have:

**Proposition 2.1.** *The algebra $\mathfrak{n}_{2,t}$ is a quasicyclic t-nilpotent Lie algebra of type 2 and any other t-nilpotent Lie algebra of type 2 is a homomorphic image of $\mathfrak{n}_{2,t}$. Moreover, the Lie algebra of derivations of $\mathfrak{n}_{2,t}$ is:*

$$\operatorname{Der}\mathfrak{n}_{2,t} = \big\{\widehat{\delta} : \delta \in \operatorname{Hom}(\mathfrak{m},\mathfrak{m})\big\} \;\oplus\; \big\{\widehat{\delta} : \delta \in \operatorname{Hom}\big(\mathfrak{m},\mathfrak{n}_{2,t}^2\big)\big\},$$

*where $\mathfrak{m} = \operatorname{span}\langle x_1, x_2\rangle$ and, for $\delta \in \operatorname{Hom}(\mathfrak{m},\mathfrak{m})$ or $\delta \in \operatorname{Hom}(\mathfrak{m},\mathfrak{n}_{2,t}^2)$ and $s \geq 2$:*

$$\widehat{\delta}\big([x_{i_1}, x_{i_2}, \dots, x_{i_j}, \dots, x_{i_s}]\big) = \sum_{j=1}^{s}\big[x_{i_1}, x_{i_2}, \dots, \delta(x_{i_j}), \dots, x_{i_s}\big]. \qquad \square \tag{3}$$

According to [11], a nilpotent Lie algebra $\mathfrak{n}$ is called *quasicyclic* if there exists a subspace $\mathfrak{u}$ such that $\mathfrak{n} = \mathfrak{n}^2 \oplus \mathfrak{u}$ (direct sum as vector spaces) and $\mathfrak{n}$ decomposes as a (finite) direct sum of subspaces $\mathfrak{u}^k = [u, u^{k-1}]$. So, from Proposition 2.1, we can assume w.l.o.g.:

$$\mathfrak{n}_{2,t} = \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \ldots \oplus \mathfrak{m}^t, \tag{4}$$

for $k \geq 2$, $\mathfrak{m}^k = \operatorname{span}\langle [\ldots[x_{i_1}x_{i_2}]\ldots x_{i_k}] : i_j \in \{1, 2\}\rangle$. The direct sum in (4) provides a graded decomposition of $\mathfrak{n}_{2,t}$ and, the dimension of each component $\mathfrak{m}^s$, is given in terms of the Möebius function $\mu$:

$$\dim \mathfrak{m}^s = \frac{1}{s} \sum_{d|s} \mu(d) 2^{\frac{s}{d}}. \tag{5}$$

We denote by $\mathfrak{gl}(\mathfrak{m})$ the general linear Lie algebra of $\mathbb{K}$-linear maps $\delta : \mathfrak{m} \to \mathfrak{m}$ and, for $j \geq 1$, $\operatorname{Der}_j \mathfrak{n}_{2,t} = \{\widehat{\delta} : \delta \in \operatorname{Hom}(\mathfrak{m}, \mathfrak{m}^j)\}$. So $\operatorname{Der}_1 \mathfrak{n}_{2,t} = \{\widehat{\delta} : \delta \in \mathfrak{gl}(\mathfrak{m})\}$, is a Lie subalgebra of $\operatorname{Der} \mathfrak{n}_{2,t}$ isomorphic to $\mathfrak{gl}(\mathfrak{m}) = \mathfrak{sl}(\mathfrak{m}) \oplus \mathbb{K} \cdot id_{\mathfrak{m}}$ (the 3-dimensional simple Lie algebra $\mathfrak{sl}(\mathfrak{m})$ on $\mathfrak{m}$ plus the identity map) and

$$\operatorname{Der} \mathfrak{n}_{2,t} = \bigoplus_{j=1}^{t} \operatorname{Der}_j \mathfrak{n}_{2,t} \tag{6}$$

with the multiplication rule $[d_i, d_k] = d_i d_k - d_k d_i \in \operatorname{Der}_{i+k-1} \mathfrak{n}_{2,t}$, $d_s \in \operatorname{Der}_s \mathfrak{n}_{2,t}$. Moreover, the derived algebra $\operatorname{Der}_1^0 \mathfrak{n}_{2,t} = [\operatorname{Der}_1 \mathfrak{n}_{2,t}, \operatorname{Der}_1 \mathfrak{n}_{2,t}]$ is just $\operatorname{Der}_1^0 \mathfrak{n}_{2,t} = \{\widehat{\delta} : \delta \in \mathfrak{sl}(\mathfrak{m})\}$, a *Levi subalgebra* of $\operatorname{Der} \mathfrak{n}_{2,t}$ which is isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. In this way,

$$\operatorname{Der}_1 \mathfrak{n}_{2,t} = \operatorname{Der}_1^0 \mathfrak{n}_{2,t} \oplus \mathbb{K} \cdot id_{2,t} \tag{7}$$

where $id_{2,t} = \widehat{id_{\mathfrak{m}}}$ is the extended derivation of $id_{\mathfrak{m}}$. Note that $id_{2,t}|_{\mathfrak{m}^s} = s \cdot id_{\mathfrak{m}^s}$ for $s \geq 1$. The solvable radical $\mathfrak{R}_{2,t}$ and the nilradical $\mathfrak{N}_{2,t}$ of $\operatorname{Der} \mathfrak{n}_{2,t}$ are given by:

$$\mathfrak{N}_{2,t} = \bigoplus_{j \geq 2}^{t} \operatorname{Der}_j \mathfrak{n}_{2,t} \tag{8}$$

$$\mathfrak{R}_{2,t} = \mathbb{K} \cdot id_{2,t} \oplus \mathfrak{N}_{2,t}. \tag{9}$$

We also note that the elements of $\mathfrak{N}_{2,t}$ are nilpotent maps and $[id_{2,t}, \widehat{\delta}] = (s-1)\widehat{\delta}$ for any $\widehat{\delta} \in \operatorname{Der}_s \mathfrak{n}_{2,t}$ $(s \geq 1)$.

Concerning the classification problem of Lie algebras having $\mathfrak{n}_{2,t}$ as solvable radical we have (cf. of [5, Proposition 3.2] and Theorem 1.1):

**Theorem 2.2.** *Up to isomorphism, $\mathfrak{g}_{2,t} = \operatorname{Der}_1^0 \mathfrak{n}_{2,t} \oplus_{id} \mathfrak{n}_{2,t}$ is the unique faithful Lie algebra with solvable radical the free nilpotent Lie algebra $\mathfrak{n}_{2,t}$. In particular, apart from $\mathfrak{g}_{2,t}$, any* *nonsolvable Lie algebra with radical $\mathfrak{n}_{2,t}$ is a direct sum of ideals of the form $\mathfrak{s} \oplus \mathfrak{n}_{2,t}$ or $\mathfrak{s} \oplus \mathfrak{g}_{2,t}$, where $\mathfrak{s}$ is an arbitrary semisimple Lie algebra.* □

By using irreducible modules $V(n)$ of $\mathfrak{sl}_2(\mathbb{K}) \cong \mathrm{Der}_1^0 \mathfrak{n}_{2,t}$ (see [9, Section II.7]) we can get, in an algorithmic way, explicit bases for $\mathfrak{n}_{2,t}$ and $\mathfrak{g}_{2,t}$ with rational structure constants. The bases are obtained through the natural action of $\mathrm{Der}_1^0 \mathfrak{n}_{2,t}$ on $\mathfrak{m} \cong V(1)$ and the induced action on the $j$-tensor vector space $\otimes^j \mathfrak{m}$ (for a computational approach see [4]), and they are closely related to *Hall bases* [8]. The next result shows how this method works:

**Proposition 2.3.** *Up to isomorphism, the nonsolvable Lie algebras with solvable radical a free nilpotent algebra of type 2 and nilindex $t \leq 5$ are:*

- *i)* *The 5-dimensional algebra $\mathfrak{g}_{2,1}$ with basis $\{e, f, h, v_0, v_1\}$ and nonzero products $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $[h, v_0] = v_0$, $[h, v_1] = -v_1$, $[e, v_1] = v_0$ and $[f, v_0] = v_1$. In this case, $\mathfrak{n}_{2,1} = \mathbb{K}v_0 \oplus \mathbb{K}v_1$ is a $V(1)$-module of the Levi subalgebra $\mathbb{K}e \oplus \mathbb{K}f \oplus \mathbb{K}h$ ($\cong \mathrm{Der}_1^0 \mathfrak{n}_{2,1}$).*
- *ii)* *The 6-dimensional algebra $\mathfrak{g}_{2,2}$ with basis $\{e, f, h, v_0, v_1, w_0\}$ and non-zero products $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $[h, v_0] = v_0$, $[h, v_1] = -v_1$, $[e, v_1] = v_0$, $[f, v_0] = v_1$ and $[v_0, v_1] = w_0$. In this case, $\mathfrak{n}_{2,2} = \mathbb{K}v_0 \oplus \mathbb{K}v_1 \oplus \mathbb{K}w_0$ is a $V(1) \oplus V(0)$-module of the Levi subalgebra $\mathbb{K}e \oplus \mathbb{K}f \oplus \mathbb{K}h$ ($\cong \mathrm{Der}_1^0 \mathfrak{n}_{2,2}$).*
- *iii)* *The 8-dimensional algebra $\mathfrak{g}_{2,3}$ with basis $\{e, f, h, v_0, v_1, w_0, z_0, z_1\}$ and nonzero products $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $[h, v_0] = v_0$, $[h, v_1] = -v_1$, $[e, v_1] = v_0$, $[f, v_0] = v_1$, $[h, z_0] = z_1$, $[h, z_0] = -z_0$, $[e, z_1] = z_0$, $[f, z_0] = z_1$, $[v_0, v_1] = w_0$, $[v_0, w_0] = z_0$ and $[v_1, w_0] = z_1$. In this case, $\mathfrak{n}_{2,3} = \mathbb{K}v_0 \oplus \mathbb{K}v_1 \oplus \mathbb{K}w_0 \oplus \mathbb{K}z_0 \oplus \mathbb{K}z_1$ is a $V(1) \oplus V(0) \oplus V(1)$-module of the Levi subalgebra $\mathbb{K}e \oplus \mathbb{K}f \oplus \mathbb{K}h$ ($\cong \mathrm{Der}_1^0 \mathfrak{n}_{2,3}$).*
- *iv)* *The 11-dimensional algebra $\mathfrak{g}_{2,4}$ with basis $\{e, f, h, v_0, v_1, w_0, z_0, z_1, x_0, x_1, x_2\}$ and nonzero products $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $[h, v_0] = v_0$, $[h, v_1] = -v_1$, $[e, v_1] = v_0$, $[f, v_0] = v_1$, $[h, z_0] = z_1$, $[h, z_0] = -z_1$, $[e, z_1] = z_0$, $[f, z_0] = z_1$, $[h, x_0] = 2x_0$, $[h, x_2] = -2x_2$, $[e, x_1] = 2x_0$, $[e, x_2] = x_1$, $[f, x_0] = x_1$, $[f, x_1] = 2x_2$, $[v_0, v_1] = w_0$, $[v_0, w_0] = z_0$, $[v_1, w_0] = z_1$, $[v_0, z_0] = x_0$, $[v_0, z_1] = [v_1, z_0] = \frac{1}{2}x_1$ and $[v_1, z_1] = x_2$. In this case, $\mathfrak{n}_{2,4} = \mathbb{K}v_0 \oplus \mathbb{K}v_1 \oplus \mathbb{K}w_0 \oplus \mathbb{K}z_0 \oplus \mathbb{K}z_1 \oplus \mathbb{K}x_0 \oplus \mathbb{K}x_1 \oplus \mathbb{K}x_2$ is a $V(1) \oplus V(0) \oplus V(1) \oplus V(2)$-module of the Levi subalgebra $\mathbb{K}e \oplus \mathbb{K}f \oplus \mathbb{K}h$ ($\cong \mathrm{Der}_1^0 \mathfrak{n}_{2,4}$).*
- *v)* *The 17-dimensional algebra $\mathfrak{g}_{2,5}$ with basis $\{e, f, h, v_0, v_1, w_0, z_0, z_1, x_0, x_1, x_2, y_0, y_1, y_2, y_3, u_0, u_1\}$ and nonzero products $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $[h, v_0] = v_0$, $[h, v_1] = -v_1$, $[e, v_1] = v_0$, $[f, v_0] = v_1$, $[h, z_0] = z_1$, $[h, z_0] = -z_1$, $[e, z_1] = z_0$, $[f, z_0] = z_1$, $[h, x_0] = 2x_0$, $[h, x_2] = -2x_2$, $[e, x_1] = 2x_0$, $[e, x_2] = x_1$, $[f, x_0] = x_1$, $[f, x_1] = 2x_2$, $[v_0, v_1] = w_0$, $[v_0, w_0] = z_0$, $[v_1, w_0] = z_1$, $[v_0, z_0] = x_0$, $[v_0, z_1] = [v_1, z_0] = \frac{1}{2}x_1$ and $[v_1, z_1] = x_2$. In this case, $\mathfrak{n}_{2,5} = \mathbb{K}v_0 \oplus \mathbb{K}v_1 \oplus \mathbb{K}w_0 \oplus \mathbb{K}z_0 \oplus \mathbb{K}z_1 \oplus \mathbb{K}x_0 \oplus \mathbb{K}x_1 \oplus \mathbb{K}x_2 \mathbb{K}y_0 \oplus \mathbb{K}y_1 \oplus \mathbb{K}y_2 \oplus \mathbb{K}y_3 \oplus \mathbb{K}u_0 \oplus \mathbb{K}u_1$ is a $V(1) \oplus V(0) \oplus V(1) \oplus V(2) \oplus V(3) \oplus V(2)$-module of the Levi subalgebra $\mathbb{K}e \oplus \mathbb{K}f \oplus \mathbb{K}h$ ($\cong \mathrm{Der}_1^0 \mathfrak{n}_{2,5}$).*

*vi)* $\mathfrak{s} \oplus \mathfrak{n}_{2,t}$ *and* $\mathfrak{s} \oplus \mathfrak{g}_{2,t}$*, where* $\mathfrak{s}$ *is an arbitrary semisimple Lie algebra and* $\oplus$ *denotes here direct sum of ideals.*

**Proof.** Let $L = \mathfrak{s} \oplus \mathfrak{n}_{2,t}$ with $\mathfrak{s}$ a Levi subalgebra. By using the adjoint (restricted) representation $\rho = \mathrm{ad}_{\mathfrak{n}_{2,t}}$ of $L$, the radical $\mathfrak{n}_{2,t}$ becomes an $\mathfrak{s}$-module. If $\rho$ is nontrivial, $\rho(\mathfrak{s})$ is a semisimple subalgebra of the Levi subalgebra of $\mathrm{Der}\,\mathfrak{n}_{2,t}$. So, $\mathfrak{s} = \mathrm{Ker}\,\rho \oplus \mathfrak{s}_1$, where $\mathfrak{s}_1$ is a 3-dimensional split simple ideal of $\mathfrak{s}$ and $\mathfrak{s}_1 \oplus \mathfrak{n}_{2,t}$ is a faithful Lie algebra. From Theorem 2.2, up to isomorphism, we can assume $\mathfrak{s}_1 \oplus \mathfrak{n}_{2,t} = \mathrm{Der}_1^0\,\mathfrak{n}_{2,t} \oplus_{id} \mathfrak{n}_{2,t}$. From now on, we fixed a standard basis $\{e, f, h\}$ of the 3-dimensional split simple algebra $\mathrm{Der}_1^0\,\mathfrak{n}_{2,t}$ so: $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Note that $\mathfrak{m}$ is a faithful module by declaring $d \cdot x = [d, x] = d(x)$. Thus $\mathfrak{m}$ is irreducible of type $V(1)$ and we can take a standard basis $\{v_0, v_1\}$ of $\mathfrak{m}$ as in [9]. If $t = 1$, *i)* follows immediately. For $s \geq 2$ the homogeneous components $\mathfrak{m}^s$ given in (4) are $\mathrm{Der}_1^0\,\mathfrak{n}_{2,t}$-submodules contained in the tensor product $\mathfrak{m} \otimes \mathfrak{m}^{s-1}$ (the Lie product $[\cdot,\cdot] : \mathfrak{m} \otimes \mathfrak{m}^{s-1} \to \mathfrak{m}^s$ is a surjective homomorphism of $\mathrm{Der}_1^0\,\mathfrak{n}_{2,t}$-modules). So, from Clebch–Gordan formula and a counting dimension argument based on (5) we have:

- $\mathfrak{m}^2 \subseteq \mathfrak{m} \otimes \mathfrak{m} = V(1) \otimes V(1) = V(2) \oplus V(0)$; so $\mathfrak{m}^2 = V(0)$ is a trivial module. Thus $\mathfrak{m}^2 = \mathrm{span}\langle [v_0, v_1]\rangle$ and for $t = 2$, defining $w_0 = [v_0, v_1]$ we have *ii)*.
- $\mathfrak{m}^3 \subseteq \mathfrak{m} \otimes \mathfrak{m}^2 = V(1) \otimes V(0) = V(1)$; so $\mathfrak{m}^3 = V(1)$ is a 2-irreducible module. Since $\mathfrak{m}^3 = [\mathfrak{m}, \mathfrak{m}^2] = \mathrm{span}\langle [v_0, w_0] = z_0, [v_1, w_0] = z_1\rangle$, for $t = 3$ we obtain *iii)*.
- $\mathfrak{m}^4 \subseteq \mathfrak{m} \otimes \mathfrak{m}^3 = V(1) \otimes V(1) = V(2) \oplus V(0)$; so $\mathfrak{m}^4 = V(2)$ is a 3-irreducible module. Since $\mathfrak{m}^4 = [\mathfrak{m}, \mathfrak{m}^3]$ and $[v_1, z_0] = [v_0, z_1]$, the set $\{[v_0, z_0] = x_0, 2[v_1, z_0] = x_1, [v_1, z_0] = x_2\}$ turns out a standard basis of $V(2)$ inside $\mathfrak{m}^4$. So, we have *iv)*.
- $\mathfrak{m}^5$ is 6-dimensional and $\mathfrak{m}^5 \subseteq \mathfrak{m} \otimes \mathfrak{m}^4 = V(1) \otimes V(2) = V(3) \oplus V(1)$, so $\mathfrak{m}^5 = V(3) \oplus V(1)$. In this case, the set $\{y_0 = [v_0, x_0], y_1 = [v_0, x_1] + [v_1, x_0], y_2 = [v_1, x_1] + [v_0, x_2], y_3 = [v_1, x_2]\}$ spans a module of type $V(3)$ (in fact it is a standard basis) and the set $\{u_0 = [v_0, x_1] - 2[v_1, x_0], u_1 = -[v_1, x_1] + 2[v_0, x_2]\}$ spans a $V(1)$ module (standard basis). In this case, $[\mathfrak{m}^i, \mathfrak{m}^j] = 0$ for $i, j \geq 3$ or $i = 2$ and $j \geq 4$, and the product relation $[w_0, z_i] = \frac{1}{2}u_i$ follows easily by using Jacobi identity. Then, $\{y_0, y_1, y_2, y_3, u_0, u_1\}$ is a basis of $\mathfrak{m}^5$ and we have *v)*.

The final item *vi)* covers the non-faithful Lie algebras with solvable radical $\mathfrak{n}_{2,t}$. □

Now, from Proposition 2.3, [6, Proposition 1.5] and [5, Theorem 3.5] we get the whole list of nilpotent Lie algebras of type 2 and nilindex $\leq 4$:

**Corollary 2.4.** *Up to isomorphism, the nilpotent Lie algebras of type* 2 *and nilindex* $t \leq 4$ *are:*

*a)* $\mathfrak{n}_{2,t}$ *for* $t = 1, 2, 3, 4$*;*
*b)* *the quotient Lie algebra* $\frac{\mathfrak{n}_{2,3}}{I}$*, where* $I$ *is any* 1*-dimensional subspace of* $\mathfrak{n}_{2,3}^3 = \mathrm{span}\langle z_0, z_1\rangle$*;*

c) *the quotient Lie algebra* $\frac{\mathfrak{n}_{2,4}}{I}$ *where* $I$ *is one of the following ideals:*
   i) *any* 1 *or* 2*-dimensional subspace of* $\mathfrak{n}_{2,3}^4 = \operatorname{span}\langle x_0, x_1, x_2\rangle$,
   ii) $I = \operatorname{span}\langle z_1 + \alpha x_0, x_1, x_2\rangle$, $\alpha \in \mathbb{K}$ *or*
   iii) $I = \operatorname{span}\langle z_0 + \alpha z_1 + \beta x_2, 2x_0 + \alpha x_1, x_1 + 2\alpha x_2\rangle$, $\alpha, \beta \in \mathbb{K}$.

*Moreover, the algebras in the previous list that admit a nontrivial Levi extension are those given in item a).*

**Proof.** From [6, Proposition 1.5], these algebras are of the form $\frac{\mathfrak{n}_{2,t}}{I}$ where the ideal $I$ is contained in $\mathfrak{n}_{2,t}^2$ and such that $\mathfrak{n}_{2,t}^t \nsubseteq I$. Now, the result for $t \leq 4$ is a straightforward computation. □

**Remark 1.** Corollary 2.4 provides the classification of filiform Lie algebras of nilindex $\leq 4$. Apart from $\mathfrak{n}_{2,1}$ and $\mathfrak{n}_{2,2}$ the algebras in this class are given in items *b*), *c*)-*ii*), *c*)-*iii*). By considering 2-subspaces $I$, item *c*)-*i*) and $\mathfrak{n}_{2,3}$ give us all quasifiliform algebras of type 2 and nilindex $\leq 4$. □

## 3. Solvable Lie algebras with nilradical $\mathfrak{n}_{2,t}$

Given a general Lie algebra $\mathfrak{g}$, and any abelian subalgebra $\mathfrak{t}$ of $\operatorname{Der}\mathfrak{g}$ (so $\mathfrak{t}^2 = [\mathfrak{t}, \mathfrak{t}] = 0$), from [10, Section 5, Chapter I] we can define the *split extension* $\mathfrak{g}(\mathfrak{t}) = \mathfrak{t} \oplus_{id} \mathfrak{g}$ by declaring $\mathfrak{g}$ and $\mathfrak{t}$ as subalgebras of $\mathfrak{g}(\mathfrak{t})$ and $[d, x] = d(x)$ for any $d \in \mathfrak{t}$. Then, by using derivations, we can get new Lie algebras easily.

We denote $\mathfrak{r}$ and $\mathfrak{n}$ the solvable and the nilpotent radicals of $\mathfrak{g}$ respectively. Because of the Levi theorem, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ where $\mathfrak{s}$ is a Levi subalgebra and $[\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{n}$. Moreover the nilradical can be described as:

$$\mathfrak{n} = \{x \in \mathfrak{r} : \operatorname{ad}_{\mathfrak{r}} x \text{ is nilpotent}\} = \{x \in \mathfrak{r} : \operatorname{ad}_{\mathfrak{g}} x \text{ is nilpotent}\}$$

(see [7, Proposition 2.3.6, Chapter 2]). So $\operatorname{ad}_{\mathfrak{r}} x$ is a non-nilpotent derivation of $\mathfrak{n}$ for any $x \in \mathfrak{r} - \mathfrak{n}$. Thus we have the direct sum decomposition $\mathfrak{r} = \mathfrak{n} \oplus \mathfrak{t}$, where $\mathfrak{t}$ is a vector subspace such that $\mathfrak{t}^2 \subseteq \mathfrak{n}$, and $\operatorname{ad}_{\mathfrak{r}} \mathfrak{t}$ are non-nilpotent derivations of $\mathfrak{n}$. Moreover, $[\operatorname{ad}_{\mathfrak{r}} \mathfrak{t}, \operatorname{ad}_{\mathfrak{r}} \mathfrak{t}] = \operatorname{ad}_{\mathfrak{r}} \mathfrak{t}^2$ is contained in $\operatorname{Inner}\mathfrak{n}$, the ideal of inner derivations of $\mathfrak{n}$.

In this section we prove that any solvable Lie algebra $\mathfrak{r}$ with nilradical $\mathfrak{n}_{2,t}$ satisfies

$$\dim \mathfrak{r} \leq \dim \mathfrak{n}_{2,t} + 2,$$

which agrees with [20, Theorem 1]. We also show that the bound $\dim \mathfrak{r} = \dim \mathfrak{n}_{2,t} + 2$ can always be obtained. Even more, we will prove that the solvable Lie algebras with nilradical $\mathfrak{n}_{2,t}$ arise from vector spaces of derivations closely related to the subalgebra $\operatorname{Der}_1 \mathfrak{n}_{2,t} = \operatorname{Der}_1^0 \mathfrak{n}_{2,t} \oplus \mathbb{K} \cdot id_{2,t}$ described in (7).

**Theorem 3.1.** *Let* $\mathfrak{r}$ *be a solvable and non-nilpotent Lie algebra with nilradical* $\mathfrak{n}_{2,t}$*. Then:*

*a) either $\mathfrak{r}$ is the split extension $\mathfrak{r}(\delta) = \mathbb{K} \cdot \delta \oplus_{id} \mathfrak{n}_{2,t}$ where $\delta$ is any non-nilpotent derivation of $\mathfrak{n}_{2,t}$,*

*b) or there are derivations $\mu_1, \mu_2$ of $\mathfrak{n}_{2,t}$, $\mu_1 = id_{2,t} + \eta_1$, $\mu_2 = \delta + \eta_2$ where $0 \neq \delta \in \operatorname{Der}_1^0 \mathfrak{n}_{2,t}$ is semisimple, $\eta_1, \eta_2 \in \mathfrak{N}_{2,t}$, the bracket $[\mu_1, \mu_2] = \mu_1\mu_2 - \mu_2\mu_1 = \operatorname{ad}_{\mathfrak{n}_{2,t}} x$ for some $x \in \mathfrak{n}_{2,t}$, and the set of derivations $\mathbb{K}\cdot\mu_1 \oplus \mathbb{K}\cdot\mu_2$ has no nilpotent elements; in this case, $\mathfrak{r}$ is the Lie algebra with underlying vector space $\mathfrak{r}(\mu_1, \mu_2; x) = \mathfrak{n}_{2,t} \oplus \mathbb{K}\cdot\mu_1 \oplus \mathbb{K}\cdot\mu_2$ and Lie product for $a, b \in \mathfrak{n}_{2,t}$ $\langle a, b\rangle_{\mu_1,\mu_2}$ the product in $\mathfrak{n}_{2,t}$, $\langle \mu_1, \mu_2\rangle_{\mu_1,\mu_2} = x$ and $\langle \mu_i, y\rangle_{\mu_1,\mu_2} = \mu_i(y)$ for any $y \in \mathfrak{n}_{2,t}$.*

*Moreover, two Lie algebras in item a) are isomorphic if and only if*

$$\theta\delta\theta^{-1} - \alpha\delta' \in \operatorname{Inner} \mathfrak{n}_{2,t},$$

*for some $0 \neq \alpha \in \mathbb{K}$ and $\theta \in \operatorname{Aut} \mathfrak{n}_{2,t}$; two Lie algebras in item b) are isomorphic if and only if there exist $\alpha_{i,j} \in \mathbb{K}$, $y_1, y_2 \in \mathfrak{n}_{2,t}$ and $\theta \in \operatorname{Aut} \mathfrak{n}_{2,t}$ such that*

$$0 \neq \Delta_{ij} = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21},$$
$$\theta\mu_i\theta^{-1} - \alpha_{i1}\mu_1' - \alpha_{i2}\mu_2' = \operatorname{ad}_{\mathfrak{n}_{2,t}} y_i \text{ and}$$
$$\theta(x) = \Delta_{ij}x' + \alpha_{11}\mu_1'(y_2) + \alpha_{12}\mu_2'(y_2) - \alpha_{21}\mu_1'(y_1) - \alpha_{22}\mu_2'(y_1) + 3[y_1, y_2].$$

**Proof.** It is a straightforward computation that $\mathfrak{r}(\delta)$ and $\mathfrak{r}(\mu_1, \mu_2; x)$ are Lie algebras. Now, let $\mathfrak{t}$ be an arbitrary complement of $\mathfrak{n}_{2,t}$ in the solvable Lie algebra $\mathfrak{r}$. From (1), we have the relation $\mathfrak{t}^2 \subseteq \mathfrak{r}^2 \subseteq \mathfrak{n}_{2,t}$. Note that, since the nilradical of $\mathfrak{r}$ is $\mathfrak{n}_{2,t}$, the vector linear space $\operatorname{ad}_{\mathfrak{n}_{2,t}} \mathfrak{t}$ is a subspace of non-nilpotent derivations of $\mathfrak{n}_{2,t}$. Consider now the adjoint representation $\operatorname{ad}_{\mathfrak{n}_{2,t}} : \mathfrak{r} \to \operatorname{Der} \mathfrak{n}_{2,t}$ and the projection homomorphism $\pi_0 : \operatorname{Der} \mathfrak{n}_{2,t} \to \operatorname{Der}_1 \mathfrak{n}_{2,t}$. Then, $\operatorname{Ker} \pi_0 \circ \operatorname{ad}_{\mathfrak{n}_{2,t}} = \{a \in \mathfrak{r} : \operatorname{ad}_{\mathfrak{n}_{2,t}} a \in \mathfrak{N}_{2,t}\} = \mathfrak{n}_{2,t}$. Thus the quotient $\frac{\mathfrak{r}}{\mathfrak{n}_{2,t}}$ is an abelian Lie algebra with the same dimension as $\mathfrak{t}$ and isomorphic to $\operatorname{Im} \pi_0 \circ \operatorname{ad}_{\mathfrak{n}_{2,t}}$. Since $\operatorname{Der}_1 \mathfrak{n}_{2,t} \cong \mathfrak{gl}_2(\mathbb{K})$, up to isomorphism, $\operatorname{Im} \pi_0 \circ \operatorname{ad}_{\mathfrak{n}_{2,t}}$ is a set of commutative linear transformations over a 2-dimensional vector space. For any $\alpha \cdot id_{\mathfrak{m}} \neq f \in \operatorname{Im} \pi_0 \circ \operatorname{ad}_{\mathfrak{n}_{2,t}}$, the centralizer of $f$, $C_{\mathfrak{gl}_2(\mathbb{K})}(f)$ is a 2-dimensional subspace and it contains $\operatorname{Im} \pi_0 \circ \operatorname{ad}_{\mathfrak{n}_{2,t}}$. So, $\dim \mathfrak{t} = 1, 2$. In case $\mathfrak{t} = \mathbb{K} \cdot x$, we get $\mathfrak{r} \cong \mathfrak{r}(\operatorname{ad}_{\mathfrak{n}_{2,t}} x)$ and item *a*) follows. For a 2-dimensional extension $\mathfrak{t}$, $\operatorname{Im} \pi_0 \circ \operatorname{ad}_{\mathfrak{n}_{2,t}} \mathfrak{t} = C_{\mathfrak{gl}_2(\mathbb{K})}(\delta) = \mathbb{K}\cdot id_{2,t} \oplus \mathbb{K}\cdot\delta$ and w.l.o.g., we can assume that $\delta$ is traceless and semisimple because $\operatorname{ad}_{\mathfrak{n}_{2,t}} \mathfrak{t}$ has not nilpotent maps. Thus there exists a basis $\{x_1, x_2\}$ of $\mathfrak{t}$ with $\operatorname{ad}_{\mathfrak{n}_{2,t}} x_i = \mu_i$ and $x = [x_1, x_2] \in \mathfrak{n}_{2,t}$ as described in item *b*). In this case, $\mathfrak{r}$ is isomorphic to the Lie algebra $\mathfrak{r}(\operatorname{ad}_{\mathfrak{n}_{2,t}} x_1, \operatorname{ad}_{\mathfrak{n}_{2,t}} x_2; x)$. The final assertion on isomorphisms is easily checked taking into account that any isomorphism between Lie algebras preserve the nilradical. □

**Example 1.** For any arbitrary $t \geq 1$, let $\mathfrak{m} = \mathbb{K}\cdot v_0 \oplus \mathbb{K}\cdot v_1$ and $h_{2,t}$ be the derivation of $\mathfrak{n}_{2,t}$ obtained by extension of the linear map $h \in \mathfrak{gl}(\mathfrak{m})$ such that $h(v_0) = v_0$ and $h(v_1) = -v_1$. Consider the abelian subalgebra $\mathfrak{t} = \mathbb{K} \cdot id_{2,t} \oplus \mathbb{K} \cdot h_{2,t}$ of derivations of $\operatorname{Der}_1 \mathfrak{n}_{2,t}$. Then, the split extensions $\mathfrak{r}(id_{2,t}) = \mathbb{K} \cdot id_{2,t} \oplus_{id} \mathfrak{n}_{2,t}$ and $\mathfrak{r}(id_{2,t}, h_{2,t}; 0) = \operatorname{span}\langle id_{2,t}, h_{2,t}\rangle \oplus_{id} \mathfrak{n}_{2,t}$ are solvable Lie algebras with nilradical the free nilpotent Lie algebra $\mathfrak{n}_{2,t}$.

**Remark 2.** Note that $Z(\mathfrak{n}_{2,t}) = \mathfrak{m}^t$, $id_{2,t}|_{\mathfrak{m}^t} = t \cdot id_{\mathfrak{m}^t}$ and any derivation of $\mathfrak{N}_{2,t}$ acts trivially on $Z(\mathfrak{n}_{2,t})$. In Theorem 3.1, item b), if the derivations $\mu_1$ and $\mu_2$ commute, the bracket derivation $[\mu_1, \mu_2] = \mu_1\mu_2 - \mu_2, \mu_1 = 0$ is equal to $\operatorname{ad}_{\mathfrak{n}_{2,t}} z$ for any $z \in Z(\mathfrak{n}_{2,t})$. So we have Lie algebras $\mathfrak{r}(\mu_1, \mu_2; z)$ $\forall_{z \in Z(\mathfrak{n}_{2,t})}$, but all of them are isomorphic to the split extension $\mathfrak{r}(\mu_1, \mu_2; 0) = \operatorname{span}\langle \mu_1, \mu_2\rangle \oplus_{id} \mathfrak{n}_{2,t}$, taking $\Phi(n) = n$ $\forall_{z \in \mathfrak{n}_{2,t}}$, $\Phi(\mu_1) = \mu_1$ and $\Phi(\mu_2) = \mu_2 - \frac{1}{t} z$, we get an isomorphism $\Phi : \mathfrak{r}(\mu_1, \mu_2; 0) \to \mathfrak{r}(\mu_1, \mu_2; z)$. Thus, from commutative 2-dimensional subspaces of derivations, up to isomorphism we get split extensions (i.e., extensions through abelian subalgebras of derivations).

### *3.1. Derivations and automorphisms*

From [19, Proposition 2.1], the set of derivations of any Lie algebra $\mathfrak{n}_{2,t}$ is completely determined by $\operatorname{Hom}(\mathfrak{m}, \mathfrak{m})$ and $\operatorname{Hom}(\mathfrak{m}, \mathfrak{n}_{d,t}^2) = \oplus_{2 \le j \le t} \operatorname{Hom}(\mathfrak{m}, \mathfrak{m}^j)$. Moreover, any automorphism appears as an extension of a map of the general linear group $GL(\mathfrak{m})$ according to [19, Proposition 3]. Along this section, the algebras of derivations and groups of automorphisms of free nilpotent Lie algebras will be represented through matrices $(\alpha_{ij})$ relative to the bases given in Proposition 2.3 ($\mathfrak{m} = \mathbb{K} \cdot v_0 \oplus \mathbb{K} \cdot v_1$, only nonzero products are displayed). The sets $\operatorname{Der} \mathfrak{n}_{2,3}$ and $\operatorname{Aut} \mathfrak{n}_{2,3}$ appear in [3]; here we present an alternative method for computing them.

$\mathfrak{n}_{2,1}$: Abelian 2-dimensional.

$\operatorname{Der} \mathfrak{n}_{2,1} = \mathfrak{gl}_2(\mathbb{K}) = \operatorname{Der}_1 \mathfrak{n}_{2,1}$:

$$D_{\mathbf{u}}^{\beta} = \begin{pmatrix} \alpha_1 + \beta & \alpha_2 \\ \alpha_3 & -\alpha_1 + \beta \end{pmatrix}, \quad \mathbf{u} = (\alpha_1, \alpha_2, \alpha_3);$$

extended derivation of the identity map $I_2$ is $I_{2,1} = I_2 = D_{\mathbf{0}}^1$;

$\mathfrak{N}_{2,1} = \operatorname{Inner} \mathfrak{n}_{2,1} = 0$;

$\operatorname{Aut} \mathfrak{n}_{2,1} = GL_2(\mathbb{K})$:

$$\theta_{\mathbf{v}} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \quad \mathbf{v} = (\alpha_1, \ldots, \alpha_4) \text{ and } \epsilon = \alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0.$$

$\mathfrak{n}_{2,2}$: Heisenberg 3-dimensional, $[v_0, v_1] = w_0$.

$$\operatorname{Der} \mathfrak{n}_{2,2} : D_{\mathbf{u}}^{\beta} = \begin{pmatrix} \alpha_1 + \beta & \alpha_2 & 0 \\ \alpha_3 & -\alpha_1 + \beta & 0 \\ \alpha_4 & \alpha_5 & 2\beta \end{pmatrix}, \quad \mathbf{u} = (\alpha_1, \ldots, \alpha_5);$$

$$\operatorname{Der}_1 \mathfrak{n}_{2,2} : D^{\beta}_{(\alpha_1,\alpha_2,\alpha_3,0,0)} = \begin{pmatrix} \alpha_1+\beta & \alpha_2 & 0 \\ \alpha_3 & -\alpha_1+\beta & 0 \\ 0 & 0 & 2\beta \end{pmatrix};$$

extended derivation of $I_2$ is the $3 \times 3$ matrix $I_{2,2} = D^1_{\mathbf{0}}$;

$$\mathfrak{N}_{2,2} = \operatorname{Inner} \mathfrak{n}_{2,2} : D^{0}_{(0,0,0,\alpha_4,\alpha_5)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_4 & \alpha_5 & 0 \end{pmatrix};$$

$$\operatorname{Aut} \mathfrak{n}_{2,2} : \theta_{\mathbf{v}} = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ \alpha_5 & \alpha_6 & \epsilon \end{pmatrix}, \quad \mathbf{v} = (\alpha_1, \ldots, \alpha_5).$$

$\mathfrak{n}_{2,3}$: $[v_0, v_1] = w_0, [v_i, w_0] = z_i, i = 0, 1$.
Der $\mathfrak{n}_{2,3}$:

$$D^{\beta}_{\mathbf{u}} = \begin{pmatrix} \alpha_1+\beta & \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & -\alpha_1+\beta & 0 & 0 & 0 \\ \alpha_4 & \alpha_5 & 2\beta & 0 & 0 \\ \alpha_6 & \alpha_7 & \alpha_5 & \alpha_1+3\beta & \alpha_2 \\ \alpha_8 & \alpha_9 & -\alpha_4 & \alpha_3 & -\alpha_1+3\beta \end{pmatrix}, \quad \mathbf{u} = (\alpha_1, \ldots, \alpha_9);$$

$\operatorname{Der}_1 \mathfrak{n}_{2,3}$:

$$D^{\beta}_{(\alpha_1,\alpha_2,\alpha_3,0,0,0,0,0,0)} = \begin{pmatrix} \alpha_1+\beta & \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & -\alpha_1+\beta & 0 & 0 & 0 \\ 0 & 0 & 2\beta & 0 & 0 \\ 0 & 0 & 0 & \alpha_1+3\beta & \alpha_2 \\ 0 & 0 & 0 & \alpha_3 & -\alpha_1+3\beta \end{pmatrix};$$

extended derivation of $I_2$ is the $5 \times 5$ matrix $I_{2,3} = D^1_{\mathbf{0}}$;

$$\mathfrak{N}_{2,3} : D^{0}_{(0,0,0,\alpha_4,\ldots,\alpha_9)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_4 & \alpha_5 & 0 & 0 & 0 \\ \alpha_6 & \alpha_7 & \alpha_5 & 0 & 0 \\ \alpha_8 & \alpha_9 & -\alpha_4 & 0 & 0 \end{pmatrix};$$

$$\operatorname{Inner} \mathfrak{n}_{2,3} : D^{0}_{(0,0,0,\alpha_4,\alpha_5,\alpha_6,0,0,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_4 & \alpha_5 & 0 & 0 & 0 \\ \alpha_6 & 0 & \alpha_5 & 0 & 0 \\ 0 & \alpha_6 & -\alpha_4 & 0 & 0 \end{pmatrix};$$

$$\operatorname{Aut}\mathfrak{n}_{2,3}:\theta_{\mathbf{v}}=\begin{pmatrix}\alpha_1 & \alpha_2 & 0 & 0 & 0\\ \alpha_3 & \alpha_4 & 0 & 0 & 0\\ \alpha_5 & \alpha_6 & \epsilon & 0 & 0\\ \alpha_7 & \alpha_8 & \alpha_1\alpha_6-\alpha_2\alpha_5 & \epsilon\alpha_1 & \epsilon\alpha_2\\ \alpha_9 & \alpha_{10} & \alpha_3\alpha_6-\alpha_4\alpha_5 & \epsilon\alpha_3 & \epsilon\alpha_4\end{pmatrix},\quad \mathbf{v}=(\alpha_1,\dots,\alpha_{10}).$$

### *3.2. Solvable* 1*-extensions of* $\mathfrak{n}_{2,t}$ *for* $t=1,2,3$

(Along Sections 3.2.1, 3.2.2 and 3.2.3 we assume that $\mathbb{K}$ is algebraically closed.)

Solvable 1-extensions $\mathfrak{r}(\delta)=\mathfrak{n}_{2,t}\oplus\mathbb{K}\cdot\delta$ follows from a no nilpotent derivation $\delta=D+O$ where $D\in\operatorname{Der}_1\mathfrak{n}_{2,t}$ and $O\in\mathfrak{N}_{2,t}$. Since $\theta\operatorname{Inner}\mathfrak{n}_{2,t}\theta^{-1}=\operatorname{Inner}\mathfrak{n}_{2,t}$, we can consider w.l.o.g. extensions by derivations with zero projection on the ideal of inner derivations (see comment on isomorphisms in Theorem 3.1).

#### *3.2.1.* 1*-extensions in case* $\mathfrak{n}_{2,1}$

From $\operatorname{Inner}\mathfrak{n}_{2,1}=0$, the solvable 1-extensions of $\mathfrak{n}_{2,1}$ are determined by Jordan forms of $2\times 2$ matrices. So, up to isomorphism, $\delta$ is one of following derivations:

1. $D^1_{(0,0,1)}=\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$.
2. $D^{\frac{1+\alpha}{2}}_{(\frac{1-\alpha}{2},0,0)}=\left(\begin{smallmatrix}1&0\\0&\alpha\end{smallmatrix}\right)$, $\alpha\in\mathbb{K}$.

($\mathfrak{r}(D^1_{(0,0,1)})$, $\mathfrak{r}(D^{\frac{1+\alpha}{2}}_{(\frac{1-\alpha}{2},0,0)})$, for any $\alpha\in\mathbb{K}$ are not isomorphic.)

#### *3.2.2.* 1*-extensions in case* $\mathfrak{n}_{2,2}$

Since $\operatorname{Inner}\mathfrak{n}_{2,2}=\mathfrak{N}_{2,2}$, the solvable 1-extensions of $\mathfrak{n}_{2,2}$: are also given by Jordan forms of $2\times 2$ matrices. As in the previous case, we have two possibilities for $\delta$:

1. $D^1_{(0,0,1,0,0)}=\left(\begin{smallmatrix}1&0&0\\1&1&0\\0&0&2\end{smallmatrix}\right)$.
2. $D^{\frac{1+\alpha}{2}}_{(\frac{1-\alpha}{2},0,0,0,0)}=\left(\begin{smallmatrix}1&0&0\\0&\alpha&0\\0&0&1+\alpha\end{smallmatrix}\right)$, $\alpha\in\mathbb{K}$.

($\mathfrak{r}(D^1_{(0,0,1,0,0)})$, $\mathfrak{r}(D^{\frac{1+\alpha}{2}}_{(\frac{1-\alpha}{2},0,0,0,0)})$, for any $\alpha\in\mathbb{K}$ are not isomorphic.)

#### *3.2.3.* 1*-extensions in case* $\mathfrak{n}_{2,3}$

Note that derivations with no projection on $\operatorname{Inner}\mathfrak{n}_{2,3}$ are of the form:

$$\begin{pmatrix}A & \mathbf{0}_{2,1} & \mathbf{0}_{2,2}\\ \mathbf{0}_{1,2} & \operatorname{tr}A & \mathbf{0}_{1,2}\\ M & \mathbf{0}_{2,1} & A+2\beta I_2\end{pmatrix}=\begin{pmatrix}A & \mathbf{0}_{2,1} & \mathbf{0}_{2,2}\\ \mathbf{0}_{1,2} & \operatorname{tr}A & \mathbf{0}_{1,2}\\ \mathbf{0}_{2,2} & \mathbf{0}_{2,1} & A+2\beta I_2\end{pmatrix}+\begin{pmatrix}\mathbf{0}_{2,2} & \mathbf{0}_{2,1} & \mathbf{0}_{2,2}\\ \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2}\\ M & \mathbf{0}_{2,1} & \mathbf{0}_{2,2}\end{pmatrix},$$

$\mathbf{0}_{m,n}$ $m \times n$ zero matrix, $A$ and $M = \left(\begin{smallmatrix} \alpha_6 & \alpha_7 \\ \alpha_8 & 0 \end{smallmatrix}\right)$ $2 \times 2$ matrices. According to (6), the second summand in previous matrix decomposition corresponds to a derivation $\delta : \mathfrak{n}_{2,3} \to Z(\mathfrak{n}_{2,3}) = \mathfrak{n}_{2,3}^3$ inside the abelian subalgebra $\operatorname{Der}_3 \mathfrak{n}_{2,3}$ without projection on $\operatorname{Inner} \mathfrak{n}_{2,3}$. This subalgebra is invariant by conjugation (i.e., $\theta \operatorname{Der}_3 \mathfrak{n}_{2,3} \theta^{-1} = \operatorname{Der}_3 \mathfrak{n}_{2,3}$) and the general form of its derivations is:

$$\begin{pmatrix} \mathbf{0}_{2,2} & \mathbf{0}_{2,1} & \mathbf{0}_{2,2} \\ \mathbf{0}_{1,2} & \mathbf{0}_{1,1} & \mathbf{0}_{1,2} \\ \begin{smallmatrix} \tau_1 & \tau_2 \\ \tau_3 & \tau_4 \end{smallmatrix} & \mathbf{0}_{2,1} & \mathbf{0}_{2,2} \end{pmatrix}.$$

Then, up to isomorphism, the extensions we are looking for depend on the Jordan form of the matrix $A$ and we have one of the following possibilities:

(a) The derivation matrix $D^1_{\mathbf{u}}$ where $\mathbf{u} = (0, 0, 1, 0, 0, \alpha_6, \alpha_7, \alpha_8, 0)$, i.e.:

$$D^1_{\mathbf{u}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \alpha_6 & \alpha_7 & 0 & 3 & 0 \\ \alpha_8 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

Denoting $\mathbf{v} = (1, 0, 0, 1, 0, 0, \frac{1}{4}(2\alpha_6 + \alpha_7), \frac{\alpha_7}{2}, -\frac{\alpha_6}{4} - \frac{\alpha_7}{4} + \frac{\alpha_8}{2}, -\frac{\alpha_7}{4})$, the automorphism $\theta_{\mathbf{v}}$ transforms previous derivation by conjugation into:

$$\theta_{\mathbf{v}}^{-1} \cdot D^1_{\mathbf{u}} \cdot \theta_{\mathbf{v}} = D^1_{\mathbf{u}_1}$$

where $\mathbf{u}_1 = (0, 0, 1, 0, 0, 0, 0, 0, 0)$. Then, we get:

$$D^1_{\mathbf{u}_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

(b) The derivation matrix $D^{\frac{1+\alpha}{2}}_{\mathbf{u}}$ with $\mathbf{u} = (\frac{1-\alpha}{2}, 0, 0, 0, 0, \alpha_6, \alpha_7, \alpha_8, 0)$, i.e.:

$$D^{\frac{1+\alpha}{2}}_{\mathbf{u}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 1+\alpha & 0 & 0 \\ \alpha_6 & \alpha_7 & 0 & 2+\alpha & 0 \\ \alpha_8 & 0 & 0 & 0 & 1+2\alpha \end{pmatrix}.$$

Now three subcases can be considered:

(b.1) $\alpha = -1$: By conjugation through the automorphism $\theta_{\mathbf{v}}$, with $\mathbf{v} = (1, 0, 0, 1, 0, 0, 0, \frac{\alpha_7}{2}, \frac{\alpha_8}{2}, 0)$, the derivation $D_{\mathbf{u}}^{\frac{1+\alpha}{2}} = D_{\mathbf{u}}^0$ becomes:

$$\theta_{\mathbf{v}}^{-1} \cdot D_{\mathbf{u}}^0 \cdot \theta_{\mathbf{v}} = D_{\mathbf{w}}^0,$$

where $\mathbf{w} = (1, 0, 0, 0, 0, \alpha_6, 0, 0, 0)$. If $\alpha_6 \neq 0$, by using $\theta_{\mathbf{v}'}$ with $\mathbf{v}' = (\frac{1}{\sqrt{\alpha_6}}, 0, 0, \frac{1}{\sqrt{\alpha_6}}, 0, 0, 0, 0, 0, 0)$, we get:

$$\theta_{\mathbf{v}'}^{-1} \cdot D_{\mathbf{w}}^0 \cdot \theta_{\mathbf{v}'} = D_{\mathbf{u}_3}^0,$$

$\mathbf{u}_3 = (1, 0, 0, 0, 0, 1, 0, 0, 0)$. So, up to isomorphism, two new possible derivations appear ($\mathbf{u}_2 = (1, 0, 0, 0, 0, 0, 0, 0, 0)$ is related to $\alpha_6 = 0$):

$$D_{\mathbf{u}_2}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad D_{\mathbf{u}_3}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

(b.2) $\alpha = 0$: Let $\mathbf{v} = (1, 0, 0, 1, 0, 0, \alpha_6, \frac{\alpha_7}{2}, 0, 0)$ be. The automorphism $\theta_{\mathbf{v}}$, transforms $D_{\mathbf{u}}^{\frac{1+\alpha}{2}} = D_{\mathbf{u}}^{\frac{1}{2}}$ into:

$$\theta_{\mathbf{v}}^{-1} \cdot D_{\mathbf{u}}^{\frac{1}{2}} \cdot \theta_{\mathbf{v}} = D_{\mathbf{w}}^{\frac{1}{2}},$$

where $\mathbf{w} = (\frac{1}{2}, 0, 0, 0, 0, 0, 0, \alpha_8, 0)$. Now, in case $\alpha_8 \neq 0$, taking $\mathbf{v}' = (\frac{1}{\sqrt{\alpha_8}}, 0, 0, \frac{1}{\sqrt{\alpha_8}}, 0, 0, 0, 0, 0, 0)$ we have:

$$\theta_{\mathbf{v}'}^{-1} \cdot D_{\mathbf{w}}^{\frac{1}{2}} \cdot \theta_{\mathbf{v}'} = D_{\mathbf{u}_5}^{\frac{1}{2}},$$

$\mathbf{u}_5 = (\frac{1}{2}, 0, 0, 0, 0, 0, 0, 1, 0)$. So, up to isomorphism, two additional derivations appear ($\mathbf{u}_4 = (\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0)$ is related to $\alpha_8 = 0$):

$$D_{\mathbf{u}_4}^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D_{\mathbf{u}_5}^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b.3) $\alpha \neq 0, -1$: From $\mathbf{v} = (1, 0, 0, 1, 0, 0, \frac{\alpha_6}{1+\alpha}, \frac{\alpha_7}{2}, \frac{\alpha_8}{2\alpha}, 0)$, we have:

$$\theta_{\mathbf{v}}^{-1} \cdot D_{\mathbf{u}}^{\frac{1+\alpha}{2}} \cdot \theta_{\mathbf{v}} = D_{\mathbf{u}_6}^{\frac{1+\alpha}{2}},$$

$\mathbf{u}_6 = (\frac{1-\alpha}{2}, 0, 0, 0, 0, 0, 0, 0, 0)$. So, we get the derivation:

$$D_{\mathbf{u}_6}^{\frac{1+\alpha}{2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 1+\alpha & 0 & 0 \\ 0 & 0 & 0 & 2+\alpha & 0 \\ 0 & 0 & 0 & 0 & 1+2\alpha \end{pmatrix}.$$

*Summarizing:* Up to isomorphism, the solvable 1-extensions of $\mathfrak{n}_{2,3}$ are $\mathfrak{r}(\delta)$ where $\delta$ is one of the following derivations:

1. $D_{\mathbf{u}_1}^1$, where $\mathbf{u}_1 = (0, 0, 1, 0, 0, 0, 0, 0, 0)$;
2. $D_{\mathbf{u}_3}^0$, where $\mathbf{u}_3 = (1, 0, 0, 0, 0, 1, 0, 0, 0)$;
3. $D_{\mathbf{u}_5}^{\frac{1}{2}}$, where $\mathbf{u}_5 = (\frac{1}{2}, 0, 0, 0, 0, 0, 0, 1, 0)$;
4. $D_{\mathbf{u}_6}^{\frac{1+\alpha}{2}}$, where $\mathbf{u}_6 = (\frac{1-\alpha}{2}, 0, 0, 0, 0, 0, 0, 0, 0)$ and $\alpha \in \mathbb{K}$.

($\mathfrak{r}(D_{\mathbf{u}_1}^1)$, $\mathfrak{r}(D_{\mathbf{u}_3}^0)$, $\mathfrak{r}(D_{\mathbf{u}_5}^{\frac{1}{2}})$ and $\mathfrak{r}(D_{\mathbf{u}_6}^{\frac{1+\alpha}{2}})$ for any $\alpha \in \mathbb{K}$ are not isomorphic.)

### *3.3. Solvable 2-extensions*

(Along Sections 3.3.1, 3.3.2 and 3.3.3 we assume $\mathbb{K}$ is algebraically closed.)

According to Theorem 3.1, the solvable 2-extensions are of the form $\mathfrak{r}(\mu_1, \mu_2; x) = \mathfrak{n}_{2,t} \oplus \mathbb{K} \cdot \mu_1 \oplus \mathbb{K} \cdot \mu_2$, $\mu_1 = I_{2,t} + O_1$, and $\mu_2 = D + O_2$ where $D \in \operatorname{Der}_1^0 \mathfrak{n}_{2,t}$ is semisimple and traceless, $O_i \in \mathfrak{N}_{2,t}$ without projection on $\operatorname{Inner} \mathfrak{n}_{2,t}$. Moreover

$$[I_{2,t} + O_1, D + O_2] = [I_{2,t}, O_2] + [O_1, D] + [O_1, O_2] \in \operatorname{Inner} \mathfrak{n}_{2,t}$$

and the span$\langle I_{2,t} + O_1, D + O_2 \rangle$ has no nilpotent derivations.

#### *3.3.1. 2-extensions in case* $\mathfrak{n}_{2,1}$

Since $\mathfrak{N}_{2,1} = 0$, the solvable extensions are just given by using abelian subalgebras of derivations $\mathbb{K} \cdot I_{2,1} \oplus \mathbb{K} \cdot D$, where $D$ is a traceless semisimple $2 \times 2$ matrix with two different nonzero eigenvalues $\alpha$ and $-\alpha$. So we can assume w.l.o.g. ($\mathfrak{n}_{2,1}$ is just a vector space with trivial product):

$$D = D_{(1,0,0)}^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From Remark 2, up to isomorphism we get the extension $\mathfrak{r}(I_{2,1}, D_{(1,0,0)}^0; 0)$.

*3.3.2. 2-extensions in case* $\mathfrak{n}_{2,2}$

Now, $I_{2,2} = D^1_{\mathbf{0}}$ and $\mathfrak{N}_{2,2} = \operatorname{Inner}\mathfrak{n}_{2,2}$. Thus, the solvable extensions are also given by abelian subalgebras $\mathbb{K}\cdot I_{2,2}\oplus\mathbb{K}\cdot D$, $D = D^0_{(\alpha_1,\alpha_2,\alpha_3,0,0)}$ semisimple and traceless derivation with eigenvalues $0 \neq \alpha$, $-\alpha$ and 0. Up to isomorphism, we can take $D = D^0_{(1,0,0,0,0)}$, i.e.:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which provides $\mathfrak{r}(I_{2,1}, D^0_{(1,0,0,0,0)}; 0)$ as unique extension from Remark 2.

*3.3.3. 2-extensions in case* $\mathfrak{n}_{2,3}$

Derivations inside $\mathfrak{N}_{2,3}$ without projection on $\operatorname{Inner}\mathfrak{n}_{2,3}$ belong to

$$\operatorname{Der}_3\mathfrak{n}_{2,3} = \{d \in \operatorname{Der}\mathfrak{n}_{2,3} : d(\mathfrak{n}_{2,3}) \subseteq Z(\mathfrak{n}_{2,3})\},$$

which is invariant by conjugation through $\operatorname{Aut}\mathfrak{n}_{2,3}$. So, up to isomorphism, we can assume $\mu_1 = I_{2,3} + O_1 = D^1_{\mathbf{u}}$ and $\mu_2 = D + O_2 = D^0_{\mathbf{w}}$ where $\mathbf{u} = (0,0,0,0,0,\alpha_6,\alpha_7,\alpha_8,0)$ and $\mathbf{w} = (1,0,0,0,0,\beta_6,\beta_7,\beta_8,0)$, i.e.:

$$D^1_{\mathbf{u}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \alpha_6 & \alpha_7 & 0 & 3 & 0 \\ \alpha_8 & 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad D^1_{\mathbf{w}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \beta_6 & \beta_7 & 0 & 1 & 0 \\ \beta_8 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now,

$$[D^1_{\mathbf{u}}, D^1_{\mathbf{w}}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2\beta_6 & 2(\beta_7 - \alpha_7) & 0 & 0 & 0 \\ 2(\alpha_8 + \beta_8) & 0 & 0 & 0 & 0 \end{pmatrix}$$

is an inner derivation if and only if $\beta_6 = 0$, $\beta_7 = \alpha_7$ and $\beta_8 = -\alpha_8$, so $\mathbb{K}\cdot\mu_1 \oplus \mathbb{K}\cdot\mu_2$ is an abelian subalgebra. Taking the automorphism $\theta_{\mathbf{v}}$, where $\mathbf{v} = (1,0,0,1,0,0,-\frac{\alpha_6}{2},-\frac{\alpha_7}{2},-\frac{\alpha_8}{2},0)$, we get the derivations:

$$\theta_{\mathbf{v}}^{-1}\cdot D^1_{\mathbf{u}}\cdot\theta_{\mathbf{v}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} = I_{2,3} \quad \text{and}$$

$$\theta_{\mathbf{v}}^{-1} \cdot D_{\mathbf{w}}^{1} \cdot \theta_{\mathbf{v}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = D_{\mathbf{u}_1}^{0},$$

where $\mathbf{u}_1 = (1, 0, 0, 0, 0, 0, 0, 0, 0)$. From Remark 2, up to isomorphism $\mathfrak{r}(I_{2,3}, D_{\mathbf{u}_1}^{0}; 0)$ is the unique 2-extension.

## 4. Non-solvable Lie algebras with nilradical $\mathfrak{n}_{2,t}$

From Theorems 2.1 and 2.2 in [21] (see also [5, Proposition 2.2]) and Theorem 2.2, the faithful nonsolvable Lie algebras with nilradical $\mathfrak{n}_{2,t}$ are of the form,

$$\mathfrak{g} = \mathfrak{g}_{2,t} \oplus \mathfrak{t}_0 \tag{10}$$

where $\mathfrak{g}_{2,t} = \operatorname{Der}_1^0 \mathfrak{n}_{2,t} \oplus_{id} \mathfrak{n}_{2,t}$ is as described in Theorem 2.2, $\mathfrak{r} = \mathfrak{n}_{2,t} \oplus \mathfrak{t}_0$ is the solvable radical of $\mathfrak{g}$, $\mathfrak{t}_0$ is a trivial $\operatorname{ad}_{\mathfrak{g}} \operatorname{Der}_1^0 \mathfrak{n}_{2,t}$-module and $\operatorname{ad}_{\mathfrak{g}} x$ is a not nilpotent map $\forall_{x \in \mathfrak{t}_0}$. So, in the Lie algebra $\mathfrak{g}$ with product $\langle a, b\rangle$, we have $\langle \operatorname{Der}_1^0 \mathfrak{n}_{2,t}, \mathfrak{t}_0\rangle = 0$. Let $(\mathfrak{n}_{2,t})_0$ be the sum of all trivial $\operatorname{ad}_{\mathfrak{g}} \operatorname{Der}_1^0 \mathfrak{n}_{2,t}$ modules in $\mathfrak{n}_{2,t}$ and note that this vector space is a subalgebra of $\mathfrak{n}_{2,t}$. Since $\mathfrak{t}_0^2 \subseteq (\mathfrak{n}_{2,t})_0$, the direct sum:

$$\mathfrak{r}_0 = \mathfrak{t}_0 \oplus (\mathfrak{n}_{2,t})_0 \tag{11}$$

is a subalgebra of $\mathfrak{r}$ that contains the (solvable) subalgebra generated by $\mathfrak{t}_0$. From the Jacobi identity in (10) and for $d \in \operatorname{Der}_1^0 \mathfrak{n}_{2,t}$, $x \in \mathfrak{r}_0$ and $a \in \mathfrak{n}_{2,t}$, we have:

$$\begin{aligned} d\big(\operatorname{ad}_{\mathfrak{g}} x(a)\big) &= \big\langle d, \langle x, a\rangle\big\rangle = \big\langle \langle d, x\rangle, a\big\rangle + \big\langle x, \langle d, a\rangle\big\rangle \\ &= \big\langle x, \langle d, a\rangle\big\rangle = \operatorname{ad}_{\mathfrak{g}} x\big(d(a)\big), \end{aligned}$$

i.e., the set of derivations of the form $\operatorname{ad}_{\mathfrak{n}_{2,t}} \mathfrak{r}_0$ centralizes $\operatorname{Der}_1^0 \mathfrak{n}_{2,t}$ in $\operatorname{Der} \mathfrak{n}_{2,t}$. These basic ideas yield to the following result:

**Theorem 4.1.** *Let $\mathfrak{g}$ be a nonsolvable Lie algebra with nilradical $\mathfrak{n}_{2,t}$. Then $\mathfrak{g}$ is one of the following Lie algebras:*

*a) $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$, a direct sum as ideals of a semisimple algebra $\mathfrak{s}$ and a solvable Lie algebra $\mathfrak{r}$ with nilradical $\mathfrak{n}_{2,t}$;*

*b) $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{g}_{2,t}$, a direct sum as ideals where $\mathfrak{s}$ is either the null algebra or any arbitrary semisimple algebra;*

c) $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{g}_{2,t}(\delta)$, *a direct sum as ideals where* $\mathfrak{s}$ *is either the null algebra or any arbitrary semisimple Lie algebra and* $\mathfrak{g}_{2,t}(\delta) = (\mathrm{Der}_1^0 \mathfrak{n}_{2,t} \oplus \mathbb{K} \cdot (id_{2,t} + \delta)) \oplus_{id} \mathfrak{n}_{2,t}$ *where* $\delta \in C_{\mathrm{Der}\, \mathfrak{n}_{2,t}}(\mathrm{Der}_1^0 \mathfrak{n}_{2,t}) \cap \mathfrak{N}_{2,t}$ *and* $\mathrm{Der}_1^0 \mathfrak{n}_{2,t} \oplus \mathbb{K} \cdot (id_{2,t} + \delta)$ *is viewed as Lie subalgebra of* $\mathrm{Der}\, \mathfrak{n}_{2,t}$.

**Proof.** Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ be a Levi decomposition of $\mathfrak{g}$. In case $[\mathfrak{s}, \mathfrak{r}] = 0$ or $\mathfrak{t}_0 = 0$ we get items $a$) and $b$). Otherwise, we have $\mathfrak{t}_0 \neq 0$ and $[\mathfrak{s}, \mathfrak{n}_{2,t}] \neq 0$ and, from Theorem 2.2 and preliminary comments in this section, $\mathfrak{g}$ decomposes as a direct sum of ideals, $\mathfrak{g} = \mathfrak{s}_1 \oplus (\mathrm{Der}_1^0 \mathfrak{n}_{2,t} \oplus_{id} \mathfrak{n}_{2,t} \oplus \mathfrak{t}_0)$, $\mathfrak{s}_1$ being either the null Lie algebra or a semisimple one. Without loss of generality, we can assume $\mathfrak{g}$ is faithful ($\mathfrak{s}_1 = 0$). Let $\mathfrak{r}_0$ the subalgebra defined in (11) and consider the (restricted) adjoint representation $\mathrm{ad}_{\mathfrak{n}_{2,t}} : \mathfrak{r}_0 \to \mathrm{Der}\, \mathfrak{n}_{2,t}$, which is both a homomorphism of Lie algebras and a $\mathrm{Der}_1^0 \mathfrak{n}_{2,t}$-module homomorphism. Since $\mathfrak{n}_{2,t}$ is the nilradical of $\mathfrak{g}$, the kernel of $\mathrm{ad}_{\mathfrak{n}_{2,t}}$ is the centralizer $C_{\mathfrak{r}_0}(\mathfrak{n}_{2,t})$ which coincides with $\mathcal{Z}(\mathfrak{n}_{2,t})_0$, the trivial module living inside the center of the nilradical. So, we can decompose $(\mathfrak{n}_{2,t})_0$ as a direct sum of subspaces: $(\mathfrak{n}_{2,t})_0 = \mathcal{Z}(\mathfrak{n}_{2,t})_0 \oplus \mathfrak{v}_0$. Then, the quotient Lie algebra $\frac{\mathfrak{r}_0}{\mathcal{Z}(\mathfrak{n}_{2,t})_0}$ ($\cong \mathfrak{v}_0 \oplus \mathfrak{t}_0$) is isomorphic to $\mathrm{ad}_{\mathfrak{n}_{2,t}}(\mathfrak{r}_0)$, a subalgebra of $\mathrm{Der}\, \mathfrak{n}_{2,t}$ contained in $C_{\mathrm{Der}\, \mathfrak{n}_{2,t}}(\mathrm{Der}_1^0 \mathfrak{n}_{2,t}) = k \cdot id_{2,t} \oplus (\mathfrak{N}_{2,t})_0$, where $(\mathfrak{N}_{2,t})_0$ is the sum of trivial $\mathrm{Der}_1^0 \mathfrak{n}_{2,t}$-modules inside $\mathfrak{N}_{2,t}$ via the adjoint representation $[\delta, \mu] = \delta\mu - \mu\delta$. But $\mathrm{ad}_{\mathfrak{n}_{2,t}} \mathfrak{t}_0$ has no nilpotent derivations so, any element inside $\mathrm{ad}_{\mathfrak{n}_{2,t}}(\mathfrak{t}_0)$ has nontrivial projection in $k \cdot id_{2,t}$. Let $x, y \in \mathfrak{t}_0$ be two nonzero elements; scaling, we can assume $\mathrm{ad}_{\mathfrak{n}_{2,t}} x = id_{2,t} \oplus \delta$ and $\mathrm{ad}_{\mathfrak{n}_{2,t}} y = id_{2,t} \oplus \mu$, where $\delta, \mu \in (\mathfrak{N}_{2,t})_0$. Then $\mathrm{ad}_{\mathfrak{n}_{2,t}}(x - y) \in (\mathfrak{N}_{2,t})_0$ and therefore, $x - y$ is a nilpotent element in $\mathfrak{t}_0$; so $x = y$, and part $c$) follows. It is immediate to check that items $a$), $b$) and $c$) provides Lie algebras. □

**Example 2.** For any arbitrary $t \geq 1$, $\mathrm{Der}_1 \mathfrak{n}_{2,t} \oplus_{id} \mathfrak{n}_{2,t}$ is a nonsolvable Lie algebra with nilradical the free nilpotent Lie algebra $\mathfrak{n}_{2,t}$ and solvable radical $\mathbb{K} \cdot id_{2,t} \oplus_{id} \mathfrak{n}_{2,t}$.

**Corollary 4.2.** *For* $t \leq 4$ *and up to isomorphism, the faithful nonsolvable Lie algebras with nilradical* $\mathfrak{n}_{2,t}$ *are* $\mathfrak{g}_{2,t} = \mathrm{Der}_1^0 \mathfrak{n}_{2,t} \oplus_{id} \mathfrak{n}_{2,t}$ *and* $\mathrm{Der}_1 \mathfrak{n}_{2,t} \oplus_{id} \mathfrak{n}_{2,t}$.

**Proof.** For free nilpotent of low index, the centralizer $C_{\mathrm{Der}\, \mathfrak{n}_{2,t}}(\mathrm{Der}_1^0 \mathfrak{n}_{2,t})$ can be easily computed. In fact, we have:

- For $t = 1, 2$, $C_{\mathrm{Der}\, \mathfrak{n}_{2,t}}(\mathrm{Der}_1^0 \mathfrak{n}_{2,t}) = \mathbb{K} \cdot id_{2,t}$;
- For $t = 3, 4$, $C_{\mathrm{Der}\, \mathfrak{n}_{2,t}}(\mathrm{Der}_1^0 \mathfrak{n}_{2,t}) = \mathbb{K} \cdot id_{2,t} \oplus \mathbb{K} \cdot \mathrm{ad}\, \omega_0$, where $\omega_0 = [v_0, v_1]$ according to Proposition 2.3.

The result follows trivially from Theorem 4.1 in cases $t = 1, 2$, and for $t = 2, 3$ we note that the elements inside $C_{\mathrm{Der}\, \mathfrak{n}_{2,t}}(\mathrm{Der}_1^0 \mathfrak{n}_{2,t})$ are of the form $\alpha \cdot id_{2,t} \oplus \beta \cdot \mathrm{ad}\, \omega_0$. But $\mathrm{ad}\, \omega_0$ is an inner derivation of $\mathfrak{n}_{2,t}$ and therefore it can be removed. □

## 5. Conclusion: Lie algebras with nilradical $\mathfrak{n}_{2,t}$ for $t = 1, 2, 3$

From previous results and comments, the classification of Lie algebras with nilradical $\mathfrak{n}_{2,t}$, reduces to determine suitable sets of derivations containing no nilpotent elements as described in Lemmas 3.1 and 4.1. These sets are completely determined for $t = 1, 2, 3$ in Section 3 (solvable extensions) by using explicit matrix expressions for derivations, inner derivations and automorphisms of $\mathfrak{n}_{2,t}$, and in Section 4 (nonsolvable extensions) by computing centralizers inside the Lie algebra of derivations $\operatorname{Der}\mathfrak{n}_{2,t}$. In this final section we summarize the results in one theorem, where the complete list of Lie algebras with nilradical $\mathfrak{n}_{2,1}, \mathfrak{n}_{2,2}$ or $\mathfrak{n}_{2,3}$ is given by their multiplication tables.

**Theorem 5.1.** *Up to isomorphism, the Lie algebras over an algebraically closed field* $\mathbb{K}$ *with nilradical a free nilpotent Lie algebra of type* 2 *and nilindex* $t \leq 3$ *are (only nonzero products involving elements not in the nilradical are given):*

*i)* *The abelian* 2*-dimensional* $\mathfrak{n}_{2,1}$*;*
*ii)* $\mathfrak{r}^1_{2,1} = \mathfrak{n}_{2,1} \oplus \mathbb{K} \cdot x$: $[x, v_0] = v_0 + v_1, [x, v_1] = v_1$;
*iii)* $\mathfrak{r}^{1,\alpha}_{2,1} = \mathfrak{n}_{2,1} \oplus \mathbb{K} \cdot x$: $[x, v_0] = v_0$ *and* $[x, v_1] = \alpha v_1$, $\alpha \in \mathbb{K}$;
*iv)* $\mathfrak{r}^2_{2,1} = \mathfrak{n}_{2,1} \oplus \mathbb{K} \cdot x \oplus \mathbb{K} \cdot y$: $[x, v_0] = [y, v_0] = v_0, [x, v_1] = -[y, v_1] = v_1$;
*v)* $\mathfrak{g}_{2,1} = \mathfrak{sl}_2(\mathbb{K}) \oplus \mathfrak{n}_{2,1}$: $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $[h, v_0] = v_0$, $[h, v_1] = -v_1$, $[e, v_1] = v_0$, $[f, v_0] = v_1$;
*vi)* $\mathfrak{g}^1_{2,1} = \mathfrak{sl}_2(\mathbb{K}) \oplus \mathfrak{r}^{1,1}_{2,1}$: $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $[h, v_0] = v_0$, $[h, v_1] = -v_1$, $[e, v_1] = v_0$, $[f, v_0] = v_1$, $[x, v_0] = v_0$, $[x, v_1] = v_1$;
*vii)* *The Heisenberg* 3*-dimensional* $\mathfrak{n}_{2,2}$: $[v_0, v_1] = w_0$;
*viii)* $\mathfrak{r}^1_{2,2} = \mathfrak{n}_{2,2} \oplus \mathbb{K} \cdot x$: $[x, v_0] = v_0 + v_1, [x, v_1] = v_1$, $[x, w_0] = 2w_0$;
*ix)* $\mathfrak{r}^{1,\alpha}_{2,2} = \mathfrak{n}_{2,2} \oplus \mathbb{K} \cdot x$: $[x, v_0] = v_0$, $[x, v_1] = \alpha v_1$, $[x, w_0] = (1 + \alpha)w_0$, $\alpha \in \mathbb{K}$;
*x)* $\mathfrak{r}^2_{2,2} = \mathfrak{n}_{2,2} \oplus \mathbb{K} \cdot x \oplus \mathbb{K} \cdot y$: $[x, v_0] = [y, v_0] = v_0$, $[x, v_1] = -[y, v_1] = v_1$, $[x, w_0] = 2w_0$;
*xi)* $\mathfrak{g}_{2,2} = \mathfrak{sl}_2(\mathbb{K}) \oplus \mathfrak{n}_{2,2}$: $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $[h, v_0] = v_0$, $[h, v_1] = -v_1$, $[e, v_1] = v_0$, $[f, v_0] = v_1$;
*xii)* $\mathfrak{g}^1_{2,2} = \mathfrak{sl}_2(\mathbb{K}) \oplus \mathfrak{r}^{1,1}_{2,2}$: $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $[h, v_0] = v_0$, $[h, v_1] = -v_1$, $[e, v_1] = v_0$, $[f, v_0] = v_1$, $[x, v_0] = v_0$, $[x, v_1] = v_1$; $[x, w_0] = 2w_0$;
*xiii)* *The* 5*-dimensional* $\mathfrak{n}_{2,3}$: $[v_0, v_1] = w_0$, $[v_i, w_0] = z_i, i = 0, 1$;
*xiv)* $\mathfrak{r}^1_{2,3} = \mathfrak{n}_{2,3} \oplus \mathbb{K} \cdot x$: $[x, v_0] = v_0 + v_1$, $[x, v_1] = v_1$, $[x, w_0] = 2w_0$, $[x, z_0] = 3z_0 + z_1$, $[x, z_1] = 3z_1$;
*xv)* $\mathfrak{r}^{1,\alpha}_{2,3} = \mathfrak{n}_{2,3} \oplus \mathbb{K} \cdot x$: $[x, v_0] = v_0$, $[x, v_1] = \alpha v_1$, $[x, w_0] = (1+\alpha)w_0$, $[x, z_0] = (2+\alpha)z_0$, $[x, z_1] = (1 + 2\alpha)z_1$, $\alpha \in \mathbb{K}$;
*xvi)* $\mathfrak{r}^2_{2,3} = \mathfrak{n}_{2,3} \oplus \mathbb{K} \cdot x$: $[x, v_0] = v_0 + z_1$, $[x, w_0] = w_0$, $[x, z_0] = 2z_0$, $[x, z_1] = z_1$;
*xvii)* $\mathfrak{r}^3_{2,3} = \mathfrak{n}_{2,3} \oplus \mathbb{K} \cdot x$: $[x, v_0] = v_0 + z_0$, $[x, v_1] = -v_1$, $[x, z_0] = z_0$, $[x, z_1] = -z_1$;
*xviii)* $\mathfrak{r}^4_{2,3} = \mathfrak{n}_{2,3} \oplus \mathbb{K} \cdot x \oplus \mathbb{K} \cdot y$: $[x, v_0] = [y, v_0] = v_0$, $[x, v_1] = -[y, v_1] = v_1$, $[x, w_0] = 2w_0$, $[x, z_0] = 3z_0$, $[x, z_1] = 3z_1$, $[y, z_0] = z_0$, $[y, z_1] = -z_1$;

*xix)* $\mathfrak{g}_{2,3} = \mathfrak{sl}_2(\mathbb{K}) \oplus \mathfrak{n}_{2,3}$*:* $[e, f] = h$*,* $[h, e] = 2e$*,* $[h, f] = -2f$*,* $[h, v_0] = v_0$*,* $[h, v_1] = -v_1$*,* $[e, v_1] = v_0$*,* $[f, v_0] = v_1$*,* $[h, z_0] = z_0$*,* $[h, z_1] = -z_1$*,* $[e, z_1] = z_0$*,* $[f, z_0] = z_1$*,* $[v_0, v_1] = w_0$*,* $[v_0, w_0] = z_0$ *and* $[v_1, w_0] = z_1$*;*

*xx)* $\mathfrak{g}^1_{2,3} = \mathfrak{sl}_2(\mathbb{K}) \oplus \mathfrak{r}^{1,1}_{2,3}$*:* $[e, f] = h$*,* $[h, e] = 2e$*,* $[h, f] = -2f$*,* $[h, v_0] = v_0$*,* $[h, v_1] = -v_1$*,* $[e, v_1] = v_0$*,* $[f, v_0] = v_1$*,* $[h, z_0] = z_0$*,* $[h, z_1] = -z_1$*,* $[e, z_1] = z_0$*,* $[f, z_0] = z_1$*,* $[v_0, v_1] = w_0$*,* $[v_0, w_0] = z_0$*,* $[v_1, w_0] = z_1$*,* $[x, v_0] = v_0$*,* $[x, v_1] = v_1$*,* $[x, w_0] = 2w_0$*,* $[x, z_0] = 3z_0$*,* $[x, z_1] = 3z_1$*;*

*xxi) the direct sum of ideals* $\mathfrak{s} \oplus \mathfrak{g}$*, where* $\mathfrak{s}$ *is a semisimple Lie algebra and* $\mathfrak{g}$ *is one of the Lie algebras in i)–xx).* □

**Remark 3.** Theorem 5.1 provides the complete classification of complex Lie algebras with nilradical the 3-dimensional Heisenberg algebra $\mathfrak{n}_{2,2}$. The classification problem in case of solvable extensions was studied in [18] including the case $\mathbb{K} = \mathbb{R}$. Our list provides the (unique) nonsolvable extensions $\mathfrak{g}_{2,2}$ and $\mathfrak{g}^1_{2,2}$. We note that the techniques used here can also be extended to provide a solution to the analogous classification problem over the field of real numbers for $\mathfrak{n}_{2,t}$, $t = 1, 2, 3, 4$. □

**Remark 4.** Extensions of $\mathfrak{n}_{2,3}$ are treated in [1, Section 3, Section 3.2]. In that paper, $\mathfrak{n}_{2,3}$ is denoted as $\mathcal{L}_{5,3}$ and introduced in Theorem 1, Section 2.2, by the multiplication table $[X_0, X_1] = X_2, [X_0, X_2] = X_3, [X_1, X_2] = X_4$. Note that the basis $\{X_0, X_1, X_2, X_3, X_4\}$ is just our basis $\{v_0, v_1, w_0, z_1, z_2\}$. The extended Lie algebras are listed in Propositions 2, 3 and 4. Proposition 2 provides the 1-solvable extensions $\mathfrak{g}^{6,1}_{5,3}$, $\mathfrak{g}^{6,2}_{5,3}$, $\mathfrak{g}^{6,3}_{5,3}$ and $\mathfrak{g}^{6,4}_{5,3}$. The first and third are just $\mathfrak{r}^{1,\alpha}_{2,3}$ and $\mathfrak{r}^2_{2,3}$. The second algebra is equal to $\mathfrak{r}^3_{2,3}$ by rescaling the basis of $\mathfrak{g}^{6,2}_{5,3}$ given in [1] in the following way: $\{-Y; X_0, X_1, X_2, X_3, -X_4\}$. The multiplication table for $\mathfrak{g}^{6,4}_{5,3}$ contains a misprint: $[Y, X_4]$ must be declared as $3X_4$ instead of $4X_4$, otherwise, $\operatorname{ad} Y$ is not a derivation (in other words, Jacobi identity $J(a, b, c) = [[a, b], c] + [[a, b], c] + [[a, b], c] = 0$ fails for $(a, b, c) = (Y, X_1, X_2)$); by doing the correction, $\mathfrak{g}^{6,4}_{5,3}$ is just $\mathfrak{r}^1_{2,3}$. Proposition 3 provides the unique 2-solvable extension $\mathfrak{g}^{7,1}_{5,3}$, which is equal to our $\mathfrak{r}^4_{2,3}$ if we consider the new basis $\{Y + Z, Y - Z; X_0, X_1, X_2, X_3, X_4\}$.

In Proposition 4, the authors give the nonsolvable extensions $\mathfrak{g}^{8,1}_{5,3}$ and $\mathfrak{g}^{9,1}_{5,3}$. The former algebra is just our $\mathfrak{g}_{2,3}$ taking into account that $\{Y, Z, X\}$ spans a split 3-dimensional simple Lie algebra with standard basis $\{h, e, f\}$ and products $[h, e] = 2e, [h, f] = -2f$, $[e, f] = h$. But $\mathfrak{g}^{9,1}_{5,3}$ has been erroneously included in the classification, because it is not a Lie algebra: $J(Y - Y', Z, X_1) = -X_0 \neq 0$ and $J(Y - Y', Z, X_4) = -X_3 \neq 0$. This is just not a misprint but a structural error: $\mathfrak{s} = \operatorname{span}\langle Y - Y', Z, X\rangle$ is a 3-split simple Lie algebra and $\mathcal{L}_{5,3}$ is $\operatorname{ad}\mathfrak{s}$-invariant, so $\mathcal{L}_{5,3}$ decomposes as $\mathfrak{s}$-irreducible modules. But the Cartan subalgebra $\mathbb{K} \cdot (Y - Y')$ acts with 5 different eigenvalues on $\mathcal{L}_{5,3}$ (namely $1, -2, -1, 0, -3$) with corresponding eigenvectors $X_0, X_1, X_2, X_3, X_4$; this fails to be an $\mathfrak{s}$-module decomposition of $\mathcal{L}_{5,3}$. The last algebra in the classification is our Lie algebra $\mathfrak{g}^1_{2,3}$ described in item *xx*) of Theorem 5.1. □

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