Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Computational and Applied Mathematics 177 (2005) 225–229
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Letter to the editor

Accelerated convergence in Newton's method for approximating square roots☆

M.A. Hernández*, N. Romero

Department of Mathematics and Computation, University of La Rioja, C/Luis de Ulloa s/n, 26004 Logroño, Spain

Received 20 April 2004; received in revised form 24 August 2004

Abstract

In this paper, we construct a modification of Newton's method to accelerate the convergence of this method to the approximation of the positive square root of a positive real number. From this modification, we can define a new iterative process with prefixed order $q \in \mathbb{N}$, $q \geqslant 2$.

MSC: 65H05; 65B99

Keywords: Iterative processes; Newton's method; Square root

1. Introduction

The problem of approximating the positive square root of a positive real number R has been widely studied by different authors [2,7]. This problem is equivalent to solve the equation

$$f(t) = 0,$$

where

$$f(t) = t^2 - R. \tag{1}$$

☆ Supported in part by a grant from the Ministry of Science and Technology (Ref. BFM 2002-00222) and a grant from the University of La Rioja (Ref. API-02/18).

* Corresponding author.
E-mail addresses: mahernan@dmc.unirioja.es (M.A. Hernández), natalia.romero@dmc.unirioja.es (N. Romero).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2004.09.025

To solve this problem, the famous Heron's formula (75 b.C approx.), $t_{n+1} = (t_n + R/t_n)/2$, $n \geqslant 0$, is well known.

Modern calculus texts treat this algorithm as a special case of Newton's method, $t_{n+1} = t_n - f(t_n)/f'(t_n)$, $n \geqslant 0$. It is an established fact that the convergence of Newton's method is quadratic, at least for t_0 sufficiently close to the solution $\alpha = \sqrt{R}$.

There are numerous studies to accelerate the convergence of Newton's method [1,5]. A modern approach is given in [3], who applies Newton's method to the function $F(t) = t^{\beta-2} f(t)$, where $\beta \in \mathbb{R}$ is looked for in order to get cubic convergence. In fact, the value $\beta = \frac{3}{2}$ is obtained as the appropriate one for this purpose. The natural extension of this idea is to modify $f(t)$ by the function $F(t) = g(t)f(t)$, and determine $g(t)$ such that the order of convergence of Newton's method applied to $F(t)$ is increased. So, Gerlach [5], for a general real function *f*, obtains that, for $g(t) = C/\sqrt{f'(t)}$, $C \in \mathbb{R}$, the Newton method has cubic convergence. Notice that, for $C = 1$, this method is a well-known third-order iteration: Halley's method or the method of tangent hyperbolas [4].

In this paper, we modify the algorithm of Newton's process instead of the function. So, we consider the modified algorithm

$$t_{n+1} = G(t_n) = t_n - h(t_n)\,\frac{f(t_n)}{f'(t_n)}, \quad n \geqslant 0, \tag{2}$$

and we determine the function *h* such that we obtain an iterative process of *q*th order of convergence, $q \in \mathbb{N}$, $q \geqslant 2$, for approximating α. This construction is realized in Section 2.

In Section 3, we analyze the monotonous convergence of these new iterative processes to the solution. Finally, in Section 4, some of these new iterative processes are compared numerically.

2. A new *q*th-order method for approximating square roots, $q \geqslant 2$

It is known, that the iterative process given in (2) is of *q*th order of convergence if $G(\alpha) = \alpha$, and $G^{(k)}(\alpha) = 0$, for $k = 1, 2, \ldots, q-1$, and $G^{(q)}(\alpha) \neq 0$. Then, if we consider the situation where $G(t) = \alpha$, for all $t \in \mathbb{R}$, we obtain an iterative process of "infinite" order of convergence. Is this possible? To do that, we consider

$$G(t) = t - h(t)\frac{f(t)}{f'(t)} = \alpha. \tag{3}$$

Therefore, we can write

$$f'(t)(t-\alpha) = h(t)f(t). \tag{4}$$

Then, by Taylor's approximation, we obtain $f'(t)(t-\alpha) = f(t) + f''(t)(\alpha - t)^2/2!$. Moreover, using (4), it follows that: $f''(t)f(t)^2h(t)^2/2! - f(t)f'(t)^2h(t) + f(t)f'(t)^2 = 0$, which is a second degree polynomial in $h(t)$. Therefore, the solutions are

$$h_-(L_f(t)) = \frac{1 - \sqrt{1 - 2L_f(t)}}{L_f(t)} \tag{5}$$

and $h_+(L_f(t)) = (1 + \sqrt{1 - 2L_f(t)})/L_f(t)$, where $L_f(t) = f(t)f''(t)/f'(t)^2$ is the degree of logarithmic convexity [8]. Notice that, for *f* given in (1), it follows that $L_f(t) \leqslant \frac{1}{2}$ for all $t \in \mathbb{R}$.

In consequence, the answer of the previous question is affirmative, but we cannot apply algorithm (2), with $h_+(L_f(t))$, to approximate the positive square root α because the function has a singularity in the root α. Then, we consider the Taylor's approximation of the function $h_-(L_f(t))$ given in (5), to define algorithm (2), and it is easy to obtain that

$$h(L_f(t)) = h_-(L_f(t)) = \sum_{j \geqslant 0} \binom{\frac{1}{2}}{j+1} (-1)^j 2^{j+1} L_f(t)^j.$$

In these conditions, we can define algorithm (2) in the following form:

$$t_{n+1} = G(t_n) = t_n - \left(\sum_{j=0}^{q-2} \binom{\frac{1}{2}}{j+1} (-1)^j 2^{j+1} L_f(t_n)^j \right) \frac{f(t_n)}{f'(t_n)}, \quad n \geqslant 0, \tag{6}$$

for $q \geqslant 2$. The question is then: Does this algorithm have q-order? The answer of this question is also affirmative. Notice that, from (6), it is easy to prove that

$$G'(t) = 1 - A_0 + \sum_{j=1}^{q-2} ((2j-1)A_{j-1} - (j+1)A_j) L_f(t)^j + (2q-3)A_{q-2} L_f(t)^{q-1},$$

where $A_j = \binom{1/2}{j+1}(-1)^j 2^{j+1}$, $0 \leqslant j \leqslant q-2$. Then, as $A_0 = 1$ and $(2j-1)A_{j-1} = (j+1)A_j$, for $j = 1, 2, \ldots, q-2$, it follows that $G'(t) = (2q-3)A_{q-2}L_f(t)^{q-1}$. Therefore, since that $L_f(\alpha) = 0$, we obtain

$$G(\alpha) = \alpha, \quad G'(\alpha) = \cdots = G^{(q-1)}(\alpha) = 0, \quad \text{and} \quad G^{(q)}(\alpha) = (2q-3)A_{q-2}(q-1)!\alpha^{1-q} \neq 0.$$

So, the algorithm given by (6), is of qth order, for $q \geqslant 2$. Thus, we have constructed a new algorithm with any prefixed order of convergence $q \geqslant 2$. Notice that, for $q = 2$, we obtain the Newton method and, for $q = 3$, we obtain a famous iterative process of third order, the Chebyshev method [6].

Remark 1. Note that if we want to increase the order of convergence in one unity, as consequence of the new sum appearing in the summatory, we only need to add the following term, $A_{j+1}L_f(t_n)^{j+1}$. Moreover, the power $L_f(t_n)^j$ has already been evaluated and $A_{j+1} = A_j(2j+1)/(j+2)$ is easily calculated. Therefore, the operational cost increases by two multiplications and one addition. The procedure is effective as far as concerns to the relation between the operational cost and the order of convergence.

3. Monotonous convergence

In this section, we study the monotonous convergence of the new algorithm given in (6). As other authors [7], we consider $t_0 > \alpha$. Then, it is easy to check that

$$t_1 - t_0 = -\left(\sum_{j=0}^{q-2} A_j L_f(t_0)^j \right) \frac{f(t_0)}{f'(t_0)} \leqslant 0,$$

since A_j, $L_f(t_0)$, $f(t_0)$ and $f'(t_0)$ are positive numbers, $j = 0, 1, \ldots, q-2$.

Table 1
Computation of values $|t_n - \sqrt{35}|$

n	Dubeau's method, $q=3$	Method (6), $q=3$	Dubeau's method, $q=4$	Method (6), $q=4$
1	$0.169\cdots\times10^{-4}$	$0.817\cdots\times10^{-5}$	$0.454\cdots\times10^{-6}$	$0.142\cdots\times10^{-6}$
2	$0.140\cdots\times10^{-15}$	$0.781\cdots\times10^{-17}$	$0.385\cdots\times10^{-27}$	$0.123\cdots\times10^{-29}$
3	$0.782\cdots\times10^{-49}$	$0.682\cdots\times10^{-53}$	$0.199\cdots\times10^{-111}$	$0.709\cdots\times10^{-122}$
4	$0.137\cdots\times10^{-148}$	$0.454\cdots\times10^{-161}$	$0.144\cdots\times10^{-448}$	$0.763\cdots\times10^{-491}$
5	$0.732\cdots\times10^{-448}$	$0.134\cdots\times10^{-485}$	$0.397\cdots\times10^{-1797}$	$0.102\cdots\times10^{-1967}$

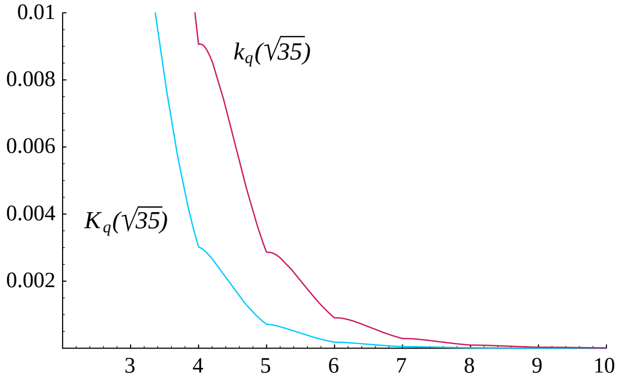

Fig. 1. Asymptotic constants.

Besides, as $G'(t)>0$ in (α, t_0), it follows that $t_1-\alpha=G'(\theta_0)(t_0-\alpha)>0$, $\theta_0\in(\alpha,t_0)$, and then $t_1>\alpha$.

By induction, if $t_{n-1}>t_n$ and $t_n>\alpha$, by an analogous reasoning as before, we obtain that $t_{n+1}<t_n$ and $t_{n+1}>\alpha$. Then there exists $l=\lim_n t_n$, and, by letting $n\to+\infty$ in (6), it follows $f(l)=0$, since that $h_-(L_f(t))>0$ for all $t\in\mathbb{R}$, and therefore $l=\alpha$.

In consequence, from the previous reasoning, the following result is obtained.

Theorem 1. *If $t_0>\alpha$, then the sequence $\{t_n\}$, given in* (6), *is monotonically decreasing and converges to the positive solution α of $f(t)=0$, with f given from* (1).

4. Numerical tests

When we want to apply the algorithm of qth order given in (6), it is interesting to note that, since

$$L_f(t)=\frac{1}{2}\left(1-\frac{R}{t^2}\right),\qquad \frac{f(t)}{f'(t)}=\frac{t}{2}\left(1-\frac{R}{t^2}\right),$$

we can write the algorithm given by (6) in the following simple form:

$$t_{n+1} = G(t_n) = t_n \sum_{j=0}^{q-1} \binom{\frac{1}{2}}{j} (-1)^j \left(1 - \frac{R}{t_n^2}\right)^j, \quad n \geqslant 0,$$

with $q \geqslant 2$. Next, we consider the application of algorithm (6) to the function $f(t) = t^2 - 35$ and we observe the improvement with regard to Dubeau's method [2]. In Table 1, the values $|t_n - 35^{1/2}|$ are shown for the computation of $\sqrt{35}$ with $t_0 = 6$.

Fig. 1, shows that the asymptotic constant of method (6): $K_q(\alpha) = (2q-3)A_{q-2}\alpha^{1-q}/q$ is smaller than the one of Dubeau's method: $k_q(\alpha) = -(q-1)2^q\binom{1/2}{q}\alpha^{1-q}$ where q is on the horizontal axis.

Moreover, if the two asymptotic constants are compared, we observe that

$$|K_q(\alpha)| < |k_q(\alpha)| \iff \frac{1}{q-1} < 1 \iff q > 2.$$

Therefore, from $q = 3$ (Chebyshev's method), processes (6) are faster than the other one given in [2].

References

[1] I.K. Argyros, On a new Newton–Mysowskii-type theorem with applications to inexact Newton-like methods and their discretizations, IMA J. Numer. Anal. 18 (1998) 37–56.

[2] F. Dubeau, Algorithms for *n*-th root approximation, Computing 57 (1996) 365–369.

[3] F. Dubeau, *n*-th Root extraction: double iteration process and Newton's method, J. Comput. Appl. Math. 91 (1998) 191–198.

[4] W. Gander, On Halley's iteration method, Amer. Math. Monthly 92 (1985) 131–134.

[5] J. Gerlach, Accelerated convergence in Newton's method, SIAM Rev. 36 (1994) 272–276.

[6] J.M. Gutiérrez, M.A. Hernández, M.A. Salanova, A family of Chebyshev–Halley type methods in Banach spaces, Bull. Austral. Math. Soc. 55 (1997) 113–130.

[7] J.M. Gutiérrez, M.A. Hernández, M.A. Salanova, Calculus of *n*-th roots and third order iterative methods, Nonlinear Anal. 47 (2001) 2875–2880.

[8] M.A. Hernández, M.A. Salanova, Index of convexity and concavity. Application to Halley method, Appl. Math. Comput. 103 (1999) 27–49.