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Variants of a classic Traub's result

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ABSTRACT

We present a generalization of a modification of a classic Traub's result, which allows constructing, from a given iterative method with order of convergence p, efficient iterative methods with order of convergence at least $pq+1$ ($q \leq p-1$, $q \in \mathbb{N}$).

1. Introduction

Let

$$f(x) = 0 \tag{1}$$

be a nonlinear equation, where $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is sufficiently differentiable in a neighborhood of a simple root α of Eq. (1). If we are interested in approximating the root α, we can do it by means of an iterative method of the form

$$x_{n+1} = \phi(x_n), \quad n \geq 0, \tag{2}$$

provided that the starting point x_0 is given. Two important features determine the choice of iterative method of type (2): the total number of iterations and the computational cost. The former is measured by the order of convergence of (2) and the latter by the necessary number of evaluations of the scalar function f and its derivatives at each step of applying (2). In the scalar case, these two features are linked by the efficiency index [1–4], EI, which is defined by $EI = p^{1/r}$, where p is the order of convergence of iteration (2) and r is the number of evaluations of f and its derivatives that are needed per iteration to apply (2). (For scalar equations, it is usually considered that the evaluation of f and its derivatives have a similar computational cost.)

To improve the efficiency of iterative method (2), we can use some modifications of (2) (see [5–8]). A well-known modification that improves the efficiency index of (2) is given by the following multipoint algorithm (see [4], Theorem 8-1, p. 166):

$$\begin{cases} x_0 \quad \text{given}, \\ z_n = \phi(x_n), \quad \text{(order of convergence } p) \\ x_{n+1} = z_n - f(z_n)/f'(x_n), \quad n \geq 0, \end{cases} \tag{3}$$

which has order of convergence $p+1$. A particular case of (3) is the two step Newton method [9]. Other situations can be seen in [10].

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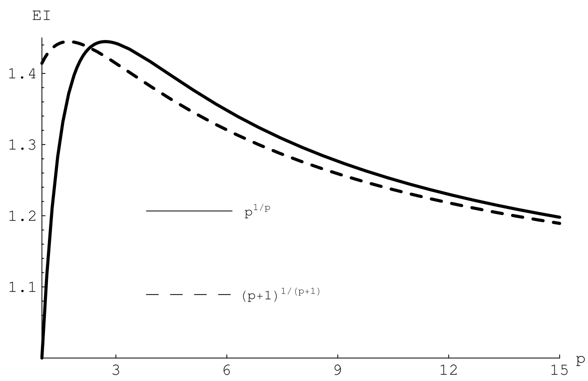

Fig. 1. Efficiency indices of methods (2) and (3).

Since (2) is a one-point iteration with order of convergence p, it is known that we need to evaluate $f, f', f'', \dots, f^{(p-1)}$ at each step and its efficiency index is then $EI = p^{1/p}$, whereas (3) has $EI = (p+1)^{1/(p+1)}$. As we can see in Fig. 1, the efficiency index of (3) is higher than that of (2) if $p = 1, 2$. (Notice that the order of convergence of one-point iterations is always a natural number.)

Following the idea of improving the efficiency of (2), the main goal of this paper is to present some modification of (2) which improves its efficiency when $p \geq 3$. In Section 2, taking into account iterative method (3) and Chebyshev's method, we construct a new modification of iterative method (2) with order of convergence at least $2p+1$ ($p \geq 3$) and higher efficiency index than those of iterative methods (2) and (3). Some numerical tests that corroborate this are also given. Finally, in Section 3, a generalization of the previous construction is presented, where iterative methods with order of convergence at least $pq+1$ ($p \geq q+1$, $q \in \mathbb{N}$) are given.

2. Preliminary analysis

In this section, we present a first modification of Traub's result given by (3), where the second step of (3) is changed by a slight modification of Chebyshev's method. Some numerical results are also included, where new iterative methods are used and commented.

2.1. A modification of Traub's result

From now on, we consider $p \geq 3$ and following the previous original idea of Traub, where a modification of Newton's method (order of convergence two) is considered in the second step of (3), we now consider the following modification of Chebyshev's method (order of convergence three) [8]: $x_{n+1} = z_n - \left(1 + L_f(x_n, z_n)/2\right) u(z_n)$, where $L_f(x_n, z_n) = f''(x_n)u(z_n)/f'(z_n)$ and $u(z_n) = f(z_n)/f'(z_n)$, and we then obtain the following modification of iterative method (2):

$$\begin{cases} x_0 \quad \text{given}, \\ z_n = \phi(x_n), \quad \text{(order of convergence } p) \\ x_{n+1} = z_n - \left(1 + L_f(x_n, z_n)/2\right) u(z_n), \quad n \geq 0. \end{cases} \tag{4}$$

As $p \geq 3$, $f(x_n)$, $f'(x_n)$ and $f''(x_n)$ have been already evaluated in the first step of (4). So, we have to evaluate two more functions, $f(z_n)$ and $f'(z_n)$, when (4) is applied.

To sum up, Traub's modification given in (3) implements a slight modification of a second-order iterative method (Newton's method) which uses one more evaluation of the function, $f(z_n)$, whereas the modification given in (4) implements a slight modification of a third-order iterative method (Chebyshev's method) which uses two more evaluations of functions, $f(z_n)$ and $f'(z_n)$.

Next, we prove that (4) has order of convergence at least $2p+1$. To do this, we consider that the error in the nth iterate of (4) is $e_n = x_n - \alpha$, so that the relation $e_{n+1} = Ce_n^m + \mathcal{O}\left(e_n^{m+1}\right)$ is called the error equation [4,11]. By substituting $e_n = x_n - \alpha$, for all n, in (4) and simplifying, we obtain the error equation for (4). The value of m obtained is then called the order of method (4); see [4,11].

Theorem 2.1. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval D and suppose that f is sufficiently differentiable in D. If f has a simple root at $\alpha \in D$ and x_0 is sufficiently close to α, then iterative method (4) has order of convergence at least $2p+1$ ($p \geq 3$).*

Proof. From Taylor's formulas, we have

$$f(z_n) = f(\alpha + E_n) = f'(\alpha)\left(E_n + A_2E_n^2 + \mathcal{O}(E_n^3)\right),$$
$$f'(z_n) = f'(\alpha + E_n) = f'(\alpha)\left(1 + 2A_2E_n + \mathcal{O}(E_n^2)\right),$$

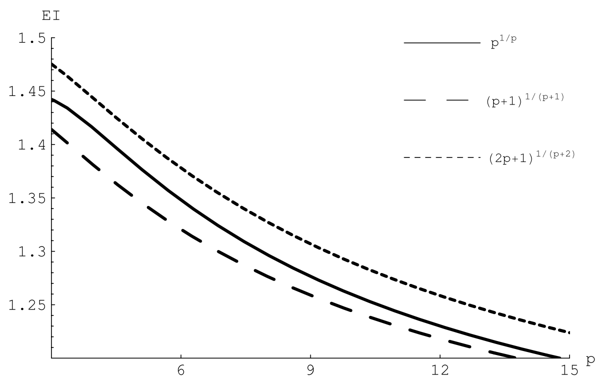

Fig. 2. Efficiency indices of methods (2)–(4).

Table 1

Computational efficiency for multi-precision arithmetic $E = p^{\frac{1-p^{-\lambda}}{r}}$.

λ	(2) with $p = 3$	(2) with $p = 4$	(4) with $p = 7$	(4) with $p = 9$
∞	$3^{1/3} \approx 1.442\ldots$	$4^{1/4} \approx 1.414\ldots$	$7^{1/5} \approx 1.476\ldots$	$9^{1/6} \approx 1.442\ldots$
2	$3^{0.2963} \approx 1.385\ldots$	$4^{0.2344} \approx 1.384\ldots$	$7^{0.1958} \approx 1.464\ldots$	$9^{0.1646} \approx 1.436\ldots$
$\log_2 3$	$3^{0.2749} \approx 1.353\ldots$	$4^{0.2222} \approx 1.361\ldots$	$7^{0.1908} \approx 1.450\ldots$	$9^{0.1615} \approx 1.426\ldots$

where $E_n = z_n - \alpha = C_0 e_n^p + \mathcal{O}\left(e_n^{p+1}\right)$ and $A_2 = f''(\alpha)/(2f'(\alpha))$. If we now develop $1/f'(z_n)$, $u(z_n)$ and $u(z_n)^2$ in powers of E_n, we obtain

$$\frac{1}{f'(z_n)} = \frac{1}{f'(\alpha)}\left(1 - 2A_2E_n + \mathcal{O}(E_n^2)\right),$$
$$u(z_n) = \left(E_n + A_2E_n^2 + \mathcal{O}(E_n^3)\right)\left(1 - 2A_2E_n + \mathcal{O}(E_n^2)\right) = E_n - A_2E_n^2 + \mathcal{O}(E_n^3),$$
$$u(z_n)^2 = E_n^2 + \mathcal{O}(E_n^3).$$

Moreover,

$$f(x_n) = f'(\alpha)\left(e_n + A_2e_n^2 + A_3e_n^3 + \mathcal{O}(e_n^4)\right), \qquad f'(x_n) = \frac{\mathrm{d}f(x_n)}{\mathrm{d}e_n}, \qquad f''(x_n) = \frac{\mathrm{d}f'(x_n)}{\mathrm{d}e_n},$$
$$\frac{u(z_n)^2}{f'(z_n)} = \frac{1}{f'(\alpha)}\left(E_n^2 + \mathcal{O}(E_n^3)\right),$$
$$L_f(x_n, z_n)u(z_n) = \frac{f''(x_n)u(z_n)^2}{f'(z_n)} = 2A_2E_n^2 + 6A_3e_nE_n^2 + \mathcal{O}(e_n^2E_n^2),$$

where $A_3 = f'''(\alpha)/(6f'(\alpha))$.

In consequence, from (4), it follows:

$$e_{n+1} = E_n - u(z_n) - L_f(x_n, z_n)u(z_n)/2 = -3A_3e_nE_n^2 + \mathcal{O}(e_n^2E_n^2),$$

and iterative method (4) has order of convergence at least $2p + 1$. □

Since iterative method (4) uses two evaluations of functions, $f(z_n)$ and $f'(z_n)$, more than iterative method (2), the efficiency index of (4) is $(2p+1)^{1/(p+2)}$, so that efficiency index of (4) is better than the ones of (2) and (3). See Fig. 2.

2.2. Numerical tests

A generalization of the computational efficiency for a multi-precision arithmetic has been given recently in [12]. That is, $E = p^{\frac{1-p^{-\lambda}}{r}}$, where p is the order of the method, r is the number of function evaluations per iteration required by the method and $\lambda = \infty, 2, \log_2 3 \approx 1.585\ldots$. When we use the classical product with double precision, we set $\lambda = \infty$. Otherwise, if we use the classical product with multi-precision (16–230 digits), we put $\lambda = 2$. And finally, when Karatsuba's method (230–3600 digits) is used, we take $\lambda = \log_2 3$. Table 1 shows the values of the efficiency for different methods and different arithmetics.

Now, we consider the following family of iterative methods with order of convergence $q + 1$, $q \in \mathbb{N}$ (see [13,14]):

$$x_{n+1} = \psi_q(x_n), \quad n \geq 0, \text{ where } \psi_q(x) = x + \gamma_1(x)f(x) + \cdots + \gamma_q(x)f(x)^q, \tag{5}$$

Table 2
Test functions, their roots and the initial points.

Function	α	x_0
$f_1(x) = x^3 - 3x^2 + x - 2$	2.893289	2.5
$f_2(x) = x^3 + \cos x - 2$	1.172578	1.5
$f_3(x) = 2\sin x + 1 - x$	2.380061	2.5
$f_4(x) = (x+1)\,\mathrm{e}^{-x} - 1$	0.557146	1.0
$f_5(x) = \mathrm{e}^{x^2+7x-30} - 1$	3.0	2.94
$f_6(x) = \mathrm{e}^{-x} + \cos(x)$	1.746140	1.5
$f_7(x) = x - 3\ln x$	1.857184	2.0

Table 3
Iteration number (Iter) and total number of function evaluations (TNFE).

Method	ψ_2	ψ_3	$\tilde{\psi}_2$	$\tilde{\psi}_3$
$f_1(x)$	9	7	5	5
$f_2(x)$	8	7	5	4
$f_3(x)$	8	6	4	4
$f_4(x)$	8	7	5	4
$f_5(x)$	9	7	5	5
$f_6(x)$	8	6	5	4
$f_7(x)$	8	6	5	4
Iter	58	46	34	30
TNFE	174	184	170	180

where $\gamma_1(x), \gamma_2(x), \ldots, \gamma_q(x)$ are q real differentiable functions such that

$$0 = 1 + \gamma_1(x)f'(x) = \gamma_1'(x) + 2\gamma_2(x)f'(x) = \cdots = \gamma_{q-1}'(x) + q\gamma_q(x)f'(x). \tag{6}$$

Notice that (5) is reduced to Newton's method if $q = 1$ and Chebyshev's method if $q = 2$.

We have tested some preceding methods with seven functions using the Maple computer algebra system. We have computed the root of each function for initial approximation x_0, and we have defined at each step of the iterative method the length of the floating point arithmetic with multi-precision by

$$\texttt{Digits} := m \times [-\log|e_n| + 1],$$

where m is the order of the method which extends the length of the mantissa of the arithmetic, and $[t]$ is the largest integer $\leq t$.

The iterative methods are stopped when $|x_{n+1} - x_n| < 10^{-2900}$ and $|f(x_{n+1})| < 10^{-2900}$. If in the last step of any iterative method it is necessary to increase the number of digits beyond 2900, then this is done. It is necessary to begin with one initial approximation x_0 for applying the methods. Table 2 shows the expression of the functions tested, the initial approximation x_0, which is the same for all the methods, and the root α with seven significant digits.

Table 3 shows the number of necessary iterations for every method and every function for computing the root with the described precision for the methods. Note that $\tilde{\psi}_i$ $(i = 2, 3)$ is the corresponding modification of the iterative function ψ_i $(i = 2, 3)$ given by (4). Observe that the efficiency index of the iterative method of order three is $3^{1/3} \approx 1.442\ldots$ and that of its modification (4) is $7^{1/5} \approx 1.476\ldots$, while the efficiency index of the iterative method of order four is $4^{1/4} \approx 1.414\ldots$ and that of its modification (4) is $9^{1/6} \approx 1.442\ldots$. As far as we know, the results seem to be excellent, the order is increased and the total number of evaluations of functions is lower for the corresponding modifications $\tilde{\psi}_i$ $(i = 2, 3)$ of ψ_i $(i = 2, 3)$ given by (4). Obviously, the best methods are the ones that improve from third and fourth orders (ψ_2 and ψ_3, respectively) to seventh and ninth orders ($\tilde{\psi}_2$ and $\tilde{\psi}_3$, respectively).

As a conclusion, we can say that these new methods of seventh and ninth orders work better than the predecessors of third and fourth orders, respectively (in accordance with the theoretical analysis of the order and the efficiency). We therefore conclude that the new methods presented in this paper are competitive with other recognized efficient equation solvers, such as some known methods of third and fourth orders. Finally, notice that using arithmetics that allow us to dynamically define the number of necessary digits for the computations is relevant.

3. Main result

The next goal is to generalize the previous ideas, which consists of using a modification of the iterative method considered in the second step of the algorithm

$$\begin{cases} x_0 \quad \text{given}, \\ z_n = \phi(x_n), \quad \text{(order of convergence } p) \\ x_{n+1} = \varphi(x_n, z_n), \quad n \geq 0, \end{cases}$$

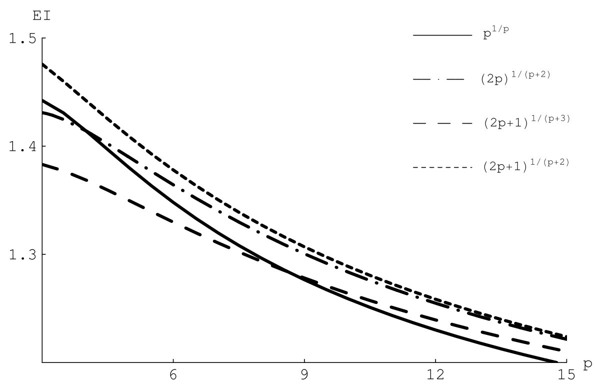

Fig. 3. Efficiency indices of methods (2), (7)–(9).

for obtaining iterative methods with higher efficiency indices than iterative method (2). The main idea is to implement a slight modification of an iterative method with order of convergence $q \geq 4$ ($q \in \mathbb{N}$) in the second step of the algorithm, $x_{n+1} = \varphi(x_n, z_n)$, so that in this step is used $f^{(q-1)}(x_n)$ and, at the most, the following new evaluations of functions: $f(z_n), f'(z_n), \ldots, f^{(q-2)}(z_n)$.

3.1. On the modification of Traub's result

Firstly, to clarify the previous generalization and its later study, we start with iterative method (4) and rewrite it in the following way:

$$\begin{cases} x_0 \quad \text{given}, \\ z_n = \phi(x_n), \quad n \geq 0, \ \text{(method of order } p) \\ y_n = z_n - f(z_n)/f'(z_n) = \mathcal{N}(z_n) = \mathcal{N}(\phi(x_n)), \quad \text{(method of order } 2p) \\ x_{n+1} = y_n - \dfrac{1}{2}\dfrac{f''(x_n)}{f'(z_n)}\left(\dfrac{f(z_n)}{f'(z_n)}\right)^2, \quad n \geq 0. \end{cases} \tag{7}$$

Observe that iteration (7) can be seen as a "modification" of the composition of iterative method (2) with order of convergence p with Newton's method. It is clear that (7) is more efficient than the simple composition of both methods,

$$x_{n+1} = \phi(x_n) - \frac{f(\phi(x_n))}{f'(\phi(x_n))} = \mathcal{N}(\phi(x_n)), \quad n \geq 0, \tag{8}$$

since its order of convergence is $2p+1$ and the number of evaluations of functions is $p+2$, whereas the order of convergence of the composition is $2p$ and the number of evaluations of functions is $p+2$. See Fig. 3.

On the other hand, if the composition of both methods is modified by classical Traub's result [4], then the following iterative method is obtained:

$$\begin{cases} x_0 \quad \text{given}, \\ z_n = \phi(x_n), \quad n \geq 0, \ \text{(method of order } p) \\ y_n = z_n - f(z_n)/f'(z_n) = \mathcal{N}(z_n) = \mathcal{N}(\phi(x_n)), \quad \text{(method of order } 2p) \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(x_n)} = y_n - \dfrac{f(\mathcal{N}(\phi(x_n)))}{f'(x_n)}, \quad n \geq 0. \end{cases} \tag{9}$$

The order of convergence of this method is also $2p + 1$ and the number of evaluations of functions is $p + 3$, so that this modification is less efficient than the one given by (7). See again Fig. 3.

Therefore, the "modification" of Traub's result given by (7) is more efficient than the two natural alternatives: the composition of the two methods and the classical Traub's modification given by (9).

Following the idea of the modification of Traub's result, we can think of generalizing this procedure from (7). Firstly, we observe the following. To construct (7), we have considered the third-order method of Chebyshev written as the second-order method of Newton plus a "remainder",

$$t_{n+1} = \mathcal{C}(t_n) = \mathcal{N}(t_n) - \frac{1}{2}\frac{f''(t_n)}{f'(t_n)}\left(\frac{f(t_n)}{f'(t_n)}\right)^2 \quad n \geq 0,$$

and taking into account that $p \geq 3$ and $f''(x_n)$ has been already evaluated to obtain z_n in (7), we avoid doing more evaluations, which do not concern the composition $\mathcal{N}(\phi)$, where $f(z_n)$ and $f'(z_n)$ have been evaluated, and the highest-order derivative which appears in the "remainder" $-\frac{1}{2}\frac{f''(x_n)}{f'(z_n)}\left(\frac{f(z_n)}{f'(z_n)}\right)^2$ is evaluated in x_n, which has been previously evaluated. Then, from this idea, we consider the following known fourth-order method [13], which can be written as Chebyshev's method (method of order three) plus a "remainder"; i.e.:

$$t_{n+1} = \mathcal{C}(t_n) - \frac{3f''(t_n)^2 - f'''(t_n)f'(t_n)}{6f'(t_n)^2}\left(\frac{f(t_n)}{f'(t_n)}\right)^3 = \psi_3(t_n), \quad n \geq 0. \tag{10}$$

So, in this case, if $p \geq 4$, we construct the following iterative method:

$$\begin{cases} x_0 \quad \text{given}, \\ z_n = \phi(x_n), \quad n \geq 0, \ \text{(method of order } p) \\ y_n = z_n - \dfrac{f(z_n)}{f'(z_n)} - \dfrac{f''(z_n)f(z_n)^2}{2f'(z_n)^3} = \mathcal{C}(z_n) = \mathcal{C}(\phi(x_n)), \quad \text{(method of order } 3p) \\ x_{n+1} = y_n - \dfrac{3f''(z_n)^2 - f'''(x_n)f'(z_n)}{6f'(z_n)^2}\left(\dfrac{f(z_n)}{f'(z_n)}\right)^3, \quad n \geq 0. \end{cases} \tag{11}$$

Note that the values $f(z_n), f'(z_n)$ and $f''(z_n)$ which appear in the "remainder" have been evaluated to define y_n and the value $f'''(x_n)$ has been evaluated to define z_n, since $z_n = \phi(x_n)$, $n \geq 0$, is a method of order $p \geq 4$ and $f'''(x_n)$ has had to be evaluated. Observe that (11) needs $p+3$ evaluations of functions. Next, we prove that method (11) has order of convergence at least $3p + 1$.

Lemma 3.1. *Let (8). Then, following Traub's notation [4], we have*

$$\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^q = (x-\alpha)^{pq}\left(C_\phi(x) - C_{\mathcal{N}(\phi)}(x)(x-\alpha)^p\right)^q, \tag{12}$$

where $C_\phi(\alpha) = \frac{\phi^{(p)}(\alpha)}{p!}$ and $C_{\mathcal{N}(\phi)}(\alpha) = \frac{f''(\alpha)}{2f'(\alpha)}\left(\frac{\phi^{(p)}(\alpha)}{p!}\right)^2$ are the asymptotic error constants of methods (2) and (8), respectively, and α is a solution of the equation $f(x) = 0$.

Proof. Since methods (2) and (8) have respectively orders of convergence p and $2p$, we have:

$$\phi(x) = \alpha + C_\phi(x)(x-\alpha)^p,$$
$$\mathcal{N}(\phi(x)) = \alpha + C_{\mathcal{N}(\phi)}(x)(x-\alpha)^{2p}.$$

Therefore,

$$\phi(x) - \mathcal{N}(\phi(x)) = (x-\alpha)^p\left(C_\phi(x) - C_{\mathcal{N}(\phi)}(x)(x-\alpha)^p\right)$$

and Eq. (12) follows immediately. □

Theorem 3.2. *Iterative method (11) has order of convergence at least $3p + 1$.*

Proof. Firstly, we write (11) as $x_{n+1} = \varphi(x_n)$, $n \geq 0$, where

$$\varphi(x) = \mathcal{C}(\phi(x)) - \frac{3f''(\phi(x))^2 - f'''(x)f'(\phi(x))}{6f'(\phi(x))^2}\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^3.$$

Secondly, from Schröder's characterization [14], we have to prove:

$$\varphi(\alpha) = \alpha, \qquad \varphi^{(k)}(\alpha) = 0, \quad \text{for } k = 1, 2, \ldots, 3p, \qquad \varphi^{(3p+1)}(\alpha) \neq 0.$$

To do this, we now write the iteration function $\varphi(x)$ as

$$\varphi(x) = \psi_3(\phi(x)) + R_3(x), \quad \text{where } R_3(x) = \frac{(-1)^3}{3!}\frac{f'''(\phi(x)) - f'''(x)}{f'(\phi(x))}\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^3$$

and ψ_3 is defined in (10).

Observe that $\psi_3(\phi(x))$ is the iteration function of a method of order $4p$ (the composition of (10) with (2)). Then,

$$\psi_3(\phi(\alpha)) = \alpha, \qquad \left.\frac{d^k\psi_3(\phi(x))}{dx^k}\right|_{x=\alpha} = 0 \quad (k = 1, 2, \ldots, 4p-1), \qquad \left.\frac{d^{4p}\psi_3(\phi(x))}{dx^{4p}}\right|_{x=\alpha} \neq 0.$$

Therefore, we have to study the derivatives of the product

$$R_3(x) = T(x)\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^3, \quad \text{where } T(x) = \frac{(-1)^3}{3!}\frac{f'''(\phi(x)) - f'''(x)}{f'(\phi(x))},$$

since $\varphi(\alpha)=\alpha$. So, from Leibniz's formula, it follows that

$$R_3^{(k)}(x)=\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(T(x)\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^3\right)=\sum_{i=0}^{k}\binom{k}{i}T^{(i)}(x)\frac{\mathrm{d}^{k-i}}{\mathrm{d}x^{k-i}}\left(\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^3\right).$$

By Lemma 3.1, it is clear that

$$\frac{\mathrm{d}^{k-i}}{\mathrm{d}x^{k-i}}\left(\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^p\right)\Bigg|_{x=\alpha}=0,\quad k-i=1,2,\ldots,3p-1,$$

and consequently, $R_3^{(k)}(\alpha)=0$, $k=1,2,\ldots,3p-1$. Moreover, since $T(\alpha)=0$,

$$R_3^{(3p)}(\alpha)=T(\alpha)(3p)!\left(C_\phi(\alpha)-C_{\mathcal{N}(\phi)}(\alpha)(\alpha-\alpha)^p\right)^3=0.$$

Finally,

$$\begin{aligned}R_3^{(3p+1)}(\alpha)&=\sum_{i=0}^{3p+1}\binom{3p+1}{i}\left(T^{(i)}(x)\frac{\mathrm{d}^{3p+1-i}}{\mathrm{d}x^{3p+1-i}}\left(\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^3\right)\Bigg|_{x=\alpha}\right)\\&=\binom{3p+1}{1}T'(\alpha)(3p)!C_\phi(\alpha)^3=(3p+1)!\left(\frac{\phi^{(p)}(\alpha)}{p!}\right)^3\frac{f''''(\alpha)}{6f'(\alpha)},\end{aligned}$$

which in general is not zero. □

3.2. A generalization of the previous modification of Traub's result

To generalize the previous result, we consider the iterative method given by (5) and rewrite it as

$$x_{n+1}=\psi_q(x_n),\quad n\geq 0,\ \text{where}\ \psi_q(x)=\psi_{q-1}(x)+\gamma_q(x)f(x)^q.$$

Note that $\psi_1(x)=\mathcal{N}(x)$ and $\psi_2(x)=\mathcal{C}(x)$, so that (7) and (11) can also be written respectively as

$$\begin{aligned}&x_{n+1}=\psi_1(\phi(x_n))+\bar\gamma_2(x_n,\phi(x_n))f(\phi(x_n))^2,\quad n\geq 0,\\&x_{n+1}=\psi_2(\phi(x_n))+\bar\gamma_3(x_n,\phi(x_n))f(\phi(x_n))^3,\quad n\geq 0,\end{aligned}$$

where $\bar\gamma_k(x_n,\phi(x_n))$ coincides with $\gamma_k(x_n)$, once $f^{(k)}(\phi(x_n))$ has been replaced with $f^{(k)}(x_n)$, for $k=2,3$, so that $f^{(k)}(x_n)$ has already been computed, since it has been used in method (2), whose order of convergence is $p\geq k+1$.

Following this interesting idea, we can then generalize the previous situation. To do this, we suppose that $p\geq q+1$, $q\in\mathbb{N}$, so that $f^{(q)}(x_n)$ is evaluated to define the method of order of convergence p given by (2), since it uses $f(x_n),f'(x_n),\ldots,f^{(p-1)}(x_n)$. In this case, we define

$$\begin{cases}x_0\quad\text{given},\\ z_n=\phi(x_n),\quad n\geq 0,\ \text{(method of order }p)\\ y_n=\psi_{q-1}(z_n)=\psi_{q-1}(\phi(x_n)),\quad\text{(method of order }pq)\\ x_{n+1}=y_n+\bar\gamma_q(x_n,z_n)f(z_n)^q,\end{cases}\tag{13}$$

where $\bar\gamma_q(x_n,z_n)$ coincides with $\gamma_q(z_n)$, once $f^{(q)}(z_n)$ has been replaced with $f^{(q)}(x_n)$.

Our next goal is to prove that method (13) has order of convergence at least $pq+1$ (with $p+q$ evaluations of functions). To see this, we follow in a way similar to the above for method (11). So, we first write (13) as

$$\begin{aligned}x_{n+1}&=\psi_{q-1}(\phi(x_n))+\bar\gamma_q(x_n,\phi(x_n))f(\phi(x_n))^q\\&=\psi_q(\phi(x_n))+\left(\bar\gamma_q(x_n,\phi(x_n))-\gamma_q(\phi(x_n))\right)f(\phi(x_n))^q\\&=\psi_q(\phi(x_n))+R_q(x_n),\end{aligned}$$

where $R_q(x_n)$ is a "remainder" that has to be written in a suitable form (observe how method (11) is written in the proof of Theorem 3.2). For this, we need the following functions (recursive definition): Let $f:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ such that $f\in C^{(m)}(I)$ and $H_k:C^{(m)}(I)\times\overset{k}{\cdots}\times C^{(m)}(I)\longrightarrow\mathbb{R}$, $m\geq k$, such that

$$\begin{cases}H_1(f'(t))=0,\\ H_k\left(f'(t),f''(t),\ldots,f^{(k)}(t)\right)=\dfrac{f^{(k)}(t)}{f'(t)}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{f'(t)^{k+1}}\right)+\dfrac{\frac{\mathrm{d}}{\mathrm{d}t}\left(H_{k-1}\left(f'(t),f''(t),\ldots,f^{(k-1)}(t)\right)\right)}{f'(t)},\\ k=2,3,\ldots,q-1.\end{cases}$$

Now, we can define γ_k $(k=1,2,\ldots,q)$, which appear in (5), as in the next lemma.

Lemma 3.3. *If (6) is satisfied, then*

$$\begin{cases} \gamma_1(t) = -1/f'(t), \\ \gamma_k(t) = \dfrac{(-1)^{k-1}}{k!}\left(\dfrac{f^{(k)}(t)}{f'(t)^{k+1}} + H_{k-1}\left(f'(t), f''(t), \ldots, f^{(k-1)}(t)\right)\right), \quad \textit{for } k = 2, 3, \ldots, q. \end{cases}$$

Proof. The expression of $\gamma_1(t)$ follows immediately from the condition given. For $\gamma_2(t)$:

$$\gamma_2(t) = -\frac{1}{2}\frac{\gamma_1'(t)}{f'(t)} = -\frac{1}{2}\frac{f''(t)}{f'(t)^3} = \frac{(-1)}{2!}\left(\frac{f''(t)}{f'(t)^3} + H_1(f'(t))\right).$$

If we assume that

$$\gamma_k(t) = \frac{(-1)^{k-1}}{k!}\left(\frac{f^{(k)}(t)}{f'(t)^{k+1}} + H_{k-1}\left(f'(t), f''(t), \ldots, f^{(k-1)}(t)\right)\right), \tag{14}$$

it follows in the same way that

$$\begin{aligned} \gamma_{k+1}(t) &= -\frac{1}{k+1}\frac{\gamma_k'(t)}{f'(t)} \\ &= \frac{(-1)^k}{(k+1)!}\frac{1}{f'(t)}\left(f^{(k)}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{f'(t)^{k+1}}\right) + \frac{f^{(k+1)}(t)}{f'(t)^{k+1}} + \frac{\mathrm{d}}{\mathrm{d}t}\left(H_{k-1}\left(f'(t), f''(t), \ldots, f^{(k-1)}(t)\right)\right)\right) \\ &= \frac{(-1)^k}{(k+1)!}\left(\frac{f^{(k+1)}(t)}{f'(t)^{k+2}} + H_k\left(f'(t), f''(t), \ldots, f^{(k)}(t)\right)\right), \end{aligned}$$

so that (14) is true for all positive integers k by mathematical induction. □

After that, it is easy to define the functions $\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_q$ in terms of the functions $\gamma_1, \gamma_2, \ldots, \gamma_q$ as

$$\begin{cases} \bar{\gamma}_1(s,t) = \gamma_1(t), \\ \bar{\gamma}_k(s,t) = \gamma_k(t) + \dfrac{(-1)^k}{k!}\dfrac{f^{(k)}(t) - f^{(k)}(s)}{f'(t)^{k+1}}, \quad \text{for } k = 2, 3, \ldots, q \end{cases}$$

and consequently,

$$R_k(x) = \frac{(-1)^k}{k!}\left(\frac{f^{(k)}(\phi(x)) - f^{(k)}(x)}{f'(\phi(x))}\right)\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^k, \quad \text{for } k = 2, 3, \ldots, q.$$

Theorem 3.4. *Iterative method (13) has order of convergence at least $pq + 1$.*

Proof. The proof also follows from Schröder's characterization [14]. So, we have to prove:

$$G(\alpha) = \alpha, \qquad G^{(k)}(\alpha) = 0, \quad \text{for } k = 1, 2, \ldots, pq, \qquad G^{(pq+1)}(\alpha) \neq 0,$$

where $G(x) = \psi_{q-1}(\phi(x)) + \bar{\gamma}_q(x, \phi(x)) f(\phi(x))^q$. To do this, we first write $G(x)$ as

$$G(x) = \psi_q(\phi(x)) + R_q(x) \quad \text{and} \quad R_q(x) = \frac{(-1)^q}{q!}\frac{f^{(q)}(\phi(x)) - f^{(q)}(x)}{f'(\phi(x))}\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^q.$$

Now, it is easy to prove that

$$\psi_q(\alpha) = \alpha \quad \text{and} \quad \psi_q^{(k)}(\alpha) = 0, \quad \text{for } k = 1, 2, \ldots, pq+1,$$

and

$$R_q^{(k)}(\alpha) = 0, \quad \text{for } k = 1, 2, \ldots, pq.$$

On the other hand, since

$$R_q^{(pq+1)}(x) = \frac{(-1)^q}{q!}\sum_{i=0}^{pq+1}\binom{pq+1}{i}\frac{\mathrm{d}^i}{\mathrm{d}x^i}\left(\frac{f^{(q)}(\phi(x)) - f^{(q)}(x)}{f'(\phi(x))}\right)\frac{\mathrm{d}^{pq+1-i}}{\mathrm{d}x^{pq+1-i}}\left(\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^q\right),$$

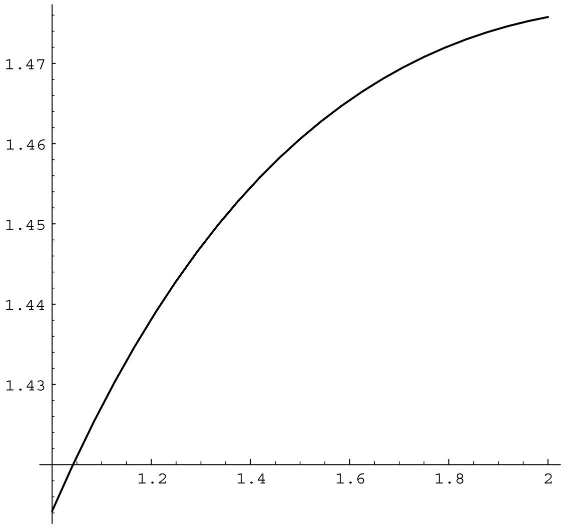

Fig. 4. $EI = (3q+1)^{1/(3+q)}$, $q \le 2$.

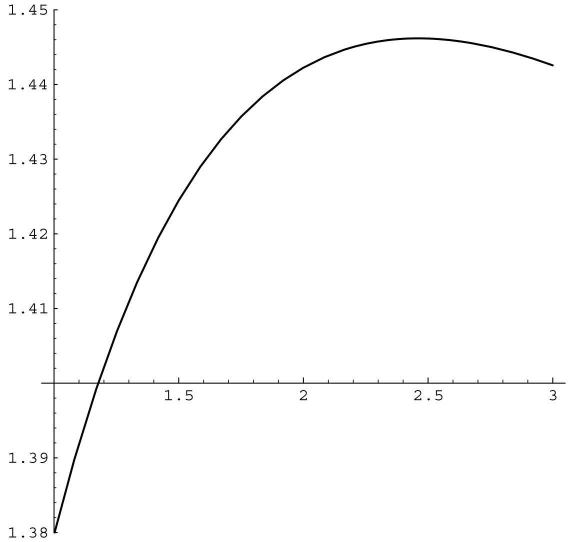

Fig. 5. $EI = (4q+1)^{1/(4+q)}$, $q \le 3$.

we have, from the previous lemma,

$$\begin{aligned} R_q^{(pq+1)}(\alpha) &= \frac{(-1)^q}{q!}\binom{pq+1}{1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f^{(q)}(\phi(x)) - f^{(q)}(x)}{f'(\phi(x))}\right)\frac{\mathrm{d}^{pq}}{\mathrm{d}x^{pq}}\left(\left(\frac{f(\phi(x))}{f'(\phi(x))}\right)^q\right)\bigg|_{x=\alpha} \\ &= \frac{(-1)^{q+1}}{q!}(pq+1)\frac{f^{(q+1)}(\alpha)}{f'(\alpha)}\left((x-\alpha)^{pq}\left(-C_{\mathcal{N}(\phi)}(\alpha)(x-\alpha)^p + C_\phi(\alpha)\right)^q\right)^{(pq)}\bigg|_{x=\alpha} \\ &= \frac{(-1)^{q+1}}{q!}(pq+1)!\frac{f^{(q+1)}(\alpha)}{f'(\alpha)}C_\phi(\alpha)^q. \end{aligned}$$

In consequence,

$$G^{(pq+1)}(\alpha) = (-1)^{q+1}\frac{(pq+1)!}{q!}\frac{f^{(q+1)}(\alpha)}{f'(\alpha)}\left(\frac{\phi^{(p)}(\alpha)}{p!}\right)^q,$$

which in general is not zero and the theorem then follows. □

3.3. Numerical tests

Note that with generalization (13), we can optimize in the following sense the numerical tests presented above. Firstly, we denote iterative method (13) as $x_{n+1} = \Phi_{p,q}(x_n)$, $n \geq 0$. According to the assumption above, if we consider that iterative method (2) is Chebyshev's method (order of convergence three and $EI = \sqrt[3]{3} = 1.4422\ldots$), then the next two modifications can be obtained from (13): $x_{n+1} = \Phi_{3,q}(x_n)$, $n \geq 0$, with $q = 1, 2$. Observe that if $q = 2$, then the last iterative method coincides with that corresponding to the iterate function $\tilde{\psi}_2$ given previously. As the efficiency indices of the two modifications are $\sqrt{2} = 1.4142\ldots$ ($q = 1$) and $\sqrt[5]{7} = 1.4757\ldots$ ($q = 2$), it is clear that the modification with $q = 2$ is more efficient for solving nonlinear scalar equations than that with $q = 1$. See Fig. 4.

After that, we consider that (2) is the iterative method with order of convergence four that is deduced from (5) (and $EI = \sqrt[4]{4} = 1.4142\ldots$). Then, three modifications can be obtained from (13): $x_{n+1} = \Phi_{4,q}(x_n)$, $n \geq 0$, with $q = 1, 2, 3$. Note that if $q = 2$, then the last iterative method coincides with that corresponding to the iterate function $\tilde{\psi}_3$ given previously. The efficiency indices of the three modifications are $\sqrt[5]{5} = 1.3797\ldots$ ($q = 1$), $\sqrt[3]{3} = 1.4422\ldots$ ($q = 2$) and $\sqrt[7]{13} = 1.4425\ldots$ ($q = 3$), we can say that the third modification ($q = 3$) is the best choice for solving nonlinear scalar equations. See Fig. 5.

Finally, following an analogous analysis, we could obtain the most efficient modification given by (13) once any pth order iterative method given by (2) is fixed.

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