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Dynamics of a higher-order family of iterative methods

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ABSTRACT

We study the dynamics of a higher-order family of iterative methods for solving non-linear equations. We show that these iterative root-finding methods are generally convergent when extracting radicals. We examine the Julia sets of these methods with particular polynomials. The examination takes place in the complex plane.

1. Introduction

One of the most important applications of an iterative fixed-point method is the search for roots of a non-linear equation

$$f(x) = 0. \tag{1}$$

An iterative method starts from an initial guess x_0 which is subsequently improved by means of an iteration function $\Phi(x)$, that is, $x_{n+1} = \Phi(x_n)$. Conditions are imposed on x_0 (and, eventually, on f, or Φ, or both) to ensure the convergence of the sequence of iterates $(x_n)_{n\geq 0}$ to a solution ζ of the equation $f(x) = 0$ and establish the order of convergence of the iterative method defined by Φ.

The processes most used for solving Eq. (1) are iterative methods. Among these, the most well-known is the classical Newton iterative method given by

$$x_{n+1} = N_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

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where x_0 is given and $n \in \mathbb{N}$. Newton's method is the iteration of the Newton map $N_f(x) = x - f(x)/f'(x)$. The importance of Newton's method comes from the following fact: if $\xi \in \mathbb{C}$ is a root of multiplicity $m \geq 1$, then $N_f(\xi) = \xi$ and $N_f'(\xi) = 1 - 1/m$. Hence, in neighborhoods of simple roots, that is, roots with multiplicity $m = 1$, Newton's method converges quadratically, and if $m > 1$, we only have linear convergence. Conversely, every finite fixed point of N_f is attracting and a root of f. In fact, this property characterizes Newton's map.

In order to increase the order of convergence, in [9], a family of Newton type methods is introduced as a characterization of iterative methods of R-order at least 3 as follows. Let $\Omega \subset \mathbb{C}$ be an open set, and let $f : \Omega \to \mathbb{C}$ be an analytic complex function; then the following family of iterative methods is considered:

$$\begin{cases} \text{given } x_0 \in \Omega, \\ x_{n+1} = x_n - \left(1 + \dfrac{1}{2}L_f(x_n) + \displaystyle\sum_{k\geq 2} A_k L_f(x_n)^k\right) f'(x_n)^{-1} f(x_n), \quad \{A_k\} \subset \mathbb{R}^+, \end{cases} \tag{3}$$

provided that $f'(x_n) \neq 0$ at each step, $\{A_k\}_{k\geq 0}$ is a real sequence such that $\sum_{k\geq 2} A_k t^k < +\infty$ for $|t| < r$, and condition $|L_f(x_n)| < r$ is required for the valid definition of (3). On the other hand, L_f is known as the degree of logarithmic convexity [11] and it is given by the expression $L_f(z) = f(z)f''(z)/f'(z)^2$. Notice that Newton's map satisfies that $L_f(z) = N_f'(z)$.

For a more detailed study and properties of this family of iterative methods, see [9]. Note that family (3) includes the Newton method and the well-known one-point iterative processes of R-order of convergence 3, Chebyshev's method, Halley's method, and the super-Halley method, among others (see [1,2,9]).

Family (3) may be written in a more general and compact form. Indeed, we define the iterative function

$$\begin{cases} G_f(x_n) = x_n + (N_f(x_n) - x_n)H(L_f(x_n)), \\ H(x) = 1 + \dfrac{1}{2}x + \displaystyle\sum_{k\geq 2} A_k x^k, \quad \{A_k\} \subset \mathbb{R}^+, \end{cases} \tag{4}$$

provided that $f'(x_n) \neq 0$ at each step, $\{A_k\}_{k\geq 0}$ is a real sequence such that $\sum_{k\geq 2} A_k t^k < +\infty$ for $|t| < r$, and condition $|L_f(x_n)| < r$ is satisfied. Using this notation, we consider $x_{n+1} = G_f(x_n)$ where x_0 is given and $n \in \mathbb{N}$. It is not difficult to see that every root of f is a finite fixed point of G_f. We will prove that the roots of f with multiplicity $m \geq 1$ are (super)attracting fixed points for each element of the family of iterative methods under study. For the study of semi-local convergence and properties of family (4), see for example [9].

In this paper, we give the local characterization of the convergence of family (4) in terms of the derivatives of the iterative function G_f at the fixed points corresponding to the roots of Eq. (1), where f is an analytic complex function. In addition, we study the general convergence of a family of iterative methods obtained by truncation of the power series defining function H in (4) when extracting radicals, because in this situation the iterative method obtained is given by a rational map of the extended complex plane into itself. Note that we can also write $z_n = G_f^{\circ n}(z_0)$, where $G_f^{\circ n}$ stands for the composition of G_f with itself n times. Thus studying the family of iterative methods (4) is equivalent to studying the map G_f as a dynamical system.

2. Preliminaries

We are interested in understanding convergence properties of the family of root-finding iterative methods given by (4) when applying it to a complex polynomial. The local convergence of an iterative method is determined by the derivative of the associated meromorphic function at the fixed points which correspond to the roots of the polynomial.

Before presenting our results for the iterative methods that we are interested in, we recall basic notions as regards the iteration of a map. Let F be a meromorphic function on a domain D of the complex plane $\mathbb{C}$. We say that ξ is a *fixed point* of F if $F(\xi) = \xi$. The *multiplier* of the fixed point ξ of F is the number $\lambda(\xi) = |F'(\xi)|$. We say that a fixed point ξ of F is, respectively, *attracting, repelling,* or *indifferent* if its multiplier $\lambda = |F'(\xi)|$ is less than, greater than, or equal to 1. We say that ξ is a super-attracting fixed point if $\lambda = 0$.

Let ζ be a (super)attracting fixed point of F. Its *basin of attraction* or *convergence region* is the set

$$B(\zeta) = \{z \in \overline{\mathbb{C}} : F^{\circ n}(z) \to \zeta \text{ as } n \to \infty\}$$

and its *immediate basin of attraction*, denoted as $B^*(\zeta)$, is the connected component of $B(\zeta)$ containing ζ.

For $z \in \overline{\mathbb{C}}$, we define its *orbit* as the set

$$\mathrm{orb}(z) = \{z, F(z), F^{\circ 2}(z), \ldots, F^{\circ k}(z), \ldots\}.$$

A *cycle* of length n is an orbit of cardinality exactly equal to n. In the case where $\mathrm{orb}(z_0)$ is an n-cycle, its multiplier is defined as $\lambda(\mathrm{orb}(z_0)) = |(F^{\circ n})'(z_j)|$ for any $z_j \in \mathrm{orb}(z_0)$. By the chain rule, this value does not depend on a chosen point on the cycle. An n-cycle $\mathrm{orb}(z_0)$ is said to be attracting, repelling, or indifferent if z_0 as a fixed point of $F^{\circ n}$ is, respectively, attracting, repelling, or indifferent. An n-cycle $\mathrm{orb}(z_0)$ is said to be a *super-attracting cycle* of F if z_0 as a fixed point of $F^{\circ n}$ is super-attracting. In the case where $\mathrm{orb}(z_0)$ is an attracting n-cycle of F, the *basin of attraction* of $\mathrm{orb}(z_0)$ is the set $B(\mathrm{orb}(z_0)) = \cup_{j=0}^{n-1} F^{\circ j}(B(z_0))$, where $B(z_0)$ is the basin of attraction of z_0 as a fixed point of $F^{\circ n}$.

The concept of normal families partitions the complex plane into completely invariant sets. Let $D \subset \mathbb{C}$ be a domain, and let $g : D \longrightarrow \mathbb{C}$ be an analytic map. We say that a point $z \in \mathbb{C}$ is in the *Fatou set* $\mathcal{F}(g)$ if the sequence of iterates $\{g^{\circ n} : n \in \mathbb{N}\}$ forms a normal family there, or in other words $z \in \mathcal{F}(g)$ if it has a neighborhood on which the sequence of iterates has a uniformly convergent subsequence. Otherwise, z is in the Julia set $\mathcal{J}(g)$.

It is well-known that if F is a meromorphic function, then the *Julia set* is the closure of the set of repelling cycles. Another well-known fact is as follows. If F has an attracting fixed point ζ, then the basin of attraction $B(\zeta)$ is contained in the Fatou set. For a comprehensive review of the theory of iteration of meromorphic functions see [3] and for rational maps see [4,14,16].

From the discussion above, we see that an important part of the convergence properties of our iterative root-finding processes is determined by the values the derivative of the associated meromorphic function at fixed points which correspond to the roots of the polynomial that we are considering.

3. Main results

When we study the convergence regions of an iterative root-finding method, we want to give a description for a broad class of functions. For example, if a function g is obtained from another function f, through a change of variables, it is desirable that the convergence regions of both maps be essentially the same, except for a change of coordinates. This is the content of the following.

Theorem 1 (*Scaling Theorem*)**.** *Let f be an analytic function on the Riemann sphere, and let $A(z) = \alpha z + \beta$, with $\alpha \neq 0$, be an affine map. If $g(z) = \lambda(f \circ A)(z)$, where $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$, then $A \circ G_g \circ A^{-1}(z) = G_f(z)$. In other words, G_f is analytically conjugated to G_g by A.*

Proof. We may assume that $g(z) = f \circ A(z)$ (that is, $\lambda = 1$). According to the Scaling Theorem for Newton's method (see [17]), and taking into account that $L_f(z) = N_f'(z)$, we have that $\alpha f'(A(z)) = g'(z)$, $A(N_g(z)) = N_f(A(z))$ and $N_g'(z) = N_f'(A(z))$, i.e., $L_g = L_f \circ A$. Hence,

$$\begin{aligned}
G_f(A(z)) &= A(z) - H(L_f(A(z)))\frac{f(A(z))}{f'(A(z))} \\
&= A(z) - \alpha H(L_g(z))\frac{g(z)}{g'(z)} \\
&= \alpha\left(z - H(L_g(z))\frac{g(z)}{g'(z)}\right) + \beta \\
&= A(G_g(z)).
\end{aligned}$$

If $g(z) = \lambda f(z)$, a straightforward calculation yields $N_g = N_f$, which completes the proof. $\square$

In order to describe the convergence regions of an iterative root-finding method, we must first describe the character, as attractor, repeller or indifferent, of the fixed points corresponding to the roots that we are trying to approximate. It is not difficult to see that for a polynomial p and for any finite fixed point ξ, we have that $N_p'(z) = L_p(z) = (k-1)/k$, where k is the multiplicity of ξ as a zero of p, and $N_p'(\infty) = d/(d-1)$, where d is the degree of p. For the converse, the following result is well-known (see [20]).

Theorem 2. *A rational function $R : \mathbb{C} \longrightarrow \overline{\mathbb{C}}$ of degree $d \geq 2$ is the Newton map of a polynomial of degree at least 2 if and only if the point at infinity is the unique repelling fixed point and for all other fixed points $\alpha_1, \dots, \alpha_d \in \mathbb{C}$, there exists a number $n_j \in \mathbb{N}$ such that $R'(\alpha_j) = (n_j - 1)/n_j < 1$.*

This result characterizes the rational functions which underlie Newton's method for a polynomial in terms of multipliers of the fixed points of the rational function.

We note that this result was also proved by Heat in 1987 and by Nishisama and Fujimura in 1992. On the other hand, Buff and Henriksen (see [5]) extended this result to the König root-finding algorithms applied to a polynomial with simple roots; also Rückert and Schleicher extended this result to entire functions (see [18]).

For the family of algorithms under study, we have the following result, which is the first step in the direction of the characterization of these algorithms.

Theorem 3. *Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ be an analytic complex function and α a root of f with multiplicity n and let $G_f : \Omega \to \overline{\mathbb{C}}$ be the map given in (4). Then α is a (super)attracting fixed point of G_f with multiplier*

$$\lambda(\alpha) = 1 - \frac{1}{n} H\left(\frac{n-1}{n}\right). \tag{5}$$

Moreover, if f is a polynomial of degree $d \geq 2$ ($\Omega = \mathbb{C}$ in this case), then G_f has a repelling fixed point at ∞ with multiplier

$$\lambda(\infty) = \frac{1}{1 - \frac{1}{d} H\left(\frac{d-1}{d}\right)}. \tag{6}$$

In particular, if $H((n-1)/n) < 2n$ for all $n \geq 1$, then the roots of f are attracting fixed points of G_f and ∞ is a repelling fixed point of G_f.

Proof. Since $L_f(z) = N_f'(z)$, where N_f is Newton's method associated with f, we have:

(a) If f has a zero α of multiplicity $n \geq 1$, then it is well-known that α is a (super)attracting fixed point of Newton's method with multiplier $L_f(\alpha) = (n-1)/n$, and therefore

$$G_f'(\alpha) = 1 - \frac{1}{n} H\left(\frac{n-1}{n}\right).$$

For $n = 1$ we have $G_f'(\alpha) = 0$, since $H(0) = 1$, i.e., simple roots are super-attracting fixed points of G_f. On the other hand, $|G_f'(\alpha)| < 1$ for $n \geq 2$, and then the multiple roots are attracting fixed points of G_f.

(b) Using the coordinate change $w \mapsto 1/w$ defined on a neighborhood of the point $z = \infty$, we see that

$$\lambda(\infty) = G_f'(\infty) = \frac{1}{1 - \frac{1}{d} H\left(\frac{d-1}{d}\right)}.$$

It is easy to check that the multiplier $\lambda(\infty)$ is larger than 1.

Moreover, it is clear that if $H((n-1)/n) < 2n$ for all $n \geq 1$, then the roots of f are attracting fixed points of G_f and ∞ is a repelling fixed point of G_f. □

Note that the simple roots of f are critical points of G_f. On the other hand, there may exist critical points which are not roots of f. We call these points *free critical points*. They are solutions, other than the roots of f, of the equation

$$1 + H'(L_f(z))L_f'(z)(N_f(z) - z) + H(L_f(z))(L_f(z) - 1) = 0. \tag{7}$$

Notice that these points depend on the chosen method in (4).

Free critical points are important due to the following classical result.

Theorem 4 (*Fatou–Julia*)**.** *Let G be an analytic map. If z_0 is an attracting periodic point, then the immediate basin of attraction $B^*(z_0)$ contains at least one critical point.*

Consequently, the existence of attracting periodic orbits interferes with our search for roots of the equation $f(z) = 0$ when we apply the iterative method G_f. In fact, in this case, the orbit of each free critical point must be computed and its set of limit points determined. If the set of limit points of the orbit of some free critical point is not a root of $f(z)$, then either it is an attracting periodic orbit, or its structure is more complicated. For example, the limit set could be either in the boundary of a Siegel disk, or in the boundary of a Herman ring, or recurrent (see [13]). Note that the existence of attracting periodic orbits interferes with the basins of attraction of the roots of $f(z)$. Further, in some cases there may exist fixed points other than the roots of $f(z)$.

A fixed point ξ of G_f is called *extraneous* if it is not a root of f. Since the unique fixed points of N_f are the roots of f and the point $z = \infty$, it follows from the expression for G_f given by (4) that ξ is an extraneous fixed point when $H(L_f(\xi)) = 0$. On the other hand, the critical points of G_f are the solutions of Eq. (7) other than the roots of f. Using the one-parameter family of cubic polynomials $p_A(z) = z^3 + (A-1)z - A$, Curry et al. [7] show the patterns of non-convergent Newton sequences in the complex domain, that is, they give a description of the set of parameters λ for which the corresponding Newton sequence $N^n_{p_\lambda}(0)$ converges to a (super)attracting periodic orbit. Similar phenomena for the same family of cubic polynomials are observed for both the Schröder and the König iteration functions. (See [22,23].)

4. General convergence for extracting radicals

In this section we show that family (4) for particular coefficients A_k is generally convergent when it is applied for extracting radicals. It is well-known that Newton's method is not generally convergent for polynomials of higher degree in any reasonable sense. To see this, it suffices to find a polynomial f such that N_f has an attracting cycle of order at least 2. For example, Smale [20] constructs the polynomial $f(z) = z^3 - 2z + 2$ for which N_f has the super-attracting cycle $\{0, 1\}$. The same situation will prevail for every polynomial or analytic map g near f. On the other hand, notice that Newton's method is generally convergent for quadratic polynomials. The following theorem was proved independently by Schröder (1870) in [19] and by Cayley (1879) in [6].

Theorem 5 (*Schröder [19]; Cayley [6]*)**.** *Let f be a quadratic polynomial with distinct roots. Then N_f and the map $Q : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ given by $Q(z) = z^2$ are conjugated by the Möbius map $T(z) = \frac{z-\alpha}{z-\beta}$, where α and β are the roots of f. In addition, let L be the bisector of the segment joining the roots α and β of f. Then the attraction region of α (respectively, β) is given by the region of the complex plane determined by L and contains α (respectively, β). Moreover, the behavior of the iterations of N_f for points in L is chaotic. In other words, all points in the complex plane, except those on the bisector L, converge under iteration N_f to the nearest root of f.*

From the work of McMullen [15] concerning the general convergence of iterations of rational functions we know that the family of iterative processes (4) is not generally convergent in the space of all polynomials of fixed degree. However, we will show that the family of the iterative root-finding methods under study is generally convergent for a special class of polynomials, namely the polynomials of the form $q(z) = z^n - c$ where $c \in \mathbb{C}$, that is, when we extract radicals. Moreover, it is proved in [8] that the iterative methods of family (4) are generally convergent for quadratic polynomials.

We first recall some definitions. (See [12,20].)

Definition 1. Let p be a polynomial of degree d greater than or equal to 2. A rational map $T_p : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ of degree $k(d)$ whose coefficients are rational functions of the coefficients of p is an iteration function for the polynomial if each root of p is a (super)attracting fixed point of T_p. If this property holds for all polynomials of degree d, we say that the map $T : p \longrightarrow T_p$ is a purely iterative algorithm.

Definition 2. An iterative function T_p associated with a polynomial p is said to be generally convergent if for almost all $z \in \mathbb{C}$, its orbit converges to a root of p.

We have the following well-known results; see [14].

Theorem 6. *Let R be a rational map. If every critical orbit of R is either finite or converges to an attracting cycle, then the Lebesgue measure of the Julia set of R is zero. Moreover, the iterations by R of any z not in the Julia set of R converge to an attracting cycle of R.*

Theorem 7. *Let T_p be an iteration function associated with a polynomial p. If every critical orbit of T_p is either finite or converges to a root of p under iteration by the map T_p, then T_p is generally convergent.*

The following result is an immediate consequence of Theorem 1 with $\lambda = c$ and $A(z) = c^{-1/n}z$.

It follows from the Scaling Theorem for the family (4) that G_q and G_p are conjugated for the polynomials $p(z) = z^n - 1$ and $q(z) = z^n - c$, where $c \in \mathbb{C}$ with $c \neq 0$. Thus the description of convergence regions of the roots of the polynomial $q(z) = z^n - c$ under iterations by G_q is equivalent to describing the convergence regions of the roots of the polynomial $p(z) = z^n - 1$ under iterations by G_p. Therefore, we consider the polynomial p. To obtain the best order of convergence for the method, we consider the iteration function obtained from the family (4) by truncation, that is,

$$G_{m,f}(z) = z - \left(1 + \frac{1}{2}L_f(z) + \sum_{k=2}^{m-2} A_k L_f(z)^k\right)\frac{f(z)}{f'(z)}, \quad \text{for } A_k \in \mathbb{R}^+ \text{ and } k \geq 2 \tag{8}$$

in other words, all A_j equal zero for all $j \geq m-1$. Note that when we apply the iterative method $G_{n,f}$ to a polynomial f, we obtain a rational map. Next we look for the parameters $A_2, A_3, \ldots, A_{m-2}$ in order to obtain an order of convergence m, with $m \geq 2$. In [10], it is shown that to obtain the best possible order of convergence, the coefficients A_j, with $j = 2, \ldots, m-2$, must be given by

$$A_j = \binom{\frac{1}{n}}{j+1}(-1)^j \frac{n^{j+1}}{(n-1)^j}.$$

Now, from $L_p(z) = \frac{(n-1)(z^n-1)}{nz^n}$, family (8) applied to the polynomial $p(z) = z^n - 1$ is reduced to

$$G_{m,p}(z) = z - \left(\sum_{j=0}^{m-2} \binom{\frac{1}{n}}{j+1}(-1)^j n \frac{(z^n-1)^j}{z^{nj}}\right)\frac{z^n-1}{nz^{n-1}}, \tag{9}$$

and hence it follows (see [10]) that

$$G'_{m,p}(z) = \left(\frac{(m-2)n}{n-1} + 1\right) A_{m-2} L_p(z)^{m-1}. \tag{10}$$

We now show that the iterative function $G_{m,\cdot}$ is generally convergent when extracting radicals.

Theorem 8. *The iterative function $G_{m,p}$, with $m \geq 2$, defined in (9) has n forward invariant Fatou components which are super-attracting and $\mathfrak{m}(\mathcal{J}(G_{m,p})) = 0$, where $\mathfrak{m}$ is the Lebesgue measure on $\mathbb{C}$.*

Proof. Let ξ be a root of $z^n - 1$. Then $G_{m,p}(\xi z) = \xi G_{m,p}(z)$. Furthermore, if $\lim_{k\to\infty} G^k_{m,p}(z) = \theta$, then $\lim_{k\to\infty} G^k_{m,p}(\xi z) = \xi\theta$.

Now, from (10) we have that the critical points of $G_{m,p}$ are the zeros of $L_p(z)$ and its poles. We have that $L_p(z) = 0$ if and only if either $p(z) = 0$ (the roots of p) or $p''(z) = 0$. But $p''(z) = 0$ if and only if $z = 0$. The poles of L_p are the critical points of p. Now $p'(z) = 0$ if and only if $z = 0$. It is not

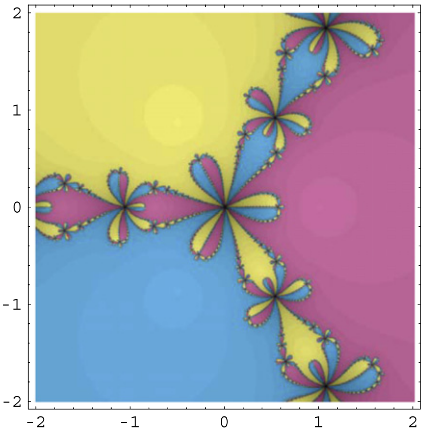

Fig. 1. Method (9) with $n = 3$ and $m = 3$.

difficult to verify that $G_{m,p}(0) = \infty$, and we know that the point $z = \infty$ is a repelling fixed point of $G_{m,p}$. Therefore, $z = 0$ belongs to $J(G_{m,p})$ and from Theorem 6, the Lebesgue measure of $\mathcal{J}(G_{m,p})$ is zero. $\square$

Finally, we show the attraction basins that methods (9) generate when they are applied to extract radicals. The attraction basins clarify the structures of the universal Julia sets associated with the corresponding iterative methods. This allows us to observe graphically the dynamical behavior of the rational maps $G_{m,p}$ for different values of m and n and compare the regions of the convergence of the methods.

We apply methods (9) with different orders of convergence, $m = 3, 4$, to obtain the n-roots of the polynomial $p(z) = z^n - 1$ with $n = 3, 4$, and we paint their attraction basins. For that, we put the dynamically interesting regions into a rectangle $R = [-2, 2] \times [-2, 2] \subset \mathbb{C}$ that contains the roots of p, and use a variation of a Mathematica code due to Varona; see [21]. In practice, we consider a grid of 1024×1024 points in R and we choose these points as z_0. In all the cases, we use a tolerance 10^{-4} and the maximum of 100 iterations. We assign a color to each attraction basin of an n-root and we make the color darker according to the number of iterations needed to reach the n-root. If we do not obtain the desired tolerance with the fixed iterations, we do not continue and we decide that the iterative method starting at z_0 does not converge to any root and assign black color to those points.

As we observe in Figs. 1 and 2, the iterative functions $G_{3,p}$ and $G_{4,p}$ with third and fourth order of convergence, respectively, applied to approximate the roots of the polynomial $p(z) = z^3 - 1$, have three forward invariant Fatou components which are super-attracting where the iterates converge to the corresponding roots.

Similarly, in Figs. 3 and 4 we show the four super-attracting forward Fatou components obtained from $G_{3,p}$ and $G_{4,p}$ when are applied to approximate the roots of the polynomial $p(z) = z^4 - 1$.

Notice that in the pictures above there is a symmetry: they are invariant under a rotation by the angle $2\pi/n$, where n is the degree of the polynomial p.

5. Conclusions

In this paper we have described the character of the fixed points which correspond to the roots of polynomial equations in the complex plane whose solutions we are looking for, using our family of iterative root-finding methods (4). The foregoing describes locally the behavior of the iterations under the corresponding family of methods. In particular, this gives an idea of what the basins of attraction of fixed points corresponding to the roots of a polynomial look like. The fixed point at infinity is of interest since, as it is a repeller, we stay in the finite part of the extended complex plane under iterations of our family applied to a polynomial. However, this does not mean that we have ensured

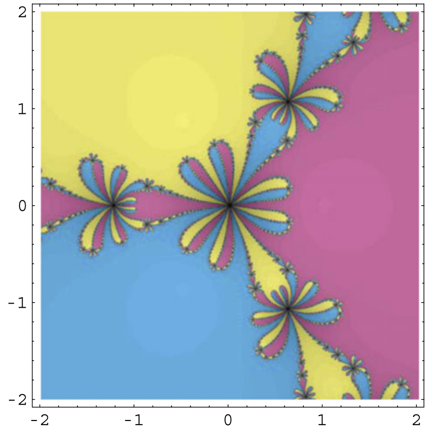

Fig. 2. Method (9) with $n = 3$ and $m = 4$.

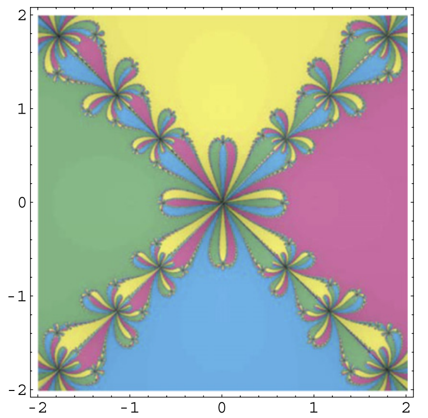

Fig. 3. Method (9) with $n = 4$ and $m = 3$.

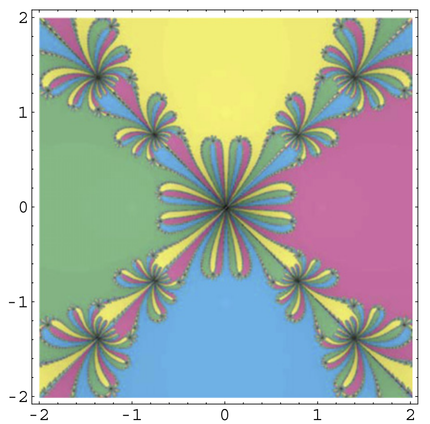

Fig. 4. Method (9) with $n = 4$ and $m = 4$.

the convergence of the iterations to the fixed points which correspond to the roots of a polynomial. As is well-known, there may be either (super)attracting cycles or (super)attracting fixed points which do not correspond to the roots of a polynomial. There may also exist indifferent cycles and/or fixed points for some methods. In addition, we have established the general convergence of methods defined in (9) when they are applied to extracting radicals.

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