THE CONTINUED FRACTION EXPANSION OF CERTAIN PIERCE SERIES

JUAN LUIS VARONA

Abstract. A Pierce series is an alternating sum of the reciprocals of an increasing sequence of positive integers, each one divisible by the previous one. In this short note we study some Pierce series that arise from certain continued fractions. Moreover, we show that the sums of these Pierce series are transcendental numbers.

1. Introduction and main results

Given an eventually increasing sequence of positive integers (x_n) such that $x_n \mid x_{n+1}$ for all n, the sum of the reciprocals is the Engel series

$$\sum_{j=1}^{\infty} \frac{1}{x_j} = \sum_{j=1}^{\infty} \frac{1}{y_1 y_2 \cdots y_j}, \tag{1}$$

where $y_1 = x_1$ and $y_{n+1} = x_{n+1}/x_n$ for $n \geq 1$, and the alternating sum of the reciprocals is the Pierce series

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{x_j} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{y_1 y_2 \cdots y_j}. \tag{2}$$

In the recent papers [3, 4], A.N.W. Hone gives some families of sequences (x_n) that are generated by certain nonlinear recurrences of second order, and such that some continued fraction expansion defined from the (x_n) coincides with the Engel series (1). Additionally, he proves, under some circumstances, that (1) is a transcendental number. We give a similar result for Pierce series, although the structure of the corresponding continued fractions is different.

To generate the sequences (x_n), let us take the initial values $x_0 = x_1 = 1$ and define (x_n) by the recurrence relation

$$x_{n+2} x_n = (1 + \alpha_{n+1} x_{n+1}) x_{n+1}^2, \quad n \geq 0, \tag{3}$$

with α_{n+1} any positive integer. If we take

$$y_n = \frac{x_{n+1}}{x_n}, \qquad z_n = \frac{y_{n+1}}{y_n} = \frac{x_{n+2} x_n}{x_{n+1}^2} = 1 + \alpha_{n+1} x_{n+1}, \tag{4}$$

it is clear that $y_0 = 1$ and $x_2 = y_1 = z_0 = 1 + \alpha_1$; moreover, by induction we can see that $x_n, y_n, z_n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ (use (4) to prove that, if $x_{n+2}, y_{n+1}, z_n \in \mathbb{Z}^+$, then $z_{n+1}, y_{n+2}, x_{n+3} \in \mathbb{Z}^+$).

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We use the standard notation [1, 5, 6, 7, 9] for continued fractions, namely

$$[a_0; a_1, a_2, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_n}}} = \frac{p_n}{q_n},$$

with $a_0 \in \mathbb{Z}$ and $a_j \in \mathbb{Z}^+$ for $j \geq 1$ (we also assume $q_n > 0$), and

$$[a_0; a_1, a_2, \dots] = \lim_{n\to\infty} [a_0; a_1, a_2, \dots, a_n],$$

a limit that always exists.

The convergents p_n/q_n of a continued fraction satisfy the recurrence relation

$$\begin{array}{llll} p_{-1} = 1, & p_0 = a_0, & p_1 = a_0 a_1 + 1, & p_k = a_k p_{k-1} + p_{k-2},\ k \geq 1, \\ q_{-1} = 0, & q_0 = 1, & q_1 = a_1, & q_k = a_k q_{k-1} + q_{k-2},\ k \geq 1. \end{array} \tag{5}$$

Among the large number of well-known properties of continued fractions, let us mention that they satisfy

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}, \qquad k \geq 1.$$

Using this property (written as $p_k/q_k - p_{k-1}/q_{k-1} = (-1)^{k-1}/(q_{k-1}q_k)$) and induction, we obtain

$$[a_0; a_1, a_2, \dots, a_n] = a_0 + \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_2 q_3} + \cdots + \frac{(-1)^{n-1}}{q_{n-1} q_n}. \tag{6}$$

This provides, of course, an alternating series, but it is not a Peirce series because, in general, $q_j \nmid q_{j+2}$ (actually, this is what happens in [2], another paper that studies continued fractions for some alternating series whose sum is a transcendental number). Anyway, the existence of this alternating series closely related to the basic theory of continued fractions is a strong motivation for searching for series of type (2) (Pierce series), perhaps more than series of type (1) (Engel series). Let us also remember that alternating series (with the general term decreasing to zero) have a great interest due to their very good approximation properties: according to Leibniz's theorem, any partial sum approximates the sum of the series with an error bounded by the first omitted term.

The proof of the following theorem can be found in Section 2.

Theorem 1. *Let (x_n) be a sequence generated from the initial values $x_0 = x_1 = 1$ by the recurrence* (3)*. The partial alternating sums of their reciprocals have the continued fraction expansions*

$$S_N := \sum_{j=1}^{N} \frac{(-1)^{j+1}}{x_j} = [a_0; a_1, a_2, \dots, a_{3N-4}] \tag{7}$$

for all $N \geq 2$, where $a_0 = 0$, $a_1 = 1$, $a_2 = x_2 - 1$, $a_3 = \alpha_2 x_2$, and

$$a_{3n+1} = 1, \quad a_{3n+2} = x_{n+1} - 1, \quad a_{3n+3} = \frac{\alpha_{n+2} x_{n+2}}{x_{n+1}} - 1 \in \mathbb{Z}^+, \quad n \geq 1. \tag{8}$$

It is interesting to note that this result is somewhat different to what happens in the case of Engel series (1), where we have a relation of the type

$$\sum_{j=1}^{N} \frac{1}{x_j} = [b_0; b_1, b_2, \dots, b_{2N-2}],$$

and where, instead of (8), only two expressions $b_{2n} = x_n$ and $b_{2n+1} = \alpha_{n+1}x_{n+1}/x_n$ appear. The details can be found in [3].

Finally, taking $N \to \infty$ in Theorem 1 and using Roth's theorem on rational approximation and transcendence, we have the following result whose proof can be seen in Section 3:

Theorem 2. *The infinite sum*

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{x_j} = [a_0; a_1, a_2, \dots, a_n, \dots] \tag{9}$$

(where the coefficients of the continued fraction are as in (8)*) is a transcendental number.*

2. Proof of Theorem 1

Let us take the sequence (a_n) as in (8), whose terms belong to $\mathbb{Z}^+$ for $n > 0$ because we can write $a_{3n+3} = \alpha_{n+2}x_{n+2}/x_{n+1} - 1 = y_{n+1} - 1$.

Now, let us denote $\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n]$. From $a_0 = p_0/q_0$, $a_0 + 1/a_1 = p_1/q_1$, $a_0 + 1/(a_1 + 1/a_2) = p_2/q_2$, and taking into account that $a_0 = 0$, $a_1 = 1$, $a_2 = x_2 - 1$, we easily obtain $p_0 = 0$, $q_0 = 1$, $p_1 = 1$, $q_1 = a_1 = 1$, $p_2 = a_2 = x_2 - 1$ and $q_2 = a_2 + 1 = x_2$. Let us prove now by induction that the denominators q_n satisfy the recurrence relation

$$q_{3n-1} = x_{n+1}, \quad q_{3n} = \frac{x_{n+2}}{x_{n+1}} - x_{n+1} + 1, \quad q_{3n+1} = \frac{x_{n+2}}{x_{n+1}} + 1, \quad n \geq 1. \tag{10}$$

For $n = 1$ we have already seen that $q_2 = x_2$; by using (5) and $(1 + \alpha_2 x_2)x_2 = x_1x_3/x_2$ (with $x_1 = 1$), we get $q_3 = a_3q_2 + q_1 = (-1+1+\alpha_2x_2)x_2 + 1 = x_3/x_2 - x_2 + 1$; and, finally, $q_4 = a_4q_3 + q_2 = 1(x_3/x_2 - x_2 + 1) + x_2 = x_3/x_2 + 1$. Let us now check the induction step. Assuming (10) for n, for $n + 1$ we have

$$\begin{aligned} q_{3n+2} &= a_{3n+2}q_{3n+1} + q_{3n} \\ &= (x_{n+1} - 1)\Big(\frac{x_{n+2}}{x_{n+1}} + 1\Big) + \frac{x_{n+2}}{x_{n+1}} - x_{n+1} + 1 = x_{n+2}, \\ q_{3n+3} &= a_{3n+3}q_{3n+2} + q_{3n+1} \\ &= \Big(\frac{-1 + 1 + \alpha_{n+2}x_{n+2}}{x_{n+1}} - 1\Big)x_{n+2} + \frac{x_{n+2}}{x_{n+1}} + 1 = \frac{x_{n+3}}{x_{n+2}} - x_{n+2} + 1, \\ q_{3n+4} &= a_{3n+4}q_{3n+3} + q_{3n+2} \\ &= 1\Big(\frac{x_{n+3}}{x_{n+2}} - x_{n+2} + 1\Big) + x_{n+2} = \frac{x_{n+3}}{x_{n+2}} + 1. \end{aligned}$$

Now, let us prove (7) by induction. For $N = 2$, (7) becomes

$$S_2 = \frac{1}{x_1} - \frac{1}{x_2} = [a_0; a_1, a_2],$$

which follows easily from the equalities $a_0 = 0$, $a_1 = 1$, $a_2 = x_2 - 1$. For the induction step, let us use (6), which allows us to write

$$[a_0; a_1, a_2, \dots, a_{3N-4}] = a_0 + \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_2 q_3} + \cdots + \frac{(-1)^{3N-5}}{q_{3N-5} q_{3N-4}}$$

and the corresponding expression for $[a_0; a_1, a_2, \dots, a_{3N-1}]$. Then, we only need to check that

$$\frac{1}{x_{N+1}} = \frac{1}{q_{3N-4} q_{3N-3}} - \frac{1}{q_{3N-3} q_{3N-2}} + \frac{1}{q_{3N-2} q_{3N-1}}.$$

According to (10), we have

$$\begin{aligned}
&\frac{1}{q_{3N-4} q_{3N-3}} - \frac{1}{q_{3N-3} q_{3N-2}} + \frac{1}{q_{3N-2} q_{3N-1}} \\
&\quad = \frac{1}{x_{N+1} - x_N^2 + x_N} - \frac{x_N^2}{(x_{N+1} - x_N^2 + x_N)(x_{N+1} + x_N)} \\
&\qquad\quad + \frac{x_N}{(x_{N+1} + x_N) x_{N+1}} \\
&\quad = \frac{x_{N+1}(x_{N+1} + x_N) - x_N^2 x_{N+1} + x_N(x_{N+1} - x_N^2 + x_N)}{(x_{N+1} - x_N^2 + x_N)(x_{N+1} + x_N) x_{N+1}} \\
&\quad = \frac{x_{N+1}(x_{N+1} - x_N^2 + x_N) + x_N(x_{N+1} - x_N^2 + x_N)}{(x_{N+1} - x_N^2 + x_N)(x_{N+1} + x_N) x_{N+1}} = \frac{1}{x_{N+1}},
\end{aligned}$$

and the proof is complete.

3. Proof of Theorem 2

The transcendence of (9) is due to the large rate of growth of the sequence (x_n). The analysis of the increasing of (x_n) is already done in [3, 4], but we repeat it here for the sake of completeness. By definition, $\alpha_n \geq 1$, so that

$$x_{n+1} = (1 + \alpha_n x_n) x_n^2 / x_{n-1} > x_n^3 / x_{n-1} \geq x_n^2, \qquad n \geq 1. \tag{11}$$

This rate of growth is not enough for our purposes, but this result can be easily improved. Let us write (11) as $x_{n-1} \leq x_n^{1/2}$, and use it in $x_{n+1} > x_n^3 / x_{n-1}$. In this way, we obtain

$$x_{n+1} > x_n^3 / x_{n-1} \geq x_n^{2+1/2}, \qquad n \geq 1. \tag{12}$$

Roth's theorem (1955, [8, 1])**.** *Let ξ be an irrational algebraic number. Then, for any arbitrary fixed $\varepsilon > 0$, there are only finitely many rational approximations p/q (we assume $q > 0$) for which*

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}}.$$

Now, Theorem 2 follows immediately from Leibnitz's error estimate for the truncation of alternating series, (12), and Roth's theorem:

$$\left| \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{x_j} - \frac{p_{3n-1}}{q_{3n-1}} \right| = \left| \sum_{j=n+2}^{\infty} \frac{(-1)^{j+1}}{x_j} \right| < \frac{1}{x_{n+2}} < \frac{1}{x_{n+1}^{2+1/2}} = \frac{1}{q_{3n-1}^{2+1/2}}.$$

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Departamento de Matemáticas y Computación, Universidad de La Rioja, 26006 Logroño, Spain

E-mail address: `jvarona@unirioja.es`

URL: `http://www.unirioja.es/cu/jvarona/`