J. London Math. Soc. (2) 98 (2018) 204–222
doi:10.1112/jlms.12129

Garling sequence spaces

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Abstract

The aim of this paper is to introduce and investigate a new class of separable Banach spaces modeled after an example of Garling from 1968. For each $1 \leqslant p < \infty$ and each nonincreasing weight $\mathbf{w} \in c_0 \setminus \ell_1$, we exhibit an ℓ_p-saturated, complementably homogeneous, and uniformly subprojective Banach space $g(\mathbf{w}, p)$. We also show that $g(\mathbf{w}, p)$ admits a unique subsymmetric basis despite the fact that for a wide class of weights it does not admit a symmetric basis. This provides the first known examples of Banach spaces where those two properties coexist.

1. *Introduction, notation, and terminology*

Let $1 \leqslant p < \infty$ and let $\mathbf{w} = (w_n)_{n=1}^\infty$ be a weight, that is, a sequence of positive scalars. Assume that $\mathbf{w}$ is decreasing. Given a sequence of scalars $f = (a_n)_{n=1}^\infty$ we put

$$\|f\|_{g(\mathbf{w},p)} = \sup_{\phi \in \mathcal{O}} \left(\sum_{n=1}^\infty |a_{\phi(n)}|^p w_n \right)^{1/p}$$

where $\mathcal{O}$ denotes the set of increasing functions from $\mathbb{N}$ to $\mathbb{N}$. We consider the normed space

$$g(\mathbf{w}, p) := \{f = (a_n)_{n=1}^\infty \colon \|f\|_{g(\mathbf{w},p)} < \infty\}.$$

A straightforward application of Fatou's lemma gives that $g(\mathbf{w}, p)$ is a Banach space. Imposing the further conditions $\mathbf{w} \in c_0$ and $\mathbf{w} \notin \ell_1$ will prevent us, respectively, from having $g(\mathbf{w}, p) = \ell_p$ or $g(\mathbf{w}, p) = \ell_\infty$. We will assume as well that $\mathbf{w}$ is normalized, that is, $w_1 = 1$. Thus, we put

$$\mathcal{W} := \{(w_n)_{n=1}^\infty \in c_0 \setminus \ell_1 : 1 = w_1 \geqslant w_2 > \cdots w_n \geqslant w_{n+1} \geqslant \cdots > 0\}.$$

The germ of this family of spaces goes back to [7], where Garling showed that for $\mathbf{w} = (n^{-1/2})_{n=1}^\infty$ the canonical unit vectors of $g(\mathbf{w}, 1)$ form a subsymmetric basic sequence that is not symmetric. Indeed, until then these two concepts were believed to be the same. Other Garling sequence spaces have appeared in past literature since then. In [11], Pujara investigated a variation of $g(\mathbf{w}, 2)$ for $\mathbf{w} = (n^{-1/2})_{n=1}^\infty$. He proved that those spaces are uniformly convex, and that their canonical basis is subsymmetric but not symmetric. Dilworth *et al.* [6] studied derived forms of $g(\mathbf{w}, 1)$ for various choices of weights $\mathbf{w}$ that yield spaces in which the canonical basis is 1-greedy and subsymmetric but not symmetric.

Of course, all these spaces are inspired in *sequence Lorentz spaces* defined as

$$d(\mathbf{w}, p) = \{f = (a_n)_{n=1}^\infty \colon \|f\|_{d(\mathbf{w},p)} < \infty\},$$

Received 31 March 2017; published online 13 April 2018.

2010 *Mathematics Subject Classification* 46B25 (primary), 46B45, 46B03 (secondary).

F. Albiac acknowledges the support of the Spanish Ministry for Economy and Competitivity Grants MTM2014-53009-P for *Análisis Vectorial, Multilineal y Aplicaciones*, and MTM2016-76808-P for *Operators, lattices, and structure of Banach spaces*. J. L. Ansorena acknowledges the support of the Spanish Ministry for Economy and Competitivity Grant MTM2014-53009-P for *Análisis Vectorial, Multilineal y Aplicaciones*.

where, if Π is the set of permutations on $\mathbb{N}$,

$$\|f\|_{d(\mathbf{w},p)} = \sup_{\sigma\in\Pi} \left(\sum_{n=1}^{\infty} |a_{\sigma(n)}|^p w_n \right)^{1/p}.$$

The main difference between Lorentz sequence spaces and Garling sequence spaces is that, unlike for $d(\mathbf{w}, p)$, the canonical basis of $g(\mathbf{w}, p)$ is not symmetric. We will tackle this issue in Section 5. The canonical basis is subsymmetric, though. Moreover, while $d(\mathbf{w}, p)$ is known to admit a unique symmetric basis, in Section 4 we prove that $g(\mathbf{w}, p)$ has a unique subsymmetric basis. Along the way to proving the uniqueness of subsymmetric basis, we will show in Section 3 that many structural results of $d(\mathbf{w}, p)$ carry over to $g(\mathbf{w}, p)$. To be precise, we will show that $g(\mathbf{w}, p)$ is complementably homogeneous and uniformly complementably ℓ_p-saturated. As a consequence of the latter, $g(\mathbf{w}, p)$ is uniformly subprojective. Both subprojectivity and complementable homogeneity have been used in the study of the closed ideal structure of the algebra of endomorphisms of a Banach space. Subprojectivity was introduced in [**13**] in order to study the closed operator ideal of strictly singular operators. For instance, if X and Y are Banach spaces, Y is subprojective, and $T \in \mathcal{L}(X, Y)$, then the dual operator T^* is strictly singular only if T is. If furthermore $X = Y$ and is reflexive, the converse holds, that is, if X is reflexive then $T \in \mathcal{L}(X)$ is strictly singular if and only if T^* is (see [**10**] for further discussion of subprojectivity). Complementable homogeneity, meanwhile, has been used in several recent papers (cf., for example, [**5**, **9**, **14**]) to show the existence of a unique maximal ideal in $\mathcal{L}(X)$. In particular, the set of all X-strictly singular operators — those for which the restrictions to subspaces isomorphic to X are never bounded below — acting on a complementably homogeneous Banach space X forms the unique maximal ideal in $\mathcal{L}(X)$ whenever it is closed under addition. By an argument in [**8**, Corollary 5.2] and the second paragraph of the proof of [**8**, Theorem 5.3], it follows that the set of $g(\mathbf{w}, p)$-strictly singular operators forms the unique maximal ideal in $\mathcal{L}(g(\mathbf{w}, p))$.

We close this introductory section by setting the notation and recalling the terminology that will be most heavily used. We will write $\mathbb{F}$ for the real or complex field and denote by $(\mathbf{e}_n)_{n=1}^\infty$ the canonical basis of $\mathbb{F}^{\mathbb{N}}$, that is, $\mathbf{e}_n = (\delta_{k,n})_{k=1}^\infty$, where $\delta_{k,n} = 1$ if $n = k$ and $\delta_{k,n} = 0$ otherwise. On occasion we will need to compare linear combinations of the vectors $(\mathbf{e}_n)_{n=1}^\infty$ with respect to different norms; thus to avoid confusion we may denote by $(\mathbf{d}_n)_{n=1}^\infty$, $(\mathbf{f}_n)_{n=1}^\infty$, and $(\mathbf{g}_n)_{n=1}^\infty$ the vectors $(\mathbf{e}_n)_{n=1}^\infty$ when seen, respectively, inside the spaces $d(\mathbf{w}, p)$, ℓ_p, or $g(\mathbf{w}, p)$. Also for simplicity, when p and $\mathbf{w}$ are clear from context the norms in the spaces $d(\mathbf{w}, p)$ and $g(\mathbf{w}, p)$ will be denoted by $\|\cdot\|_d$ and $\|\cdot\|_g$, respectively. As it is customary, we write c_{00} for the space of all scalar sequences with finitely many nonzero entries. If $(\mathbf{x}_n)_{n=1}^\infty$ is a basis for a Banach space X, and $f = \sum_{n=1}^\infty a_n \mathbf{x}_n \in X$, then the *support* of f is the set $\operatorname{supp}(f) := \{n \in \mathbb{N} : a_n \neq 0\}$ of indices corresponding to its nonzero entries. Given a basic sequence $(\mathbf{z}_n)_{n=1}^\infty$ in X, $\langle \mathbf{z}_n \rangle_{n=1}^\infty$ denotes its linear span and $[\mathbf{z}_n]_{n=1}^\infty$ will be its closed linear span.

Given families of positive real numbers $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$, the symbol $\alpha_i \lesssim \beta_i$ for $i \in I$ means that $\sup_{i\in I} \alpha_i/\beta_i < \infty$, while $\alpha_i \approx \beta_i$ for $i \in I$ means that $\alpha_i \lesssim \beta_i$ and $\beta_i \lesssim \alpha_i$ for $i \in I$. Now suppose $(\mathbf{x}_n)_{n=1}^\infty$ and $(\mathbf{y}_n)_{n=1}^\infty$ are basic sequences in X and Y, respectively. If for some positive C,

$$\left\| \sum_{n=1}^{\infty} a_n \mathbf{x}_n \right\|_X \leqslant C \left\| \sum_{n=1}^{\infty} a_n \mathbf{y}_n \right\|_Y, \quad (a_n)_{n=1}^\infty \in c_{00},$$

we say that $(\mathbf{y}_n)_{n=1}^\infty$ *C-dominates* $(\mathbf{x}_n)_{n=1}^\infty$, and write $(\mathbf{x}_n)_{n=1}^\infty \lesssim_C (\mathbf{y}_n)_{n=1}^\infty$. When the value of the constant is irrelevant we will simply say that $(\mathbf{y}_n)_{n=1}^\infty$ *dominates* $(\mathbf{x}_n)_{n=1}^\infty$, and write $(\mathbf{x}_n)_{n=1}^\infty \lesssim (\mathbf{y}_n)_{n=1}^\infty$. Whenever $(\mathbf{x}_n)_{n=1}^\infty \lesssim_C (\mathbf{y}_n)_{n=1}^\infty$ and $(\mathbf{y}_n)_{n=1}^\infty \lesssim_C (\mathbf{x}_n)_{n=1}^\infty$, we say that $(\mathbf{x}_n)_{n=1}^\infty$ and $(\mathbf{y}_n)_{n=1}^\infty$ are *C-equivalent*, and write $(\mathbf{x}_n)_{n=1}^\infty \approx_C (\mathbf{y}_n)_{n=1}^\infty$ or, simply, $(\mathbf{x}_n)_{n=1}^\infty \approx (\mathbf{y}_n)_{n=1}^\infty$.

By a *sign* we mean a scalar of modulus one. A basic sequence $(\mathbf{x}_n)_{n=1}^\infty$ is called *unconditional* if $(\mathbf{x}_{\sigma(n)})_{n=1}^\infty$ is a basic sequence for any $\sigma \in \Pi$. It is well known that a basic sequence $(\mathbf{x}_n)_{n=1}^\infty$ is unconditional if and only if there exists a constant $C \geqslant 1$ so that $(\mathbf{x}_n)_{n=1}^\infty \approx_C (\epsilon_n \mathbf{x}_n)_{n=1}^\infty$ for any choice of signs $(\epsilon_n)_{n=1}^\infty$. A basic sequence $(\mathbf{x}_n)_{n=1}^\infty$ is called *subsymmetric* if it is unconditional and equivalent to all its subsequences. It is called *symmetric* whenever it is equivalent to each of its permutations. If $(\mathbf{x}_n)_{n=1}^\infty$ is a subsymmetric basic sequence then there is a constant $C \geqslant 1$ such that $(\mathbf{x}_n)_{n=1}^\infty \approx_C (\epsilon_n \mathbf{x}_{\phi(n)})_{n=1}^\infty$ for any $\phi \in \mathcal{O}$ and any choice of signs $(\epsilon_n)_{n=1}^\infty$. In this case we say that $(\mathbf{x}_n)_{n=1}^\infty$ is *C-subsymmetric*. Similarly, if $(\mathbf{x}_n)_{n=1}^\infty$ is symmetric then there is $C \geqslant 1$ such that $(\mathbf{x}_n)_{n=1}^\infty \approx_C (\epsilon_n \mathbf{x}_{\sigma(n)})_{n=1}^\infty$ for any choice of signs $(\epsilon_n)_{n=1}^\infty$ and any $\sigma \in \Pi$, in which case we say that $(\mathbf{x}_n)_{n=1}^\infty$ is *C-symmetric*. Note that C-symmetry implies C-subsymmetry, which in turn implies C-unconditionality. Note also that every subsymmetric basis $(\mathbf{x}_n)_{n=1}^\infty$ is *semi-normalized*, that is, $\|\mathbf{x}_n\| \approx 1$ for $n \in \mathbb{N}$.

Given a function ϕ we denote by $\mathrm{R}(\phi)$ its range. Let $\mathcal{O}_f$ be the set of increasing functions from an integer interval $[1, \dots, r] \cap \mathbb{N}$ into $\mathbb{N}$. Given $\phi \in \mathcal{O}_f$ we denote by $r(\phi)$ the largest integer in its domain. A function in $\mathcal{O}_f$ is univocally determined by its range. For $\mathbf{w} = (w_n)_{n=1}^\infty$, $1 \leqslant p < \infty$, and $f = (a_n)_{n=1}^\infty \in \mathbb{F}^\mathbb{N}$ we have

$$\|f\|_g^p = \sup_{\phi \in \mathcal{O}_f} \sum_{n=1}^{r(\phi)} |a_{\phi(n)}|^p w_n. \tag{1.1}$$

A Banach space with an unconditional (respectively, subsymmetric or symmetric) basis is said to have a *unique unconditional* (respectively, *unique subsymmetric or unique symmetric basis*) *if* any two semi-normalized unconditional (respectively, subsymmetric or symmetric) bases of X are equivalent.

Given infinite-dimensional Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X into Y, and we will put $\mathcal{L}(X)$ to denote the space $\mathcal{L}(X, X)$. The symbol $X \approx Y$ means that X and Y are isomorphic. The space X is said to be *Y-saturated* (or *hereditarily Y*) if every infinite-dimensional closed subspace of X admits a further subspace $Z \approx Y$. If, in addition, there is a constant C such that Z can be chosen C-complemented in X we say that X is *uniformly complementably Y-saturated*. A Banach space X is said to be *complementably homogeneous* whenever for every closed subspace Y of X such that $Y \approx X$, there exists another closed subspace Z such that Z is complemented in X, $Z \approx X$, and $Z \subseteq Y$. A Banach space X is said to be *uniformly subprojective* (see [**10**]) if there is a constant C such that for every infinite-dimensional subspace $Y \subseteq X$ there is a further subspace $Z \subseteq Y$ that is C-complemented in X.

Most other terminology is standard, such as might appear, for instance, in [**1**].

2. *Preliminaries*

In order to exhibit some embeddings involving Garling sequence spaces we will invoke the *weak Lorentz sequence space* $d_\infty(\mathbf{w}, p)$ for $\mathbf{w} \in \mathcal{W}$ and $1 \leqslant p < \infty$, consisting of all sequences $f = (a_n)_{n=1}^\infty \in c_0$ so that

$$\|f\|_{d_\infty(\mathbf{w},p)} = \sup_n \left(\sum_{k=1}^n w_k \right)^{1/p} a_n^* < \infty,$$

where $(a_n^*)_{n=1}^\infty$ denotes the decreasing rearrangement of f. Note that $d_\infty(\mathbf{w}, p)$ is a quasi-Banach space (see [**3**]).

Lemma 2.1. *Suppose $1 \leqslant p < \infty$ and $\mathbf{w} = (w_n)_{n=1}^\infty \in \mathcal{W}$. A sequence $f = (a_n)_{n=1}^\infty \in \mathbb{F}^\mathbb{N}$ belongs to the space $d_\infty(\mathbf{w}, p)$ if and only if $D_f := \sup_n \| \sum_{k=1}^n a_k \mathbf{e}_k \|_{d_\infty(\mathbf{w},p)} < \infty$, in which case $\|f\|_{d_\infty(\mathbf{w},p)} = D_f$.*

Proof. We may suppose $\{n \in \mathbb{N} \colon a_n \neq 0\}$ is infinite (otherwise there is nothing to prove). Put $s_n = (\sum_{k=1}^n w_k)^{1/p}$. Assume $D_f < \infty$. In order to show that $f \in c_0$ it suffices to see that for each $\varepsilon > 0$ the set $\{k \in \mathbb{N} \colon |a_k| \geqslant \varepsilon\}$ is finite. Let $m \in \mathbb{N}$ be such that $D_f < \varepsilon s_m$. For a given j, denote by $(b_n^*)_{n=1}^\infty$ the decreasing rearrangement of $\sum_{n=1}^j a_n \mathbf{e}_n$. We have $b_m^* < \varepsilon$ and so $|\{n \leqslant j \colon |a_n| \geqslant \varepsilon\}| \leqslant m-1$. Since j is arbitrary, we are done.

Now we consider the decreasing rearrangement $(a_n^*)_{n=1}^\infty$ of f. Given $n \in \mathbb{N}$, pick $r \in \mathbb{N}$ such that $|a_k| < a_n^*$ for $k > r$. We have $a_n^* = c_n^*$, where $(c_n^*)_{n=1}^\infty$ is the decreasing rearrangement of $\sum_{n=1}^r a_n \mathbf{e}_n$. Therefore $a_n^* s_n \leqslant D_f$. In other words, $f \in d_\infty(\mathbf{w}, p)$ and $\|f\|_{d_\infty(\mathbf{w},p)} \leqslant D_f$.

The reverse inequality and the converse implication are obvious. □

Proposition 2.2. *Let $1 \leqslant p < \infty$ and $\mathbf{w} \in \mathcal{W}$. Then*

$$\ell_p \subsetneq d(\mathbf{w}, p) \subseteq g(\mathbf{w}, p) \subseteq d_\infty(\mathbf{w}, p),$$

with norm-one inclusions.

Proof. The inclusion $\ell_p \subsetneq d(\mathbf{w}, p)$ is clear and well known.

Let $f = (a_n)_{n=1}^\infty \in \mathbb{F}^\mathbb{N}$ and $\phi \in \mathcal{O}_f$. Pick $\sigma \in \Pi$ extending ϕ. Then

$$\sum_{n=1}^{r(\phi)} |a_{\phi(n)}|^p w_n = \sum_{n=1}^{r(\phi)} |a_{\sigma(n)}|^p w_n \leqslant \sum_{n=1}^{\infty} |a_{\sigma(n)}|^p w_n \leqslant \|f\|_{d(\mathbf{w},p)}^p.$$

Taking the supremum on ϕ we get $\|f\|_{g(\mathbf{w},p)} \leqslant \|f\|_{d(\mathbf{w},p)}$.

Now let $f = (a_n)_{n=1}^\infty \in c_{00}$. For a given $n \in \mathbb{N}$ there is $\phi \in \mathcal{O}_f$ with $r(\phi) = n$ and $|a_{\phi(k)}| \geqslant a_n^*$ for $1 \leqslant k \leqslant r(\phi) = n$. We have

$$a_n^* \left(\sum_{k=1}^n w_k \right)^{1/p} \leqslant \left(\sum_{k=1}^n |a_{\phi(k)}|^p w_k \right)^{1/p} \leqslant \|f\|_{d(\mathbf{w},p)},$$

so that $\|f\|_{d_\infty(\mathbf{w},p)} \leqslant \|f\|_{g(\mathbf{w},p)}$. An appeal to Lemma 2.1 concludes the proof. □

Before proceeding with the proof of the aforementioned subsymmetry of the canonical basis of Garling sequence spaces, let us introduce some linear maps that will be handy.

- Given a sequence of signs $\epsilon = (\epsilon_n)_{n=1}^\infty$ we define $M_\epsilon \colon \mathbb{F}^\mathbb{N} \to \mathbb{F}^\mathbb{N}$ by

$$M_\epsilon((a_n)_{n=1}^\infty) = (\epsilon_n a_n)_{n=1}^\infty.$$

- The coordinate projection on $A \subseteq \mathbb{N}$ is defined by

$$P_A \colon \mathbb{F}^\mathbb{N} \to \mathbb{F}^\mathbb{N}, \quad P_A((a_n)_{n=1}^\infty) = (\lambda_n a_n)_{n=1}^\infty,$$

where $\lambda_n = 1$ if $n \in A$ and $\lambda_n = 0$ otherwise.

- Given $\phi \in \mathcal{O}$ we define $V_\phi \colon \mathbb{F}^\mathbb{N} \to \mathbb{F}^\mathbb{N}$ by $V_\phi((a_n)_{n=1}^\infty) = (a_{\phi(n)})_{n=1}^\infty$, and
- $U_\phi \colon \mathbb{F}^\mathbb{N} \to \mathbb{F}^\mathbb{N}$ by $U_\phi((a_n)_{n=1}^\infty) = (b_n)_{n=1}^\infty$, where

$$b_n = \begin{cases} a_k & \text{if } n = \phi(k), \\ 0 & \text{if } n \notin \mathrm{R}(\phi). \end{cases}$$

Remark 1. Given $\phi \in \mathcal{O}$ and $n \in \mathbb{N}$, we have $U_\phi \circ V_\phi = \mathrm{Id}_{\mathbb{F}^\mathbb{N}}$ and $V_\phi(\mathbf{e}_n) = \mathbf{e}_{\phi(n)}$.

Proposition 2.3. *Let* $\mathbf{w} \in \mathcal{W}$ *and* $1 \leqslant p < \infty$. *Let* $\epsilon = (\epsilon_n)_{n=1}^\infty$ *be a sequence of signs, let* A *be a subset of* $\mathbb{N}$, *and let* ϕ *be a map in* $\mathcal{O}$.

(i) M_ϵ *and* P_A *are norm-one operators from* $g(\mathbf{w}, p)$ *into* $g(\mathbf{w}, p)$.
(ii) V_ϕ *and* U_ϕ *are norm-one operators from* $g(\mathbf{w}, p)$ *into* $g(\mathbf{w}, p)$.
(iii) V_ϕ *is an isometric embedding from* $g(\mathbf{w}, p)$ *into* $g(\mathbf{w}, p)$.
(iv) *The standard unit vectors form a* 1*-subsymmetric basic sequence in* $g(\mathbf{w}, p)$.

Proof. (i) is clear; (iii) is a consequence of (ii) and Remark 1; (iv) is a consequence of (i), (iii), and Remark 1. Thus we must only care to show (ii). Let $f = (a_n)_{n=1}^\infty \in \mathbb{F}^\mathbb{N}$. From (i) we know that $\|M_\epsilon(f)\|_g = \|f\|_g$ and that $\|P_A(f)\|_g \leqslant \|f\|_g$. Since $\phi \circ \psi \in \mathcal{O}$ for all $\psi \in \mathcal{O}$,

$$\|V_\phi(f)\|_g^p = \sup_{\psi \in \mathcal{O}} \sum_{n=1}^\infty |a_{\phi(\psi(n))}|^p w_n \leqslant \|f\|_g.$$

Let $U_\phi(f) = (b_n)_{n=1}^\infty$. Pick $\psi \in \mathcal{O}$. The function $\phi^{-1} \circ \psi$ is an increasing map from a set $A \subseteq \mathbb{N}$ to $\mathbb{N}$. Put $J = \{n \in \mathbb{N} \colon n \leqslant |A|\}$ and choose $\gamma \colon J \to A$ increasing and bijective. Since $\rho := \phi^{-1} \circ \psi \circ \gamma \in \mathcal{O} \cup \mathcal{O}_f$ and $n \leqslant \gamma(n)$ for all $n \in J$, we have

$$\begin{aligned}\sum_{n=1}^\infty |b_{\psi(n)}|^p w_n &= \sum_{n \in A} |a_{\phi^{-1} \circ \psi(n)}|^p w_n = \sum_{n \in J} |a_{\rho(n)}|^p w_{\gamma(n)} \\ &\leqslant \sum_{n \in J} |a_{\rho(n)}|^p w_n \leqslant \|f\|_g^p.\end{aligned}$$

Taking the supremum on ψ yields $\|U_\phi(f)\|_g \leqslant \|f\|_g$. □

Definition 1. Let $(\mathbf{x}_n)_{n=1}^\infty$ be a basis for a Banach space X and let $g = \sum_{n=1}^\infty b_n \mathbf{x}_n$ be a vector in X. Given another $f = \sum_{n=1}^\infty a_n \mathbf{x}_n \in X$ we write $f \prec g$ (with respect to the basis $(\mathbf{x}_n)_{n=1}^\infty$) if

- $\operatorname{supp}(f) \subseteq \operatorname{supp}(g)$,
- $a_n = b_n$ for all $n \in \operatorname{supp}(f)$, and
- $\|f\| = \|g\|$.

We say that g is *minimal* in X (with respect to $(\mathbf{x}_n)_{n=1}^\infty$) if it is minimal in X equipped with the partial order $\prec$, that is, if $f \prec g$ implies $f = g$.

Remark 2. Note that if g is finitely supported with respect to a basis $(\mathbf{x}_n)_{n=1}^\infty$ then there exists a minimal $f \prec g$.

Remark 3. Every sequence in $d(\mathbf{w}, p)$, $1 \leqslant p < \infty$, is minimal. The situation is quite different for $g(\mathbf{w}, p)$. For instance, if $(1 - w_2)t^p \geqslant 1$, then the vector $\mathbf{g}_1 + t\mathbf{g}_2$ is not minimal.

Lemma 2.4. *Let* $\mathbf{w} = (w_n)_{n=1}^\infty \in \mathcal{W}$ *and* $1 \leqslant p < \infty$. *For any* $A \subseteq \mathbb{N}$ *finite we have:*

(i) $\|\sum_{n \in A} \mathbf{e}_n\|_g = \sum_{n=1}^{|A|} w_n$;
(ii) *the vector* $\sum_{n \in A} \mathbf{e}_n$ *is minimal in* $g(\mathbf{w}, p)$ (*with respect to its canonical basis*).

Proof. Part (i) can be obtained, for example, from Proposition 2.2. To prove (ii), let $f \prec \sum_{n \in A} \mathbf{e}_n$. Then $f = \sum_{n \in B} \mathbf{e}_n$ for some $B \subseteq A$, and

$$\sum_{n=1}^{|B|} w_n = \|f\|_g = \left\| \sum_{n \in A} \mathbf{e}_n \right\|_g = \sum_{n=1}^{|A|} w_n.$$

Consequently $|B| = |A|$ and so $B = A$. □

LEMMA 2.5. *Let* $\mathbf{w} \in \mathcal{W}$ *and* $1 \leqslant p < \infty$. *Suppose* $f \in c_{00}$ *is minimal in* $g(\mathbf{w}, p)$ *with respect to the canonical basis. Then*

$$\|f\|_g = \left(\sum_{n=1}^{r(\psi)} |a_{\psi(n)}|^p w_n \right)^{1/p}, \tag{2.1}$$

where $\psi \in \mathcal{O}_f$ *is determined by* $\mathrm{R}(\psi) = \operatorname{supp} f$.

Proof. Given $\phi \in \mathcal{O}_f$, let ψ be the map in $\mathcal{O}_f$ determined by $\mathrm{R}(\psi) = \mathrm{R}(\phi) \cap \operatorname{supp} f$. Let γ be the inverse function of ϕ restricted to $\mathrm{R}(\psi)$. We have

$$\sum_{n=1}^{r(\phi)} |a_{\phi(n)}|^p w_n = \sum_{n \in \mathrm{R}(\psi)} |a_n|^p w_{\gamma(n)} \leqslant \sum_{n \in \mathrm{R}(\psi)} |a_n|^p w_{\psi^{-1}(n)} = \sum_{n=1}^{r(\psi)} |a_{\psi(n)}|^p w_n.$$

Therefore, in order to compute the supremum in (1.1) we can restrict ourselves to $\{\phi \in \mathcal{O}_f \colon \mathrm{R}(\phi) \subseteq \operatorname{supp} f\}$, so that this supremum is attained. Let ψ be the element in $\mathcal{O}_f$ with $\mathrm{R}(\psi) \subseteq \operatorname{supp}(f)$ where the supremum in (1.1) is attained. Let h be the projection of f onto $\mathrm{R}(\psi)$. Then

$$\|f\|_g^p = \sum_{n=1}^{r(\psi)} |a_{\psi(n)}|^p w_n \leqslant \|h\|_g^p,$$

hence $h = f$ and $\mathrm{R}(\psi) = \operatorname{supp} f$. □

Recall that a *block basic sequence* of a basic sequence $(\mathbf{x}_n)_{n=1}^\infty$ in a Banach space X is a sequence $(\mathbf{y}_n)_{n=1}^\infty$ of nonzero vectors of the form

$$\mathbf{y}_n = \sum_{i=p_n}^{p_{n+1}-1} a_i \mathbf{x}_i$$

for some (unique) $(a_n)_{n=1}^\infty \in \mathbb{F}^\mathbb{N}$ and some increasing sequence of integers $(p_n)_{n=1}^\infty$ with $p_1 = 1$.

DEFINITION 2. Let $\mathbf{y}_n = \sum_{i=p_n}^{p_{n+1}-1} a_i \mathbf{x}_i$, $n \in \mathbb{N}$, be a block basic sequence of a basic sequence $(\mathbf{x}_n)_{n=1}^\infty$ in a Banach space X. Then:

(a) $(\mathbf{y}_n)_{n=1}^\infty$ is said to be *uniformly null* if $\lim_n a_n = 0$;
(b) $(\mathbf{y}_n)_{n=1}^\infty$ is said to verify the *gliding hump property* if

$$\inf_{n \in \mathbb{N}} \sup_{p_n \leqslant i \leqslant p_{n+1}-1} |a_i| > 0.$$

REMARK 4. A subsequence of a block basic sequence also is a block basic sequence. Moreover, we have the following dichotomy: A block basic sequence is either uniformly null or has a subsequence verifying the gliding hump property.

LEMMA 2.6. *Let* $(\mathbf{x}_n)_{n=1}^\infty$ *be a semi-normalized basis in a Banach space* X. *If* $(\mathbf{y}_n)_{n=1}^\infty$ *is a semi-normalized uniformly null block basic sequence of* $(\mathbf{x}_n)_{n=1}^\infty$, *then* $\lim_n |\operatorname{supp}(\mathbf{y}_n)| = \infty$.

Proof. Assume that the claim fails. Then there is an infinite subset $A \subseteq \mathbb{N}$ with $D = \sup_{n \in A} |\operatorname{supp}(\mathbf{y}_n)| < \infty$. Write $\mathbf{y}_n = \sum_{i=p_n}^{p_{n+1}-1} a_i \mathbf{x}_i$ and put

$$C_n = \max_{p_n \leqslant i \leqslant p_{n+1}-1} |a_i|, \quad C = \sup_n \|\mathbf{x}_n\|, \quad \text{and} \quad B = \inf_n \|y_n\|.$$

We have $B \leqslant CDC_n$ for all $n \in A$. Letting n tend to infinity through A we obtain $B \leqslant 0$, an absurdity. □

We shall also need two properties on block basic sequences of subsymmetric bases, which we gather in the following proposition.

PROPOSITION 2.7. *Let X be a Banach space with a subsymmetric basis $(\mathbf{x}_n)_{n=1}^\infty$ and suppose that $(\mathbf{y}_n)_{n=1}^\infty$ is a semi-normalized block basic sequence of $(\mathbf{x}_n)_{n=1}^\infty$.*

(i) *If $\sup_n |\operatorname{supp}(\mathbf{y}_n)| < \infty$, then $(\mathbf{y}_n)_{n=1}^\infty$ is equivalent to $(\mathbf{x}_n)_{n=1}^\infty$.*
(ii) *If $(\mathbf{y}_n)_{n=1}^\infty$ verifies the gliding hump property, then $(\mathbf{x}_n)_{n=1}^\infty \lesssim (\mathbf{y}_n)_{n=1}^\infty$.*

Proof. The proofs follow the steps of [**2**, Propositions 3 and 4] and so we leave them out for the reader. □

To make this paper as self-contained as possible we record the following well-known fact.

PROPOSITION 2.8 (cf. [**2**, Proposition 1]). *Every subsymmetric basic sequence in a Banach space is either weakly null or else equivalent to the canonical basis of ℓ_1.*

COROLLARY 2.9. *Let $(\mathbf{x}_n)_{n=1}^\infty$ be a basic sequence in a Banach space X. If $(\mathbf{x}_n)_{n=1}^\infty$ is equivalent to the unit vector basis of $g(\mathbf{w}, p)$ for $\mathbf{w} \in \mathcal{W}$ and $1 \leqslant p < \infty$, then it is weakly null.*

Proof. It is straightforward from Propositions 2.2, 2.3, and 2.8. □

We put an end to this preliminary section recalling a specific version of Bessaga–Pelczyński Selection Principle that we will need below.

THEOREM 2.10 (cf. [**1**, Proposition 1.3.10]). *Let X be a Banach space with a basis $(\mathbf{x}_n)_{n=1}^\infty$. Let $(\mathbf{y}_n)_{n=1}^\infty$ be a normalized weakly null sequence in X and let $\epsilon > 0$. Then there exists a subsequence $(\mathbf{y}_{n_k})_{n=1}^\infty$ of $(\mathbf{y}_n)_{n=1}^\infty$ that is $(1+\epsilon)$-equivalent to a normalized block basic sequence $(\mathbf{z}_k)_{k=1}^\infty$ of $(\mathbf{x}_n)_{n=1}^\infty$. Moreover, if $T\colon [\mathbf{z}_k]_{k=1}^\infty \to [\mathbf{y}_{n_k}]_{n=1}^\infty$ is the isomorphism given by $T(\mathbf{z}_k) = \mathbf{y}_{n_k}$, we have that whenever $Z \subseteq [\mathbf{z}_k]_{k=1}^\infty$ is C-complemented in X, then $T(Z)$ is $C(1+\epsilon)$-complemented in X.*

3. *Geometric properties of Garling sequence spaces*

Our ultimate goal in this section is to establish the following set of structural results about the spaces $g(\mathbf{w}, p)$.

THEOREM 3.1. *Let $1 \leqslant p < \infty$ and $\mathbf{w} = (w_n)_{n=1}^\infty \in \mathcal{W}$.*

(i) *The unit vectors in $g(\mathbf{w}, p)$ span the entire space and form a boundedly complete basis.*
(ii) *Every basic sequence in $g(\mathbf{w}, p)$ equivalent to its canonical basis admits a subsequence spanning a complemented subspace in $g(\mathbf{w}, p)$.*
(iii) *The space $g(\mathbf{w}, p)$ is reflexive if $1 < p < \infty$ and nonreflexive if $p = 1$.*
(iv) *No subspace of ℓ_p is isomorphic to $g(\mathbf{w}, p)$.*
(v) *For every $\epsilon > 0$ and every infinite-dimensional closed subspace Y of $g(\mathbf{w}, p)$, there exists a further subspace $Z \subseteq Y$ that is $(1+\epsilon)$-isomorphic to ℓ_p and $(1+\epsilon)$-complemented in $g(w, p)$.*
(vi) *The identity operator on ℓ_p factors through the inclusion map $I_{d,g}\colon d(\mathbf{w}, p) \to g(\mathbf{w}, p)$.*

To that end, let us begin with the following proposition.

PROPOSITION 3.2. *Let $1 \leqslant p < \infty$ and $\mathbf{w} = (w_n)_{n=1}^\infty \in \mathcal{W}$. Every normalized block basic sequence of the canonical basis of $g(\mathbf{w}, p)$ is 1-dominated by the canonical basis of ℓ_p.*

Proof. Let

$$\mathbf{y}_n = \sum_{i=p_n}^{p_{n+1}-1} a_i \mathbf{g}_i, \quad n \in \mathbb{N},$$

be a normalized block basic sequence. For $i \in \mathbb{N}$, define $n = n(i) \in \mathbb{N}$ by $p_n \leqslant i \leqslant p_{n+1} - 1$. Let $(b_n)_{n=1}^\infty \in c_{00}$. Denote $(c_i)_{i=1}^\infty := \sum_{n=1}^\infty b_n \mathbf{y}_n$. We have $c_i = b_{n(i)} a_i$ for all $i \in \mathbb{N}$. Let $\phi \in \mathcal{O}$. For each $n \in \mathbb{N}$, $J_n := \{i \in \mathbb{N} \colon p_n \leqslant \phi(i) \leqslant p_{n+1} - 1\}$ is an integer interval and, then, for some q_n, $r_n \in \mathbb{N} \cup \{0\}$, $J_n = \{j \in \mathbb{N} \colon 1 + q_n \leqslant j \leqslant q_n + r_n\}$. We have

$$\begin{aligned}
\sum_{n=1}^\infty |c_{\phi(n)}|^p w_n &= \sum_{n=1}^\infty |b_n|^p \sum_{i \in J_n} |a_{\phi(i)}|^p w_i \\
&= \sum_{n=1}^\infty |b_n|^p \sum_{j=1}^{r_n} |a_{\phi(j+q_n)}|^p w_{j+q_n} \\
&\leqslant \sum_{n=1}^\infty |b_n|^p \sum_{j=1}^{r_n} |a_{\phi(j+q_n)}|^p w_j \\
&\leqslant \sum_{n=1}^\infty |b_n|^p \, \|\mathbf{y}_n\|_g^p \\
&= \sum_{n=1}^\infty |b_n|^p.
\end{aligned}$$

Taking the supremum on ϕ yields $\| \sum_{n=1}^\infty b_n \mathbf{y}_n \|^p \leqslant \sum_{n=1}^\infty |b_n|^p$. □

We are now able to state a powerful result. In its proof we will make use of the following construction.

DEFINITION 3. Let $(\mathbf{x}_n)_{n=1}^\infty$ be a basis in a Banach space X. Given a block basic sequence $(\mathbf{y}_n)_{n=1}^\infty$ of $(\mathbf{x}_n)_{n=1}^\infty$,

$$\mathbf{y}_n = \sum_{i=p_n}^{p_{n+1}-1} a_i \mathbf{x}_i, \quad n \in \mathbb{N},$$

its *left-shifted block basic sequence* $(\hat{\mathbf{y}}_n)_{n=1}^\infty$ is the block basic sequence of $(\mathbf{x}_n)_{n=1}^\infty$ constructed from $(\mathbf{y}_n)_{n=1}^\infty$ as follows: Consider $\phi \in \mathcal{O}$ with range $\{i \in \mathbb{N} \colon a_i \neq 0\}$ and put

$$\hat{\mathbf{y}}_n = \sum_{i=\hat{p}_n}^{\hat{p}_{n+1}-1} a_{\phi(i)} \mathbf{x}_i, \quad n \in \mathbb{N},$$

where $\hat{p}_1 = 1$, and $\hat{p}_{n+1} - \hat{p}_n = |\operatorname{supp} \mathbf{y}_n|$ for all $n \in \mathbb{N}$.

THEOREM 3.3. *Let $1 \leqslant p < \infty$ and $\mathbf{w} \in \mathcal{W}$. Suppose $(\mathbf{y}_n)_{n=1}^\infty$ is a uniformly null normalized block basic sequence of the canonical basis of $g(\mathbf{w}, p)$. Then for each $\epsilon > 0$, there exists a* *subsequence $(\mathbf{y}_{n_k})_{k=1}^\infty$ of $(\mathbf{y}_n)_{n=1}^\infty$ that is $(1+\epsilon)$-equivalent to the canonical basis of ℓ_p and such that $[\mathbf{y}_{n_k}]_{k=1}^\infty$ is $(1+\epsilon)$-complemented in $g(\mathbf{w},p)$.*

Proof. By Proposition 2.3(i), without loss of generality we can assume that each $\mathbf{y}_n$ has nonnegative coefficients with respect to $(\mathbf{g}_i)_{i=1}^\infty$. For each $n\in\mathbb{N}$ pick a minimal $\tilde{\mathbf{y}}_n \prec \mathbf{y}_n$. Let

$$\hat{\mathbf{y}}_n = \sum_{i=p_n}^{p_{n+1}-1} a_i \mathbf{g}_i, \quad n\in\mathbb{N},$$

be the left-shifted block basic sequence of $(\tilde{\mathbf{y}}_n)_{n=1}^\infty$, where $(a_i)_{i=1}^\infty$ is a sequence of positive scalars converging to zero and $(p_n)_{n=1}^\infty$ is an increasing sequence of integers with $p_1=1$. By Lemma 2.6 we have $\lim_n(p_{n+1}-p_n)=\infty$. Let $\mathbf{w}=(w_n)_{n=1}^\infty$ and pick $\alpha=(1+\epsilon)^{-p}\in(0,1)$. We claim that there exist increasing sequences of positive integers $(q_k)_{k=1}^\infty$ and $(n_k)_{k=1}^\infty$ with $q_1=1$ so that for each $k\in\mathbb{N}$:

- $q_{k+1}-q_k = p_{1+n_k}-p_{n_k}$,
- $A_k := \sum_{i=q_k}^{q_{k+1}-1} a^p_{i+p_{n_k}-q_k} w_i \geqslant \alpha$.

To prove the claim we proceed recursively. Suppose that q_k and n_{k-1} have been constructed (for the case $k=1$ we put $n_0=0$). Since $\mathbf{w}\in c_0$ we can find $L\in\mathbb{N}$ so that

$$\sum_{n=L+1}^{L+q_k-1} w_n < \frac{1-\alpha}{2}.$$

Then we choose $M\in\mathbb{N}$ such that for any $i\geqslant M$

$$a_i < \left(\frac{1-\alpha}{2L}\right)^{1/p}.$$

Select $j=n_k\in\mathbb{N}$ so that $j>n_{k-1}$, $p_j\geqslant M$, and $p_{j+1}-p_j>L+q_k$. Set $q_{k+1}=p_{j+1}-p_j+q_k$. Let $\phi\in\mathcal{O}$ determined by

$$\mathrm{R}(\phi)=A:=\cup_{n=1}^\infty \operatorname{supp}(\tilde{\mathbf{y}}_n).$$

We have $\tilde{\mathbf{y}}_j = V_\phi(\hat{\mathbf{y}}_j)$. Hence, by Lemma 2.3(iii), $\hat{\mathbf{y}}_j$ is minimal. Then, by Lemma 2.5,

$$1=\|\hat{\mathbf{y}}_j\|_g^p = \sum_{n=1}^{p_{j+1}-p_j} a^p_{p_j+n-1} w_n = \sum_{i=q_k}^{q_{k+1}-1} a^p_{p_j-q_k+i} w_{i-q_k+1},$$

and so

$$\begin{aligned}
1-\sum_{i=q_k}^{q_{k+1}-1} a^p_{i+p_j-q_k} w_i &= \sum_{i=q_k}^{q_{k+1}-1} a^p_{i+p_j-q_k}(w_{i-q_k+1}-w_i)\\
&\leqslant \sum_{i=q_k}^{q_{k+1}-1} \frac{1-\alpha}{2L}(w_{i-q_k+1}-w_i)\\
&\leqslant \sum_{i=q_k}^{q_k+L-1} \frac{1-\alpha}{2L} + \sum_{i=q_k+L}^{q_{k+1}-1}(w_{i-q_k+1}-w_i)\\
&= \frac{1-\alpha}{2} + \sum_{n=L+1}^{q_{k+1}-q_k} w_n - \sum_{n=L+q_k}^{q_{k+1}-1} w_n
\end{aligned}$$

$$= \frac{1-\alpha}{2} + \sum_{n=L+1}^{L+q_k-1} w_n - \sum_{n=q_{k+1}-q_k+1}^{q_{k+1}-1} w_n$$

$$\leqslant \frac{1-\alpha}{2} + \frac{1-\alpha}{2} = 1-\alpha.$$

This completes the proof of our claim.

Consider the linear map $S\colon \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$ given by

$$S((c_i)_{i=1}^\infty) = \left(\frac{1}{A_k} \sum_{i=q_k}^{q_{k+1}-1} a_{i+p_{n_k}-q_k}^{p-1} w_i c_i \right)_{k=1}^\infty .$$

Let $f = (c_i)_{i=1}^\infty \in \mathbb{F}^{\mathbb{N}}$. Hölder's inequality gives

$$\begin{aligned}
\|S(f)\|_p^p &= \sum_{k=1}^\infty \frac{1}{A_k^p} \left| \sum_{i=q_k}^{q_{k+1}-1} c_i a_{i+p_{n_k}-q_k}^{p-1} w_i \right|^p \\
&\leqslant \sum_{k=1}^\infty \frac{1}{A_k^p} \left(\sum_{i=q_k}^{q_{k+1}-1} |c_i|^p w_i \right) \left(\sum_{i=q_k}^{q_{k+1}-1} a_{i+p_{n_k}-q_k}^{p} w_i \right)^{p-1} \\
&= \sum_{k=1}^\infty \frac{1}{A_k} \left(\sum_{i=q_k}^{q_{k+1}-1} |c_i|^p w_i \right) \\
&\leqslant \alpha^{-1} \sum_{k=1}^\infty \left(\sum_{i=q_k}^{q_{k+1}-1} |c_i|^p w_i \right) \\
&= \alpha^{-1} \|f\|_g^p.
\end{aligned}$$

That is, S is an $\alpha^{-1/p}$-bounded operator from $g(\mathbf{w},p)$ into ℓ_p.

The left-shifted block basic sequence of $(\hat{\mathbf{y}}_{n_k})_{k=1}^\infty$, which we denote by $(\mathbf{z}_k)_{k=1}^\infty$, is given by the formula

$$\mathbf{z}_k = \sum_{i=q_k}^{q_{k+1}-1} a_{i+p_{n_k}-q_k} \mathbf{g}_i, \quad k \in \mathbb{N}.$$

Since the left-shifted block basic sequences of $(\tilde{\mathbf{y}}_{n_k})_{n=1}^\infty$ and $(\hat{\mathbf{y}}_{n_k})_{k=1}^\infty$ coincide, there is $\psi \in \mathcal{O}$ such that $V_\psi(\mathbf{z}_k) = \tilde{\mathbf{y}}_{n_k}$ for all $k \in \mathbb{N}$. Let $T = S \circ U_\psi \circ P_A$. By Proposition 2.3(i) and (ii), T is a $(1+\epsilon)$-bounded operator from $g(\mathbf{w},p)$ into ℓ_p.

By Proposition 3.2, $(\mathbf{y}_{n_k})_{k=1}^\infty \lesssim_1 (\mathbf{f}_k)_{k=1}^\infty$. In other words, there is a norm-one operator $R\colon \ell_p \to g(\mathbf{w},p)$ with $R(\mathbf{f}_k) = \mathbf{y}_{n_k}$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ we have

$$\begin{aligned}
(T \circ R)(\mathbf{f}_k) &= S(U_\psi(P_A(R(\mathbf{f}_k)))) \\
&= S(U_\psi(P_A(\mathbf{y}_{n_k}))) \\
&= S(U_\psi(\tilde{\mathbf{y}}_{n_k})) \\
&= S(U_\psi(V_\psi(\mathbf{z}_k))) \\
&= S(\mathbf{z}_k)
\end{aligned}$$

$$= \frac{1}{A_k}\left(\sum_{i=q_k}^{q_{k+1}-1} a_{i+p_{n_k}-q_k} a_{i+p_{n_k}-q_k}^{p-1} w_i\right)\mathbf{f}_k$$

$$= \mathbf{f}_k.$$

Hence, $T\circ R = \mathrm{Id}_{\ell_p}$. We infer that $R\circ T$ is a projection from $g(\mathbf{w},p)$ onto $[\mathbf{y}_{n_k}]_{k=1}^\infty$ and that $(\mathbf{f}_k)_{k=1}^\infty \lesssim_{1+\epsilon} (\mathbf{y}_{n_k})_{k=1}^\infty$. □

REMARK 5. If we consider the natural lattice structure both in $g(\mathbf{w},p)$ and in ℓ_p, the maps R and T constructed in the proof of Theorem 3.3 are lattice homomorphisms.

COROLLARY 3.4. *Every normalized block basic sequence of $(\mathbf{g}_n)_{n=1}^\infty$ admits a subsequence that dominates $(\mathbf{g}_n)_{n=1}^\infty$.*

Proof. This is an immediate consequence of Remark 4, Proposition 2.7(ii), Theorem 3.3, and the inclusion $\ell_p \subseteq g(\mathbf{w},p)$ provided by Proposition 2.2, which reads as $(\mathbf{g}_n)_{n=1}^\infty \lesssim (\mathbf{f}_n)_{n=1}^\infty$. □

Before proving Theorem 3.1 we need a couple more results.

PROPOSITION 3.5. *Let $1\leqslant p<\infty$ and $\mathbf{w}\in\mathcal{W}$. If $(\mathbf{y}_n)_{n=1}^\infty$ is a semi-normalized block basic sequence of the canonical basis of $g(\mathbf{w},p)$ then*

$$\lim_{m\to\infty}\left\|\sum_{n=1}^m \mathbf{y}_n\right\|_g = \infty.$$

Proof. From Remark 4 in combination with Proposition 2.7(ii) and Theorem 3.3, we infer that we can find a subsequence $(\mathbf{y}_{n_k})_{k=1}^\infty$ that either dominates $(\mathbf{g}_n)_{n=1}^\infty$ or else is equivalent to $(\mathbf{f}_n)_{n=1}^\infty$. In either case, since, by Proposition 2.2, $(\mathbf{f}_n)_{n=1}^\infty$ dominates $(\mathbf{g}_n)_{n=1}^\infty$, we have $(\mathbf{g}_n)_{n=1}^\infty \lesssim (\mathbf{y}_{n_k})_{k=1}^\infty$. Put $\mathbf{w}=(w_n)_{n=1}^\infty$. Appealing to the 1-unconditionality of $(\mathbf{g}_n)_{n=1}^\infty$ and to Lemma 2.4,

$$\sum_{k=1}^m w_k = \left\|\sum_{k=1}^m \mathbf{g}_k\right\|_g \lesssim \left\|\sum_{k=1}^m \mathbf{y}_{n_k}\right\|_g \leqslant \left\|\sum_{n=1}^j \mathbf{y}_n\right\|_g, \quad m\in\mathbb{N},\ j\geqslant n_m.$$

Since $\mathbf{w}\notin\ell_1$ we are done. □

PROPOSITION 3.6. *Let $1\leqslant p<\infty$ and $\mathbf{w}=(w_n)_{n=1}^\infty\in\mathcal{W}$. Suppose $(\mathbf{y}_n)_{n=1}^\infty$ is a block basic sequence of the canonical basis of $g(\mathbf{w},p)$. Then for any $\epsilon>0$, there exists a further block basic sequence of $(\mathbf{y}_n)_{n=1}^\infty$ that is $(1+\epsilon)$-equivalent to the canonical basis of ℓ_p and $(1+\epsilon)$-complemented in $g(\mathbf{w},p)$.*

Proof. Without loss of generality we assume that $(\mathbf{y}_n)_{n=1}^\infty$ is normalized. By Proposition 3.5 we can recursively construct an increasing sequence $(p_n)_{n=1}^\infty$ of positive integers such that $\|\sum_{i=p_n}^{p_{n+1}-1}\mathbf{y}_i\|_g\geqslant n$ for all $n\in\mathbb{N}$. Then the block basic sequence

$$\mathbf{z}_n = \frac{\sum_{i=p_n}^{p_{n+1}-1}\mathbf{y}_i}{\|\sum_{i=p_n}^{p_{n+1}-1}\mathbf{y}_i\|_g}, \quad n\in\mathbb{N},$$

is uniformly null, and so by Theorem 3.3 we can find a subsequence $(\mathbf{z}_{n_k})_{k=1}^\infty$ that is $(1+\epsilon)$-equivalent to the canonical basis of ℓ_p and $(1+\epsilon)$-complemented in $g(\mathbf{w},p)$. □

Completion of the Proof of Theorem 3.1.

(i) By Proposition 3.5, no block basic sequence of $(\mathbf{g}_n)_{n=1}^\infty$ is equivalent to the canonical basis of c_0. Appealing to [1, Theorem 3.3.2] we infer that $(\mathbf{g}_n)_{n=1}^\infty$ is a boundedly complete basic sequence.

Let us prove that $[\mathbf{g}_n]_{n=1}^\infty = g(\mathbf{w},p)$. Take $f=(a_n)_{n=1}^\infty \in g(\mathbf{w},p)$. Then $\sup_m \| \sum_{n=1}^m a_n \mathbf{g}_n\|_g < \infty$. Since $(\mathbf{g}_n)_{n=1}^\infty$ is boundedly complete, the series $\sum_{n=1}^\infty a_n \mathbf{g}_n$ converges to some $h \in g(\mathbf{w},p)$. Obviously, $f=h$.

(ii) Let $(\mathbf{y}_n)_{n=1}^\infty$ be a basic sequence in $g(\mathbf{w},p)$ equivalent to its canonical basis. By Corollary 2.9, $(\mathbf{y}_n)_{n=1}^\infty$ is weakly null. Then, applying Theorem 2.10 and passing to a subsequence, we can assume that $(\mathbf{y}_n)_{n=1}^\infty$ is a block basic sequence of $(\mathbf{g}_n)_{n=1}^\infty$. By Theorem 3.3 and Remark 4, since $(\mathbf{g}_n)_{n=1}^\infty$ has no subsequence equivalent to the ℓ_p-basis, passing to a further subsequence we have that $(\mathbf{y}_n)_{n=1}^\infty$ has the gliding hump property. Let $K\geqslant 1$ be such that $(\mathbf{y}_n)_{n=1}^\infty \lesssim_K (\mathbf{g}_n)_{n=1}^\infty$. Write

$$\mathbf{y}_n = \sum_{i=p_n}^{p_{n+1}-1} a_i \mathbf{g}_i, \quad n\in\mathbb{N},$$

for some $(a_n)_{n=1}^\infty \in \mathbb{F}^\mathbb{N}$ and some increasing sequence $(p_n)_{n=1}^\infty$ with $p_1=1$. For each $n\in\mathbb{N}$ pick an integer sequence $(i_n)_{n=1}^\infty$ with $p_n \leqslant i_n < p_{n+1}$ and so that $\inf_{n\in\mathbb{N}} |a_{i_n}| := c > 0$. Observe that for any $(b_n)_{n=1}^\infty \in c_{00}$, by 1-subsymmetry of $(\mathbf{g}_n)_{n=1}^\infty$ we have

$$\left\| \sum_{n=1}^\infty \frac{b_{i_n}}{a_{i_n}} \mathbf{y}_n \right\|_g \leqslant K \left\| \sum_{n=1}^\infty \frac{b_{i_n}}{a_{i_n}} \mathbf{g}_{i_n} \right\|_g \leqslant \frac{K}{c} \left\| \sum_{n=1}^\infty b_{i_n} \mathbf{g}_{i_n} \right\|_g \leqslant \frac{K}{c} \left\| \sum_{n=1}^\infty b_n \mathbf{g}_n \right\|_g .$$

Thus we can define a bounded linear map $P\colon g(\mathbf{w},p) \to [\mathbf{y}_n]_{n=1}^\infty$ by the rule

$$\sum_{n=1}^\infty b_n \mathbf{e}_n \mapsto \sum_{n=1}^\infty \frac{b_{i_n}}{a_{i_n}} \mathbf{y}_n.$$

Since $P(\mathbf{y}_n)=\mathbf{y}_n$ for all $n\in\mathbb{N}$ we are done.

(iii) Let us suppose that $(\mathbf{g}_n)_{n=1}^\infty$ fails to be shrinking. Then due to the unconditionality of $(\mathbf{g}_n)_{n=1}^\infty$, there exists a block basic sequence $(\mathbf{y}_n)_{n=1}^\infty$ of $(\mathbf{g}_n)_{n=1}^\infty$ that is equivalent to the canonical ℓ_1-basis (see, for example, [1, Theorem 3.3.1]). Now, by Proposition 3.6, we can find a further block basic sequence equivalent to the canonical ℓ_p-basis. However, this is impossible if $p\neq 1$. Thus, for $p>1$, $(\mathbf{g}_n)_{n=1}^\infty$ is shrinking, and since it is also boundedly complete by part (i), and unconditional, it must span a reflexive space by a theorem of James (see, for example, [1, Theorem 3.2.19]). This proves the first part of (iii). The part about nonreflexivity in case when $p=1$ follows since $g(\mathbf{w},1)$ contains a (nonreflexive) copy of ℓ_1 by Proposition 3.6.

(iv) Assume that $g(\mathbf{w},p)$ embeds into ℓ_p. Then ℓ_p contains a basic sequence $(\mathbf{y}_n)_{n=1}^\infty \approx (\mathbf{g}_n)_{n=1}^\infty$. By Corollary 2.9, $(\mathbf{y}_n)_{n=1}^\infty$ is weakly null. Hence (see, [1, Proposition 2.1.3]) $(\mathbf{y}_n)_{n=1}^\infty$ has a sequence equivalent to the canonical basis $(\mathbf{f}_n)_{n=1}^\infty$ of ℓ_p. By Proposition 2.3, $(\mathbf{g}_n)_{n=1}^\infty \approx (\mathbf{f}_n)_{n=1}^\infty$, which is false by Proposition 2.2.

(v) Fix $\epsilon>0$, and let Y be an infinite-dimensional closed subspace of $g(\mathbf{w},p)$. By a classical result of Mazur (see, for example, [1, Theorem 1.4.5]), we can find a normalized basic sequence $(\mathbf{y}_n)_{n=1}^\infty$ in Y. In the case when $p>1$, this basic sequence must be weakly null by the reflexivity of $g(\mathbf{w},p)$. For the case $p=1$, we can assume by Rosenthal's ℓ_1 theorem (see, for example, [1, Theorem 11.2.1]) together with James' ℓ_1 Distortion theorem (see, for example, [1, Theorem 11.3.1]) that $(\mathbf{y}_n)_{n=1}^\infty$ admits a weakly Cauchy subsequence. By passing to a further subsequence if necessary, the sequence

$$\left(\frac{\mathbf{y}_{2n+1}-\mathbf{y}_{2n}}{\|\mathbf{y}_{2n+1}-\mathbf{y}_{2n}\|_g} \right)_{n=1}^\infty$$

is normalized and weakly null. Let us relabel if necessary so that in all cases $1 \leqslant p < \infty$ we have found a normalized and weakly null basic sequence $(\mathbf{y}_n)_{n=1}^\infty$ in Y. By Theorem 2.10 we can pass to a subsequence if necessary so that $(\mathbf{y}_n)_{n=1}^\infty$ is $(\sqrt{1+\epsilon})$-equivalent to a normalized block basic sequence $(\mathbf{y}'_n)_{n=1}^\infty$ of the canonical basis of $g(\mathbf{w},p)$. Moreover, if $Z \subseteq [\mathbf{y}'_n]_{n=1}^\infty$ is C-complemented in $g(\mathbf{w},p)$ and T is the operator defined by $T(\mathbf{y}'_n) = \mathbf{y}_n$, then $T(Z)$ is $(\sqrt{1+\epsilon})C$-complemented in $g(\mathbf{w},p)$. Now we apply Proposition 3.6 to find a block basic sequence $(\mathbf{z}'_n)_{n=1}^\infty$ of $(\mathbf{y}'_n)_{n=1}^\infty$, which is $(\sqrt{1+\epsilon})$-equivalent to the canonical basis $(\mathbf{f}_n)_{n=1}^\infty$ of ℓ_p such that $[\mathbf{z}'_n]_{n=1}^\infty$ is $(\sqrt{1+\epsilon})$-complemented in $g(\mathbf{w},p)$. For $n \in \mathbb{N}$, let $\mathbf{z}_n = T(\mathbf{z}'_n)$. It is clear that $(\mathbf{z}_n)_{n=1}^\infty$ is $(\sqrt{1+\epsilon})$-equivalent to $(\mathbf{z}'_n)_{n=1}^\infty$, and hence $(1+\epsilon)$-equivalent to $(\mathbf{f}_n)_{n=1}^\infty$. It follows that $[\mathbf{z}_n]_{n=1}^\infty$ is a subspace of Y that is $(1+\epsilon)$-isomorphic to ℓ_p and $(1+\epsilon)$-complemented in $g(\mathbf{w},p)$.

(vi) The block basic sequence

$$\mathbf{y}_n = \frac{\sum_{i=2^{n-1}}^{2^n-1} \mathbf{g}_i}{\sum_{i=1}^{2^{n-1}} w_i}, \quad n \in \mathbb{N}.$$

is normalized and uniformly null. By Lemma 2.4, each $\mathbf{y}_n$ is minimal. Thus, given $\epsilon > 0$, by Theorem 3.3 we can find a subsequence $(\mathbf{y}_{n_k})_{k=1}^\infty$ of $(\mathbf{y}_n)_{n=1}^\infty$ that is $(1+\epsilon)$-equivalent to the canonical ℓ_p-basis and such that $[\mathbf{y}_{n_k}]_{k=1}^\infty$ is $(1+\epsilon)$-complemented in $g(\mathbf{w},p)$. Let $P\colon g(\mathbf{w},p) \to [\mathbf{y}_{n_k}]_{k=1}^\infty$ be a projection with $\|P\| \leqslant 1+\epsilon$ and let $S\colon [\mathbf{y}_{n_k}]_{k=1}^\infty \to \ell_p$ be given by $S(\mathbf{y}_{n_k}) = \mathbf{f}_k$. We have $\|S\| \leqslant 1+\epsilon$. Pick $(\mathbf{z}_n)_{n=1}^\infty$ in $d(\mathbf{w},p)$ so that $\mathbf{y}_n = I_{d,g}(\mathbf{z}_n)$. The sequence $(\mathbf{z}_n)_{n=1}^\infty$ is also normalized. Then, by the $d(\mathbf{w},p)$-analog of Proposition 3.2 (see [**2**, Proposition 5]) $(\mathbf{z}_{n_k})_{n=1}^\infty$ is 1-dominated by the canonical basis of ℓ_p, that is, the operator $T \in \mathcal{L}(\ell_p, d(\mathbf{w},p))$ given by $T(\mathbf{f}_k) = \mathbf{z}_{n_k}$ for $k \in \mathbb{N}$ verifies $\|T\| \leqslant 1$. The composition operator $S \circ P \circ I_{d,g} \circ T$ is the identity map on ℓ_p. □

The longed-for features of Garling sequence spaces that were advertised in Section 1 follow now readily from Theorem 3.1.

Corollary 3.7. *Let $\mathbf{w} \in \mathcal{W}$ and $1 \leqslant p < \infty$. The space $g(\mathbf{w},p)$ is complementably homogeneous and uniformly complementably ℓ_p-saturated.*

Remark 6. There are $d(\mathbf{w},p)$-analogs of Theorem 3.1(ii) and Theorem 3.1(v) (cf. [**2**, Theorem 1; **4**, Corollary 12], respectively). Then, for $\mathbf{w} \in \mathcal{W}$ and $1 \leqslant p < \infty$, $d(\mathbf{w},p)$ is also complementably homogeneous and uniformly complementably ℓ_p-saturated.

4. *Uniqueness of subsymmetric basis in $g(\mathbf{w},p)$*

The spaces $g(\mathbf{w},p)$ do not have a unique unconditional basis since they are not isomorphic to any of the only three spaces that enjoy that property, namely c_0, ℓ_1, or ℓ_2 (see, for example, [**1**, Theorem 9.3.1]). The aim of this section is to show that they do have a unique subsymmetric basis.

Theorem 4.1. *Let $1 \leqslant p < \infty$ and $\mathbf{w} \in \mathcal{W}$. Suppose $(\mathbf{y}_n)_{n=1}^\infty$ is a subsymmetric basic sequence in $g(\mathbf{w},p)$. Then every subsymmetric basis for $Y = [\mathbf{y}_n]_{n=1}^\infty$ is equivalent to $(\mathbf{y}_n)_{n=1}^\infty$. In particular, every subsymmetric basis in $g(\mathbf{w},p)$ is equivalent to its canonical basis.*

Proof. Assume that Y is not isomorphic to ℓ_p, else we are done. Now let $(\mathbf{u}_n)_{n=1}^\infty$ be any subsymmetric basis for Y. Since neither $(\mathbf{y}_n)_{n=1}^\infty$ nor $(\mathbf{u}_n)_{n=1}^\infty$ are equivalent to the canonical basis of ℓ_1, Proposition 2.8 gives that both sequences are weakly null. Thus, we may apply Theorem 2.10 to obtain $(\mathbf{u}'_n)_{n=1}^\infty$ a normalized block basic sequence of $(\mathbf{y}_n)_{n=1}^\infty$ and $(\mathbf{y}'_n)_{n=1}^\infty$ a normalized block basic sequence of $(\mathbf{u}_n)_{n=1}^\infty$ with $(\mathbf{u}'_n)_{n=1}^\infty \approx (\mathbf{u}_n)_{n=1}^\infty$ and $(\mathbf{y}'_n)_{n=1}^\infty \approx (\mathbf{y}_n)_{n=1}^\infty$. By Theorem 3.3 neither $(\mathbf{u}'_n)_{n=1}^\infty$ nor $(\mathbf{y}'_n)_{n=1}^\infty$ are uniformly null since, otherwise, either $(\mathbf{u}_n)_{n=1}^\infty$ or $(\mathbf{y}_n)_{n=1}^\infty$ would be equivalent to the canonical basis of ℓ_p. Combining subsymmetry with Remark 4 it follows that, passing to subsequences if necessary, we may assume that both $(\mathbf{u}'_n)_{n=1}^\infty$ and $(\mathbf{y}'_n)_{n=1}^\infty$ verify the gliding hump property. Proposition 2.7(ii) yields

$$(\mathbf{y}_n)_{n=1}^\infty \lesssim (\mathbf{u}'_n)_{n=1}^\infty \approx (\mathbf{u}_n)_{n=1}^\infty \lesssim (\mathbf{y}'_n)_{n=1}^\infty \approx (\mathbf{y}_n)_{n=1}^\infty. \qquad \square$$

REMARK 7. It is also easy to check that Theorem 4.1 holds as well for $d(\mathbf{w},p)$ in place of $g(\mathbf{w},p)$ and so $d(\mathbf{w},p)$ admits a unique subsymmetric basis (which is also symmetric).

5. *Nonexistence of symmetric basis in* $g(\mathbf{w},p)$

In 2004, Sari [**12**, § 6] proved that there exist Banach spaces, called Tirilman spaces, admitting a subsymmetric basis but failing to admit a symmetric one (in fact, those spaces do not even contain symmetric basic sequences!). The aim of this section is, on the one hand, to characterize when $g(\mathbf{w},p)$ admits a symmetric basis, and on the other hand, to show that for a wide class of weights $\mathbf{w} \in \mathcal{W}$, the space $g(\mathbf{w},p)$ does not have such a basis despite having a unique subsymmetric basis.

THEOREM 5.1. *Let* $1 \leqslant p < \infty$ *and* $\mathbf{w} = (w_n)_{n=1}^\infty \in \mathcal{W}$. *The following are equivalent.*

(i) *The canonical basis is a symmetric basis in* $g(\mathbf{w},p)$.
(ii) $g(\mathbf{w},p)$ *admits a symmetric basis.*
(iii) *The canonical bases of* $g(\mathbf{w},p)$ *and* $d(\mathbf{w},p)$ *are equivalent, that is* $g(\mathbf{w},p) = d(\mathbf{w},p)$ *up to an equivalent norm.*
(iv) $g(\mathbf{w},p)$ *and* $d(\mathbf{w},p)$ *are isomorphic.*
(v) $d(\mathbf{w},p)$ *contains a subspace isomorphic to* $g(\mathbf{w},p)$.
(vi) *The inclusion map* $I_{d,g}\colon d(\mathbf{w},p) \to g(\mathbf{w},p)$ *preserves a copy of* $d(\mathbf{w},p)$ *that is complemented in* $g(\mathbf{w},p)$. *More precisely, there exists a subspace* Y *of* $d(\mathbf{w},p)$ *such that* $Y \approx d(\mathbf{w},p)$, $I_{d,g}|_Y$ *is bounded below, and* $I_{d,g}(Y)$ *is complemented in* $g(\mathbf{w},p)$.
(vii) *There is an operator* $T \in \mathcal{L}(d(\mathbf{w},p), g(\mathbf{w},p))$ *such that* $T \circ I_{d,g}$ *preserves a copy of* $d(\mathbf{w},p)$.

Proof. The implications (i) ⇒ (ii), (iii) ⇒ (i), (iii) ⇒ (iv), (iv) ⇒ (v), and (iii) ⇒ (vi) are all trivial.

(ii) ⇒ (i): A symmetric basis for $g(\mathbf{w},p)$ is also subsymmetric and so, by Theorem 4.1, equivalent to $(\mathbf{g}_n)_{n=1}^\infty$. Hence $(\mathbf{g}_n)_{n=1}^\infty$ is symmetric.

(i) ⇒ (iii): There is a uniform constant K such that $(\mathbf{g}_{\sigma(n)})_{n=1}^\infty \approx_K (\mathbf{g}_n)_{n=1}^\infty$ for all $\sigma \in \Pi$. Given $f = (a_n)_{n=1}^\infty \in c_0$ and $\sigma \in \Pi$ we have

$$\sum_{n=1}^\infty |a_{\sigma(n)}|^p w_n \leqslant \|(a_{\sigma(n)})_{n=1}^\infty\|_g^p \leqslant K\|f\|_g^p.$$

Taking the supremum over $\sigma \in \Pi$ we get $\|f\|_d \leqslant \|f\|_g$. In other words, $(\mathbf{d}_n)_{n=1}^\infty \lesssim_K (\mathbf{g}_n)_{n=1}^\infty$. The estimate $(\mathbf{g}_n)_{n=1}^\infty \lesssim_1 (\mathbf{d}_n)_{n=1}^\infty$ is a consequence of Proposition 2.2.

(v) ⇒ (iii): Let $(\mathbf{y}_n)_{n=1}^\infty$ be a basic sequence in $d(\mathbf{w},p)$ equivalent to the canonical basis of $g(\mathbf{w},p)$. By Corollary 2.9 $(\mathbf{y}_n)_{n=1}^\infty$ is weakly null. Then, via Theorem 2.10 and passing to a subsequence, we obtain that $(\mathbf{y}_n)_{n=1}^\infty$ is equivalent to a (semi-normalized) block basic sequence of $(\mathbf{d}_n)_{n=1}^\infty$. By [**2**, Corollary 2] and passing to a further subsequence, we have $(\mathbf{d}_n)_{n=1}^\infty \lesssim (\mathbf{y}_n)_{n=1}^\infty$. Since $(\mathbf{g}_n)_{n=1}^\infty \lesssim_1 (\mathbf{d}_n)_{n=1}^\infty$ we are done.

(vi) ⇒ (vii): Let P be a projection from $g(\mathbf{w},p)$ onto $I_{d,g}(Y)$ and S be the inverse map of $I_{d,g}|_Y$. We have that $S\circ P\circ I_{d,g}$ preserves a copy of $d(\mathbf{w},p)$.

(vii) ⇒ (iii): By [8, Theorem 5.5], the sequence $((T\circ I_{d,g})(\mathbf{d}_n))_{n=1}^\infty$ does not converge to zero in norm. This means that we can find a subsequence $(T(\mathbf{g}_{n_k}))_{k=1}^\infty$ that is semi-normalized. As $(\mathbf{g}_n)_{n=1}^\infty$ is weakly null by Corollary 2.9, so is $(T(\mathbf{g}_{n_k}))_{k=1}^\infty$. Pass to a further subsequence if necessary so that, by applying Theorem 2.10, $(T(\mathbf{g}_{n_k}))_{k=1}^\infty$ is (semi-normalized) and equivalent to a block basic sequence of $(\mathbf{d}_n)_{n=1}^\infty$. Again by [2, Corollary 2] we may assume, passing to a further subsequence, that $(T(\mathbf{g}_{n_k}))_{k=1}^\infty \gtrsim (\mathbf{d}_n)_{n=1}^\infty$. Hence,

$$(\mathbf{d}_n)_{n=1}^\infty \lesssim (T(\mathbf{g}_{n_k}))_{k=1}^\infty \lesssim (\mathbf{g}_{n_k})_{k=1}^\infty \approx (\mathbf{g}_n)_{n=1}^\infty \lesssim (\mathbf{d}_n)_{n=1}^\infty. \qquad \square$$

Remark 8. Note that, by Theorem 3.1(viii), the inclussion map $I_{d,g}\colon d(\mathbf{w},p)\to g(\mathbf{w},p)$ is never strictly singular.

Remark 9. Observe that $g(\mathbf{w},p)$ is never isometric in the natural way to $d(\mathbf{w},p)$. Just consider the minimum $k\in\mathbb{N}$ such that $w_{k+1}<1$, so that, letting

$$0<\alpha<\left(\frac{1-w_{k+1}}{k}\right)^{1/p}$$

and $f=\mathbf{e}_{k+1}+\alpha\sum_{n=1}^k \mathbf{e}_n$ we have $\|f\|_g = 1 < \|f\|_d$.

Before constructing weights for which $(\mathbf{g}_n)_{n=1}^\infty$ is not symmetric we need a few definitions.

Definition 4. Let $\mathbf{w}=(w_n)_{n=1}^\infty$ be a weight.

(a) $\mathbf{w}$ is said to be *essentially decreasing* if $\sup_{k\leqslant n} w_n/w_k<\infty$.

(b) $\mathbf{w}$ is said to be *regular* if

$$\sup_m \frac{1}{mw_m}\sum_{n=1}^m w_n<\infty.$$

(c) The *conjugate weight* $\mathbf{w}^*=(w_n^*)_{n=1}^\infty$ is defined by

$$w_n^*=\frac{1}{nw_n},\qquad n\in\mathbb{N}.$$

(d) $\mathbf{w}$ is said to be *bi-regular* if both $\mathbf{w}$ and $\mathbf{w}^*$ are regular weights.

Lemma 5.2. *A weight $\mathbf{w}$ is essentially decreasing if and only if there exists a normalized nonincreasing weight $\mathbf{v}$ such that $\mathbf{w}\approx\mathbf{v}$.*

Proof. If $(w_n)_{n=1}^\infty$ is an essentially decreasing weight, then the weight $\mathbf{v}=(v_n)_{n=1}^\infty$ given by $v_n=(\inf_{k\leqslant n} w_k)/w_1$ does the trick. The converse is (also) trivial. $\square$

Lemma 5.3. *If a weight $\mathbf{w}$ is regular then $\mathbf{w}\notin\ell_1$ and $\mathbf{w}^*\in c_0$.*

Proof. Assume, by contradiction, that $\mathbf{w}=(w_n)_{n=1}^\infty\in\ell_1$, that is, $\sum_{n=1}^\infty w_n\approx 1$. Then, $1/m\lesssim w_m$ for $m\in\mathbb{N}$, and so $\mathbf{w}\notin\ell_1$. Now assume, again by contradiction, that $\mathbf{w}^*=(w_n)_{n=1}^\infty\notin c_0$. Then there is an infinite subset $A\subseteq\mathbb{N}$ with $nw_n\approx 1$ for $n\in A$. Consequently, $\sum_{n=1}^m w_n\lesssim 1$ for $m\in A$ and so $\mathbf{w}\in\ell_1$. $\square$

Lemma 5.4. *Let $\mathbf{w}=(w_n)_{n=1}^\infty$ be an essentially decreasing regular weight. Then:*

(i) *$mw_m\approx\sum_{n=1}^m w_n$ for $m\in\mathbb{N}$;*

(ii) $\mathbf{w}^*$ *also is essentially decreasing.*

Proof. (i) Let $C = \sup_{k \leqslant n} w_n/w_k$. For any $m \in \mathbb{N}$ we have

$$m w_m \leqslant C \sum_{n=1}^{m} w_n.$$

(ii) By (i), $\mathbf{w}^* \approx (1/(\sum_{n=1}^{m} w_n))_{m=1}^{\infty}$. We infer from Lemma 5.3 that $\mathbf{w}^*$ is essentially decreasing. □

LEMMA 5.5. *A bi-regular weight is essentially decreasing if and only if* $m w_m \lesssim \sum_{n=1}^{m} w_n$ *for* $m \in \mathbb{N}$.

Proof. The 'only if' part is a consequence of Lemma 5.4. Assume $m w_m \approx \sum_{n=1}^{m} w_n$ for $m \in \mathbb{N}$. By Lemma 5.3 $\mathbf{w}^*$ is essentially decreasing. Then, by Lemma 5.4, $\mathbf{w} = \mathbf{w}^{**}$ is essentially decreasing. □

PROPOSITION 5.6. *Let* $\mathbf{w} = (w_n)_{n=1}^{\infty}$ *be a weight. The following are equivalent:*

(i) $\mathbf{w}$ *is essentially decreasing and bi-regular;*
(ii) $\mathbf{w}$ *is bi-regular and* $\mathbf{w}^*$ *is essentially decreasing;*
(iii) $\mathbf{w}$ *is essentially decreasing and*

$$\sup_m \sum_{n=1}^{m} \frac{w_{m+1-n}}{n w_n} < \infty. \tag{5.1}$$

(iv) $\mathbf{w}^*$ *is essentially decreasing and* (5.1) *holds.*

Proof. (i) ⇒ (ii) is straightforward from Lemma 5.4(ii), while (ii) ⇒ (i) is a consequence of (i) ⇒ (ii) and the fact that $\mathbf{w}^{**} = \mathbf{w}$.

(i) ⇒ (iii): Let $\mathbf{w}^* = (w_n^*)_{n=1}^{\infty}$ and put

$$C = \sup_{k \geqslant n} \frac{w_k}{w_n} < \infty, \quad D = \sup_m w_m^* \sum_{n=1}^{m} w_n < \infty,$$

$$C^* = \sup_{k \geqslant n} \frac{w_k^*}{w_n^*} < \infty, \quad \text{and} \quad D^* = \sup_m w_m \sum_{n=1}^{m} w_n^* < \infty.$$

Let $m \in \mathbb{N}$ and choose $r \in \mathbb{N}$ such that $2r - 1 \leqslant m \leqslant 2r$. We have

$$\begin{aligned}
\sum_{n=1}^{m} \frac{w_{m+1-n}}{n w_n} &= \sum_{n=1}^{r} w_{m+1-n} w_n^* + \sum_{n=r+1}^{m} w_{m+1-n} w_n^* \\
&\leqslant C w_r \sum_{n=1}^{r} w_n^* + C^* w_r^* \sum_{n=r+1}^{m} w_{m+1-n} \\
&\leqslant C w_r \sum_{n=1}^{r} w_n^* + C^* w_r^* \sum_{n=1}^{r} w_n \\
&= C D^* + C^* D.
\end{aligned}$$

as desired.

(iii) ⇒ (i): Let $\mathbf{w}^* = (w_n^*)_{n=1}^\infty$ and put

$$C = \sup_{k \geqslant n} \frac{w_k}{w_n} < \infty \quad \text{and} \quad D = \sup_m w_m^* \sum_{n=1}^m w_n < \infty.$$

Let us prove that $\mathbf{w}^*$ is essentially decreasing. Let k and n be integers with $1 \leqslant n \leqslant k$. Since the function η given by $\eta(x) := \log(1+x)/x$ is decreasing on $(0,\infty)$,

$$\begin{aligned} D &\geqslant \sum_{j=1}^{n+k-1} \frac{w_{n+k-j}}{j w_j} \geqslant \sum_{j=k}^{n+k-1} \frac{w_{n+k-j}}{j w_j} \geqslant \frac{1}{C^2} \frac{w_n}{w_k} \sum_{j=k}^{n+k-1} \frac{1}{j} \\ &\geqslant \frac{1}{C^2} \frac{w_n}{w_k} \int_k^{n+k} \frac{dx}{x} = \frac{1}{C^2} \eta(n/k) \frac{w_k^*}{w_n^*} \geqslant \frac{1}{C^2} \eta(1) \frac{w_k^*}{w_n^*}. \end{aligned}$$

Hence

$$w_k^* \leqslant (\log 2)^{-1} D C^2 w_n^*, \quad n \leqslant k.$$

Now, for any $m \in \mathbb{N}$,

$$\frac{1}{m w_m} \sum_{n=1}^m w_n = \frac{1}{m w_m} \sum_{n=1}^m w_{m+1-n} \leqslant \frac{DC^2}{\log 2} \sum_{n=1}^m \frac{w_{m+1-n}}{n w_n} \leqslant \frac{D^2 C^2}{\log 2}$$

and

$$\frac{1}{m w_m^*} \sum_{n=1}^m w_n^* = w_m \sum_{n=1}^m \frac{1}{n w_n} \leqslant C \sum_{n=1}^m \frac{w_{m+1-n}}{n w_n} \leqslant DC.$$

The equivalence (ii) ⇔ (iv) is a consequence of (i) ⇔ (iii) and the simple fact that

$$\sum_{n=1}^m \frac{w_{m+1-n}}{n w_n} = \sum_{n=1}^m \frac{w_{m+1-n}^*}{n w_n^*},$$

where $\mathbf{w}^* = (w_n^*)_{n=1}^\infty$. □

THEOREM 5.7. *Let* $\mathbf{w}$ *be a normalized nonincreasing bi-regular weight and* $1 \leqslant p < \infty$. *Then* $\mathbf{w} \in \mathcal{W}$ *and the canonical basis is not a symmetric basis for* $g(\mathbf{w}, p)$.

Proof. The fact that $\mathbf{w} \in \mathcal{W}$ is straightforward from Lemma 5.3. For each $r \in \mathbb{N}$, consider sequences $f^{(r)} = (a_{r,n})_{n=1}^\infty$ and $g^{(r)} = (b_{r,n})_{n=1}^\infty$ given by

$$\begin{cases} a_{r,n} = b_{r,n} = 0 & \text{if } n > r, \\ a_{r,n} = b_{r,m+1-n} = (n w_n)^{-1/p} & \text{if } n \leqslant r. \end{cases}$$

Note that $g^{(r)}$ is a rearrangement of $f^{(r)}$. We have

$$\sup_{r \in \mathbb{N}} \|f^{(r)}\|_g \geqslant \sup_{r \in \mathbb{N}} \|f^{(r)}\|_p = \sup_{r \in \mathbb{N}} \left(\sum_{n=1}^r \frac{1}{n} \right)^{1/p} = \infty.$$

By Lemma 5.4(ii), $\mathbf{w}^* = (w_n^*)_{n=1}^\infty$ is an essentially decreasing sequence. Let $r \in \mathbb{N}$ and $\phi \in \mathcal{O}$. Denote by $m(r,\phi)$ the largest integer m such that $\phi(m) \leqslant r$. We have $\phi(n) \leqslant n + r - m$ for $1 \leqslant n \leqslant m$. Therefore,

$$\sup_{r \in \mathbb{N}} \|g^{(r)}\|_g^p = \sup_{r \in \mathbb{N},\, \phi \in \mathcal{O}} \sum_{n=1}^\infty |b_{r,\phi(n)}|^p w_n$$

$$= \sup_{r\in\mathbb{N},\,\phi\in\mathcal{O}} \sum_{n=1}^{m(r,\phi)} w_n w^*_{r+1-\phi(n)}$$

$$\lesssim \sup_{r\in\mathbb{N},\,\phi\in\mathcal{O}} \sum_{n=1}^{m(r,\phi)} w_n w^*_{m(r,\phi)+1-n}$$

$$= \sup_{m\in\mathbb{N}} \sum_{n=1}^{m} w_n w^*_{m+1-n}$$

$$= \sup_{m\in\mathbb{N}} \sum_{n=1}^{m} \frac{w_{m+1-n}}{n w_n}.$$

Proposition 5.6 yields that the canonical basis is not symmetric. □

To give relief to Theorem 5.7 we need to exhibit examples of normalized nonincreasing bi-regular weights. To that end, in light of Lemma 5.3, if suffices to give examples of essentially decreasing bi-regular weights. Next, we give a very general procedure for constructing this kind of weights. For instance, we will prove that, for $0 < a < 1$ and $b \in \mathbb{R}$, $((\log(1+n))^b n^{-a})_{n=1}^\infty$ is an essentially decreasing bi-regular weight.

Definition 5. A weight $(u_n)_{n=1}^\infty$ is said to be *asymptotically constant* if

$$\lim_j \sup_{2^{j-1}\leqslant k,n\leqslant 2^j} \frac{u_n}{u_k} = 1.$$

Proposition 5.8. *Let $0 < a < 1$ and $(u_n)_{n=1}^\infty$ be an asymptotically constant weight. Then $\mathbf{w} = (n^{-a}u_n)_{n=1}^\infty$ is an essentially decreasing bi-regular weight.*

Proof. Let $b = 1 - a$ and consider $\mathbf{v} = (v_n)_{n=1}^\infty$ given by $v_n = 1/u_n$. We have $\mathbf{w}^* = (n^{-b}v_n)_{n=1}^\infty$, with $0 < b < 1$ and $\mathbf{v}$ asymptotically constant. Then, taking into account Lemma 5.5 it suffices to prove

$$\sum_{n=1}^{m} n^{-a}u_n \approx t_m := m^{1-a}u_m, \quad m \in \mathbb{N}.$$

To that end, put

$$c_j = \sup_{2^{j-1}\leqslant n<2^j} u_n, \quad j \in \mathbb{N},$$

and let $(a_n)_{n=1}^\infty$ be the unique weight such that $a_{2^{j-1}} = c_j$ for $j \in \mathbb{N}$ and $(a_n)_{n=2^{j-1}}^{2^j}$ is in arithmetic progression. Define $(s_n)_{n=0}^\infty$ by $s_0 = 0$ and $s_n = n^{1-a}a_n$ for $n \in \mathbb{N}$. Since

$$\lim_j \frac{c_{j+1}}{c_j} = 1,$$

we have $t_n \approx s_n$ for $n \in \mathbb{N}$. Thus we need only to show that

$$s_n - s_{n-1} \approx n^{-a}u_n, \quad n \in \mathbb{N}.$$

Given $n \geqslant 2$, pick $j = j(n) \in \mathbb{N}$ such that $2^{j-1} < n \leqslant 2^j$. We have

$$\frac{s_n - s_{n-1}}{n^{-a}u_n} = \frac{n(a_n - a_{n-1})}{u_n} + \frac{n^{1-a} - (n-1)^{1-a}}{n^{-a}} \frac{a_{n-1}}{u_n}$$

$$= \frac{n}{2^{j-1}} \frac{c_j}{u_n} + n \left(1 - \left(1 - \frac{1}{n}\right)^{1-a}\right) \frac{a_{n-1}}{u_n}$$
$$\approx 1. \qquad \square$$

We close with a most natural question that our work leaves open:

QUESTION 1. Do there exist $\mathbf{w} \in \mathcal{W}$ and $1 \leqslant p < \infty$ so that $g(\mathbf{w}, p)$ admits a symmetric basis?

Acknowledgement. Ben Wallis thanks Bünyamin Sarı for his helpful comments.

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