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ON THE DERIVABILITY OF LIPSCHITZ FUNCTIONS

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We show that given any Borel measure on $\mathbb{R}$, every Lipschitz function is μ-a.e. differentiable with respect to μ.

1. Introduction.

One of the earliest applications of measure theory to the differentiation of functions is Lebesgue's result to the effect that any real valued measurable function of bounded variation on $\mathbb{R}$, is differentiable almost everywhere (with respect to Lebesgue measure). In particular, this is the case for Lipschitz functions. On the other hand, given any set $E \subset \mathbb{R}$ of measure zero, there exists a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that E is contained in the set of points where f is not differentiable (more precisely, Z. Zahorski proved in [4], Theorems 3 and 4, that A is the set of points of nondifferentiability of a Lipschitz function iff $A \subset \mathbb{R}$ is a $G_{\delta\sigma}$ set of Lebesgue measure zero). The situation however is markedly different in the plane; while Lipschitz functions are differentiable a.e. in $\mathbb{R}^n$ by Rademacher's Theorem, D. Preiss has shown (cf. [2]) that there exists a measure zero G_δ subset B of the plane such that every Lipschitz function has a point of differentiability on B.

Here we shall study the situation in the one dimensional case, but replacing the derivative df/dx of f with respect to Lebesgue measure, with the derivative $df/d\mu$ of f with respect to an arbitrary Borel measure μ on

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$\mathbb{R}$. Due to the fact that $\mathbb{R}$ is an ordered topological space, the general case can be reduced to the same type of argument used for Lebesgue measure; hence, if f is Lipschitz then $df/d\mu$ exists μ-a.e.. In the other direction, it is not true that if $\mu(E) = 0$, then one can find a Lipschitz function f such that on E the derivative $df/d\mu$ does not exist. In fact, given any countable set $E \subset \mathbb{R}$, there is a locally finite, continuous Borel measure μ (so $\mu(E) = 0$) such that for every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, $df/d\mu \equiv 0$ on E.

2. Definitions and results.

Let μ be a Borel measure on $\mathbb{R}$, and let D be either $\mathbb{R}$ or $[a, b]$. A Borel function $f : D \to \mathbb{R}$ is differentiable at $x \in D$ with respect to μ if both limits

$$\lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{\mu([x, x+h])} \text{ and } \lim_{h \downarrow 0} \frac{f(x) - f(x-h)}{\mu([x-h, x])}$$

exist and are equal. When $x = a$ or $x = b$ only the corresponding one-sided limit is considered. By the usual conventions, if $\mu([x, x+h]) = \infty$ then $(f(x+h) - f(x))/\mu([x, x+h]) = 0$, while if $\mu([x, x+h]) = 0$, $(f(x+h) - f(x))/\mu([x, x+h])$ is undefined, and similarly for the left limits. A Borel measure on $X \subset \mathbb{R}$ is *continuous* if for every point $x \in X$, $\mu(\{x\}) = 0$. The outer measure generated by μ is denoted by μ^*.

The following version of Vitali's lemma is due to de Guzmán (cf.[1]). By an interval, we always mean a nondegenerate interval.

LEMMA 2.1 (Vitali's covering lemma for finite measures). *Let μ be a finite Borel measure on $[a, b]$, let $E \subset [a, b]$, and let $\mathcal{V}$ be a Vitali covering of E by closed intervals. Then for every $\epsilon > 0$ there exists a finite disjoint collection $I_1, \dots, I_n$ of intervals from $\mathcal{V}$ such that $\mu^*(E \setminus \cup_1^n I_j) < \epsilon$.*

THEOREM 2.2. *Let $f : D \to \mathbb{R}$ be continuous and monotone, where D is either $\mathbb{R}$ or interval $[a, b]$. Then for every Borel measure μ on D, $df/d\mu$ exists μ -a.e..*

Proof: Since a countable union of measure zero sets has measure zero, it suffices to consider the case $D := [a, b]$. Set $U := \cup\{I \subset D : I$ is an interval with $\mu(I) < \infty\}$, and $B := \{x \in D :$ there is a relatively open interval $J \subset D$ with $x \in J$ and $\mu(J) < \infty\}$. Note first that if $y \subset U^c$, then for every $h > 0$ we have

$$\mu([y-h, y]) = \mu([y, y+h]) = \infty,$$

so on U^c every real valued Borel function g satisfies $dg/d\mu \equiv 0$. We claim that $U \setminus B$ is countable. Write $U \setminus B = L \cup R$, where $L := \{y \in U \setminus B :$ there exists an $h_y > 0$ with $\mu([y-h_y, y]) < \infty\}$, and $R := \{y \in U \setminus B :$ there exists an $h_y > 0$ with $\mu([y, y+h_y]) < \infty\}$. If x, y are different points in L (resp. in R), then $[x-h_x, x] \cap [y-h_y, y] = \emptyset$ (resp. $[x, x+h_x] \cap [y, y+h_y] = \emptyset$). Hence both L and R are countable. Now if $x \in L \cup R$ is such that $\mu(\{x\}) > 0$, then by the continuity of f, $\frac{df}{d\mu}(x) = 0$, while $\mu((L \cup R) \setminus \{x \in L \cup R : \mu(\{x\}) > 0\}) = 0$. Hence all is left to do is to show that $df/d\mu$ exists μ-a.e. on B. Since μ is locally finite on B, it is enough to consider the case where μ is a finite measure on an interval I. Set $C := \{J \subset I : J$ is an interval with $\mu(J) = 0\}$ and $A := \cup C$. We claim that $\mu(A) = 0$. Write $A' := \cup\{\text{ int } J : J \in C\}$, where int J denotes the interior of J. Then there is a countable subcollection of open intervals $\{\text{int } J_n\}$ such that $A' = \cup_n \text{int } J_n$. Hence $\mu(A') = 0$. Suppose next that $x, y \in A \setminus A'$ are left endpoints of different intervals form C, say J_x and J_y respectively. Then int $J_x \cap$ int $J_y = \emptyset$. Using an analogous argument for points in $A \setminus A'$ which are right endpoints of intervals from C, we conclude that $A \setminus A'$ is countable, thereby proving the claim.

By the definition of A, for every $x \in I \setminus A$ and every $h > 0$, $\mu([x-h, x]) > 0$ and $\mu([x, x+h]) > 0$. Assume f is increasing. Now the fact that $df/d\mu$ exists μ-a.e. on $I \setminus A$, and hence on I, is proven following the same steps as in the case of Lebesgue measure (cf. for instance [3], Thm. 2, pp. 96-98), but using Lemma 2.1 rather than the usual Vitali's covering lemma, and recalling that on the set $\{x \in I : \mu(\{x\}) > 0\}$ the derivative $df/d\mu$ exists (since it is identically zero). ■

In terms of measures, the preceding result tells us that if v is a continuous, locally finite measure, then $dv/d\mu$ exists μ-a.e., for any such v can be obtained from a continuous increasing function f by setting $v((x, y]) := f(y) - f(x)$, and then applying Caratheodory's outer measure procedure.

Since a Lipschitz function on $[a, b]$ is a continuous function of bounded variation, it can be expressed as the difference of two monotone continuous functions. So we obtain the following corollary.

COROLLARY 2.3. *Let $f : D \to \mathbb{R}$ be Lipschitz if $D = [a, b]$, or locally Lipschitz if $D = \mathbb{R}$. Then for every Borel measure μ on D, $df/d\mu$ exists μ-a.e..*

EXAMPLE 2.4. *Let I be an interval and let $E \subset I$ be countable. Then there exists a locally finite, continuous Borel measure μ on I, such that for every Lipschitz function $f : I \to \mathbb{R}, df/d\mu \equiv$ on E.*

Proof: Set $g(0) = 0$ and $g(x) = 1/\sqrt{|x|}$ on $\mathbb{R} \setminus \{0\}$. Let $\{x_n\}_{n=1}^{\infty}$ be a countable subset of the interval I, and write $g_n(x) := 2^{-n} g(x - x_n)\chi_I(x)$. Clearly $w := \sum_1^{\infty} g_n \in L^1_{loc}(I)$ (and if I is bounded, $w \in L^1(I)$), so the measure μ defined by $\mu(B) := \int_B w(x)dx$ for every Borel set $B \subset I$, is absolutely continuous and locally finite. Let $f : I \to \mathbb{R}$ be Lipschitz. Then for every $n \geq 1$,

$$\left| \lim_{h \downarrow 0} \frac{f(x_n + h) - f(x_n)}{\mu([x_n, x_n + h])} \right| \leq \lim_{h \downarrow 0} \frac{h \mathrm{Lip}(f)}{\int_{x_n}^{x_n+h} g_n} = 0.$$

Likewise, the left limits are also zero at each x_n. ∎

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