

# Fourier-Padé Approximants for Nikishin Systems

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**Abstract** We study type I Fourier–Padé approximation for certain systems of functions formed by the Cauchy transform of finite Borel measures supported on bounded intervals of the real line. This construction is similar to type I Hermite–Padé approximation. Instead of power series expansions of the functions in the system, we take their development in a series of orthogonal polynomials. We give the exact rate of convergence of the corresponding approximants. The answer is expressed in terms of the extremal solution of an associated vector-valued equilibrium problem for the logarithmic potential.

**Keywords** Rational approximation  $\cdot$  Multipoint Padé approximation  $\cdot$  Fourier–Padé approximation  $\cdot$  Rate of convergence  $\cdot$  Potential theory

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## 1 Introduction

We study type I Fourier–Padé approximation for vector-valued analytic functions formed by Nikishin systems. Fourier–Padé approximation of general analytic functions was first considered by S.P. Suetin. In [9, 10], he obtained convergence for

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row sequences of Fourier–Padé approximants, extending to this setting the classical Montessus de Ballore Theorem. Diagonal sequences of Fourier–Padé approximants of Cauchy transforms of measures supported on the real line were studied by A.A. Gonchar, E.A. Rakhmanov, and S.P. Suetin. In [4], they describe the rate of convergence of such approximants in terms of the equilibrium measure of an associated potential theoretic problem.

M. Bello, G. López and J. Mínguez studied in [1] type II Fourier–Padé approximation for Angelesco systems of functions. Angelesco systems are formed by the Cauchy transform of a finite set of Borel measures supported on non-intersecting intervals of the real line. They obtained the rate of convergence of the error of approximation by means of linear and non-linear Fourier–Padé approximants. Here, the solution depends on a vector-valued equilibrium problem for the logarithmic potential.

Let  $S = (s_1, ..., s_m)$  be a system of finite Borel measures with constant sign and bounded support consisting of infinitely many points contained in the real line. Let  $\widehat{S} = (\widehat{s_1}, ..., \widehat{s_m})$  be the corresponding system of Markov functions, that is,

$$\widehat{s}_k(z) = \int \frac{ds_k(x)}{z - x}, \quad k = 1, \dots, m.$$
(1)

When the support of these measures lie in non-intersecting intervals, one obtains an Angelesco system.

Another special system of Markov functions was introduced by E.M. Nikishin in [5]. They constitute an important model class of functions in the theory of multiple orthogonal polynomials and simultaneous rational approximations, since many classical results of these theories have found their corresponding analogues, and thus have attracted increasing attention in recent decades. Let us define them.

Let  $\sigma_1$ ,  $\sigma_2$  be two measures with constant sign supported on  $\mathbb{R}$ , and let  $\Delta_1$ ,  $\Delta_2$  denote the smallest intervals containing their supports, supp( $\sigma_1$ ) and supp( $\sigma_2$ ), respectively. We write Co(supp( $\sigma_i$ )) =  $\Delta_i$ . Assume that  $\Delta_1 \cap \Delta_2 = \emptyset$ , and define

$$\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x-t} d\sigma_1(x) = \widehat{\sigma}_2(x) d\sigma_1(x).$$

Therefore,  $\langle \sigma_1, \sigma_2 \rangle$  is a measure with constant sign and support equal to that of  $\sigma_1$ .

For a system of intervals  $\Delta_1, \ldots, \Delta_m$  contained in  $\mathbb{R}$  satisfying  $\Delta_j \cap \Delta_{j+1} = \emptyset$ ,  $j = 1, \ldots, m-1$ , and finite Borel measures  $\sigma_1, \ldots, \sigma_m$  with constant sign in  $\text{Co}(\text{supp}(\sigma_j)) = \Delta_j$  and such that each one has infinitely many points in its support, we define recursively

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_j \rangle = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_j \rangle \rangle, \quad j = 2, \ldots, m.$$

We say that  $S = (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ , where

$$s_1 = \langle \sigma_1 \rangle = \sigma_1, \qquad s_2 = \langle \sigma_1, \sigma_2 \rangle, \dots, \qquad s_m = \langle \sigma_1, \dots, \sigma_m \rangle$$

is the Nikishin system of measures generated by  $(\sigma_1, \ldots, \sigma_m)$ . In the sequel, the system  $(\sigma_1, \ldots, \sigma_m)$  is such that  $\Delta_j \cap \Delta_{j+1} = \emptyset$ ,  $j = 1, \ldots, m - 1$ .

Notice that all the measures in a Nikishin system have the same support, namely  $supp(\sigma_1)$ . Take an arbitrary Nikishin system of measures  $S = (s_1, \ldots, s_m)$ , and let  $\widehat{S} = (\widehat{s_1}, \ldots, \widehat{s_m})$  be the corresponding Nikishin system of Markov functions.

Let  $\sigma_0$  be a finite Borel measure with constant sign and bounded support consisting of infinitely many points contained in an interval  $\Delta_0$  such that

$$\Delta_0 \cap \Delta_1 = \emptyset.$$

Consider the sequence  $\{\ell_j\}$ ,  $j \in \mathbb{Z}_+ = \{0, 1, 2, ...\}$ , of orthonormal polynomials with respect to  $\sigma_0$  with positive leading coefficient.

For  $\mathbf{n} = (n_0, n_1, \dots, n_m) \in \mathbb{Z}_+^{m+1}$  we denote  $|\mathbf{n}| = n_0 + n_1 + \dots + n_m$ . Let  $A_{\mathbf{n},0}, A_{\mathbf{n},1}, \dots, A_{\mathbf{n},m}$  be polynomials such that:

(i)  $\deg(A_{\mathbf{n},j}) \le n_j - 1, j = 0, \dots, m$ , not all identically equal to zero.

(ii) For  $k = 0, ..., |\mathbf{n}| - 2$ 

$$\int \left( A_{\mathbf{n},0}(x) + \sum_{j=1}^{m} A_{\mathbf{n},j}(x) \widehat{s}_j(x) \right) \ell_k(x) \, d\sigma_0(x) = 0. \tag{2}$$

Finding  $A_{\mathbf{n},0}, \ldots, A_{\mathbf{n},m}$  reduces to solving a homogeneous linear system of  $|\mathbf{n}| - 1$  equations on  $|\mathbf{n}|$  unknowns, so a non-trivial solution is guaranteed. The solution may not be unique. We call  $(A_{\mathbf{n},0}, A_{\mathbf{n},1}, \ldots, A_{\mathbf{n},m})$  a type I Fourier–Padé approximant of  $(\hat{s}_1, \ldots, \hat{s}_m)$  with respect to the multi-index  $\mathbf{n}$ .

Uniqueness is a desirable condition. A multi-index  $\mathbf{n} = (n_0, n_1, ..., n_m) \in \mathbb{Z}_+^{m+1}$  is said to be normal if every solution to (i)–(ii) satisfies deg  $A_{\mathbf{n},j} = n_j - 1, j = 0, ..., m$ . If an index is normal, it is easy to verify that these polynomials are uniquely determined (except for a common factor). Set

$$\mathbb{Z}^{m+1}_+(\bullet) = \left\{ \mathbf{n} \in \mathbb{Z}^{m+1}_+ : n_0 \ge n_1 \ge n_2 \ge \cdots \ge n_m \right\}.$$

In Proposition 1, we prove that all multi-indices in  $\mathbb{Z}^{m+1}_+(\bullet)$  are normal. We normalize  $(A_{\mathbf{n},0}, A_{\mathbf{n},1}, \ldots, A_{\mathbf{n},m})$  so that  $A_{\mathbf{n},m}$  is monic.

Theorem 1 gives the rate of convergence of the  $|\mathbf{n}|$ -th root of the linear forms

$$L_{\mathbf{n},0}(z) = A_{\mathbf{n},0}(z) + \sum_{j=1}^{m} A_{\mathbf{n},j}(z)\widehat{s}_{j}(z)$$

under mild conditions on the sequence of multi-indices, assuming that the measures  $\sigma_j$ , j = 0, ..., m, belong to the class **Reg** of regular measures. For different equivalent forms of defining regular measures, see Sects. 3.1 to 3.3 in [8]. In particular,  $\sigma_0 \in \mathbf{Reg}$  if and only if

$$\lim_{n} \left| \ell_n(z) \right|^{1/n} = \exp \left\{ g_{\Omega_0}(z;\infty) \right\},\,$$

uniformly on compact subsets of the complement of the smallest interval containing supp( $\sigma_0$ ), where  $g_{\Omega_0}(\cdot; \infty)$  denotes the Green's function for the region  $\Omega_0 = \mathbb{C} \setminus \text{supp}(\sigma_0)$  with singularity at  $\infty$ . Analogously, one defines regularity for the other measures  $\sigma_1, \ldots, \sigma_m$ . In the sequel, we write  $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \mathbf{Reg}$  to mean that  $\sigma_k \in \mathbf{Reg}, k = 0, \ldots, m$ . The system  $(\sigma_1, \ldots, \sigma_m)$  will be used to construct the Nikishin system of functions, whereas  $\sigma_0$  will determine the system of orthogonal polynomials with respect to which the Fourier expansions are taken. Before stating Theorem 1, we need to introduce some notation and results from potential theory.

Let  $F_k$ , k = 0, 1, ..., N, be (not necessarily distinct) closed bounded intervals of the real line and  $C = (c_{j,k})$  a real, positive definite, symmetric matrix of order N + 1. C will be called the interaction matrix. By  $\mathcal{M}(F_k)$  we denote the class of all finite, positive, Borel measures with compact support consisting of an infinite set of points contained in  $F_k$ , and  $\mathcal{M}_1(F_k)$  is the subclass of probability measures of  $\mathcal{M}(F_k)$ . Set

$$\mathcal{M}_1 = \mathcal{M}_1(F_0) \times \cdots \times \mathcal{M}_1(F_N).$$

Given a vector measure  $\mu = (\mu_0, \mu_1, \dots, \mu_N) \in \mathcal{M}_1$  and  $j = 0, 1, \dots, N$ , we define the combined potential

$$W_{j}^{\mu}(x) = \sum_{k=0}^{N} c_{j,k} V^{\mu_{k}}(x), \quad x \in \Delta_{j},$$
(3)

where

$$V^{\mu_k}(x) = \int \log \frac{1}{|x-t|} d\mu_k(t)$$

denotes the standard logarithmic potential of  $\mu_k$ . We denote

$$\omega_j^{\mu} = \inf \{ W_j^{\mu}(x) : x \in F_j \}, \quad j = 0, 1, \dots, N.$$

In Chap. 5 of [7], the authors prove (we state the result in a form convenient for our purpose):

**Lemma 1** Let C be a real, positive definite, symmetric matrix of order N + 1. If there exists  $\overline{\mu} = (\overline{\mu}_0, \overline{\mu}_1, \dots, \overline{\mu}_N) \in \mathcal{M}_1$  such that for each  $j = 0, 1, \dots, N$ 

$$W_i^{\overline{\mu}}(x) = \omega_i^{\overline{\mu}}, \quad x \in supp(\overline{\mu}_j),$$

then  $\overline{\mu}$  is unique. Moreover, if  $c_{j,k} \ge 0$  when  $F_j \cap F_k \ne \emptyset$ , then  $\overline{\mu}$  exists.

An explanation of why this lemma follows from the results in [7] is contained in Sect. 4 of [1]. The vector measure  $\overline{\mu} \in \mathcal{M}_1$  is called the equilibrium solution for the vector potential problem determined by the interaction matrix C on the system of intervals  $F_j$ , j = 0, 1, ..., N.

Let  $\Lambda = \Lambda(p_0, p_1, ..., p_m) \subset \mathbb{Z}_+^{m+1}(\bullet)$  be an infinite sequence of distinct multiindices such that

$$\lim_{\mathbf{n}\in\Lambda}\frac{n_j}{|\mathbf{n}|}=p_j\in(0,1),\quad j=0,\ldots,m.$$

Obviously,  $p_0 \ge p_1 \ge \cdots \ge p_m$  and  $\sum_{j=0}^m p_j = 1$ .

Set

$$P_j = \sum_{k=j}^m p_k$$

Let us define the interaction matrix C, which is relevant for the rest of the paper. Take

$$C = \begin{pmatrix} 1 & -\frac{P_1}{2} & 0 & \dots & 0 & 0 \\ -\frac{P_1}{2} & P_1^2 & -\frac{P_1P_2}{2} & \dots & 0 & 0 \\ 0 & -\frac{P_1P_2}{2} & P_2^2 & \dots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & P_{m-1}^2 & -\frac{P_{m-1}P_m}{2} \\ 0 & 0 & 0 & \dots & -\frac{P_{m-1}P_m}{2} & P_m^2 \end{pmatrix}$$
(4)

This matrix satisfies all the assumptions of Lemma 1 on the system of intervals  $F_j = \Delta_j, j = 0, ..., m$ , including  $c_{j,k} \ge 0$  when  $F_j \cap F_k \ne \emptyset$ , and it is positive definite because the principal section  $C_r, r = 0, ..., m$  of C satisfies

$$\det(\mathcal{C}_r) = P_1^2 \dots P_r^2 \det \begin{pmatrix} 1 & -\frac{1}{2} & 0 & \dots & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \dots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \dots & -\frac{1}{2} & 1 \end{pmatrix}_{(r+1)\times(r+1)} > 0.$$

Let  $\overline{\mu}(\mathcal{C})$  be the equilibrium solution for the corresponding vector potential problem. We have

**Theorem 1** Let  $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \text{Reg}, (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ , and consider a sequence of multi-indices  $\Lambda = \Lambda(p_0, \ldots, p_m)$ . Let  $(A_{\mathbf{n},0}, A_{\mathbf{n},1}, \ldots, A_{\mathbf{n},m})$ ,  $\mathbf{n} \in \Lambda$ , be the associated sequence of type I Fourier–Padé with respect to  $\sigma_0$  for the Nikishin system of functions  $(\widehat{s}_1, \ldots, \widehat{s}_m)$ . Then,

$$\lim_{\mathbf{n}\in\Lambda} \left| L_{\mathbf{n},0}(z) \right|^{1/|\mathbf{n}|} = G_0(z),\tag{5}$$

uniformly on each compact subset of  $\mathbb{C} \setminus (\Delta_0 \cup \Delta_1)$ , where

$$G_0(z) = \exp\left(P_1 V^{\overline{\mu}_1}(z) - V^{\overline{\mu}_0}(z) - 2\sum_{k=1}^m \frac{\omega_k^{\overline{\mu}}}{P_k}\right).$$

Here  $\overline{\mu} = \overline{\mu}(\mathcal{C}) = (\overline{\mu}_0, \dots, \overline{\mu}_m)$  is the equilibrium vector measure, and  $(\omega_0^{\overline{\mu}}, \dots, \omega_m^{\overline{\mu}})$  is the system of equilibrium constants for the vector potential problem determined by the interaction matrix  $\mathcal{C}$  defined in (4) on the system of intervals  $\Delta_j$ ,  $j = 0, \dots, m$ .

The corresponding result for Hermite–Padé approximants of Nikishin systems appears in Sect. 7, Chap. 5 of [7] (see also [6]).

Besides normality, in Sect. 2 we obtain the orthogonality relations satisfied by the different polynomials involved in the construction. Section 3 is devoted to the study of an extremal problem that allows us to prove Theorem 3 in Sect. 4, of which Theorem 1 is a corollary.

#### 2 Normality and Orthogonality Relations

Set

$$s_{j,k} = \langle \sigma_j, \ldots, \sigma_k \rangle, \quad 1 \le j < k \le m, \qquad s_{j,j} = \sigma_j.$$

We denote

$$L_{\mathbf{n},j}(z) = \sum_{k=j}^{m} A_{\mathbf{n},k}(z)\widehat{s}_{j+1,k}(z), \quad j = 0, \dots, m,$$

 $(\widehat{s}_{j+1,j}(z) \equiv 1, L_{\mathbf{n},m} \equiv A_{\mathbf{n},m}).$ 

In [5], E.M. Nikishin introduced the following definition:

**Definition 1** A set of continuous real functions  $u_0(x), \ldots, u_m(x)$ , defined on an interval  $\Delta$ , is called an AT-system for the index  $\mathbf{n} \in \mathbb{Z}^{m+1}_+$  if for any polynomials  $h_0, \ldots, h_m$  such that  $\deg(h_i) \leq n_i - 1$ ,  $i = 0, \ldots, m$ , not all simultaneously identically equal to zero, the function

$$H(x) = h_0(x)u_0(x) + \dots + h_m(x)u_m(x)$$

has at most  $|\mathbf{n}| - 1$  zeros on  $\Delta$  (deg  $h_i \leq -1$  means that  $h_i \equiv 0$ ).

Let  $\mathbb{Z}^{m+1}_+(*)$  be the set of multi-indices given by

$$\mathbb{Z}^{m+1}_+(*) = \left\{ \mathbf{n} \in \mathbb{Z}^{m+1}_+ : \nexists i < k < j \text{ such that } n_i < n_j < n_k \right\}.$$

In connection with AT-systems, in [2] U. Fidalgo and G. López proved:

**Lemma 2** Let  $\mathbf{n} \in \mathbb{Z}_{+}^{m+1}(*)$  and  $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ . Then  $(1, \hat{s}_1, \ldots, \hat{s}_m)$  defines an AT-system with respect to  $\mathbf{n}$  on any interval disjoint from  $\Delta_1$ .

Notice that for each  $j \in \{0, ..., m-1\}, (s_{j+1,j+1}, ..., s_{j+1,m}) = \mathcal{N}(\sigma_{j+1}, ..., \sigma_m)$ , and using Lemma 2 it follows that for  $\mathbf{n} \in \mathbb{Z}_+^{m+1}(\bullet) \subset \mathbb{Z}_+^{m+1}(*)$  the linear form  $L_{\mathbf{n},j}$  cannot have more that  $N_j - 1$  zeros on  $\Delta_j$ , where

$$N_j = n_j + \dots + n_m.$$

Obviously, the same is true for the polynomial  $L_{\mathbf{n},m} \equiv A_{\mathbf{n},m}$ .

The definition of Fourier-Padé approximant implies that

$$\int x^k L_{\mathbf{n},0}(x) \, d\sigma_0(x) = 0, \quad k = 0, \dots, |\mathbf{n}| - 2.$$
(6)

Since the function  $L_{\mathbf{n},0}(x)$  is continuous on  $\Delta_0$ , from (6) we have that  $L_{\mathbf{n},0}(x)$  has at least  $|\mathbf{n}| - 1$  sign changes in the interior of  $\Delta_0$ . This and the previous remark indicate that it has exactly  $|\mathbf{n}| - 1$  sign changes in the interior of  $\Delta_0$ ; thus, all the zeros of  $L_{\mathbf{n},0}(x)$  in  $\Delta_0$  are simple and lie in its interior. In connection with intervals of the real line, the interior refers to the Euclidean topology of  $\mathbb{R}$ . In short, we shall see that  $L_{\mathbf{n},0}(x)$  has no other zeros in  $\overline{\mathbb{C}} \setminus \Delta_1$ . Before proving this, let us turn to the question of normality.

**Proposition 1** Let  $\mathbf{n} \in \mathbb{Z}^{m+1}_+(\bullet)$  and  $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ . Then  $\mathbf{n}$  is normal, and  $(A_{\mathbf{n},0}, \ldots, A_{\mathbf{n},m})$  is uniquely determined except for a constant factor.

*Proof* Let us assume that there exists  $j \in \{0, ..., m\}$  such that deg  $A_{\mathbf{n},j} \le n_j - 2$ . Then  $\mathbf{n} - \mathbf{e}^j \in \mathbb{Z}^{m+1}_+(*)$ , where  $\mathbf{e}^j$  denotes the (m + 1)-dimensional unit vector with all components equal to zero except the component j + 1, which equals 1. According to Lemma 2, the linear form  $L_{\mathbf{n},0}$  has at most  $|\mathbf{n}| - 2$  zeros on  $\Delta_0$ , but we pointed out previously that it has at least  $|\mathbf{n}| - 1$  sign changes on this interval. This contradiction yields that for all  $j \in \{0, ..., m\}$ , deg  $A_{\mathbf{n}, j} = n_j - 1$ , which implies normality.

Now let us assume that  $(A_{\mathbf{n},0}, \ldots, A_{\mathbf{n},m})$  and  $(A'_{\mathbf{n},0}, \ldots, A'_{\mathbf{n},m})$  solve (i)–(ii), and that these vectors are not collinear. According to what we just proved, for all  $j \in \{0, \ldots, m\}$ , deg  $A_{\mathbf{n},j} = \deg A'_{\mathbf{n},j} = n_j - 1$ . Take  $\lambda \in \mathbb{C} \setminus \{0\}$  such that deg $(A_{\mathbf{n},0} - \lambda A'_{\mathbf{n},0}) \leq n_0 - 2$ . Obviously, the vector  $(A_{\mathbf{n},0} - \lambda A'_{\mathbf{n},0}, \ldots, A_{\mathbf{n},m} - \lambda A'_{\mathbf{n},0})$  is not identically equal to zero and also solves (i)–(ii), which is not possible since all non-trivial solutions must have all components of maximal degree.

Because of Proposition 1, we can assume that  $(A_{n,0}, A_{n,1}, \ldots, A_{n,m})$  is normalized so that  $A_{n,m}$  is a monic polynomial of degree  $n_m - 1$ . In the rest of the paper, we take this normalization in order to determine the linear forms  $L_{n,j}$  in a unique manner.

In the sequel, we assume that  $\mathbf{n} \in \mathbb{Z}^{m+1}_+(\bullet)$ . For j = 0, ..., m, let  $Q_{\mathbf{n},j}$  be the monic polynomial whose zeros are those of the linear form  $L_{\mathbf{n},j}$  in the region  $\overline{\mathbb{C}} \setminus \Delta_{j+1}$  counting multiplicities  $(\Delta_{m+1} = \emptyset)$ . In particular,  $L_{\mathbf{n},m} = A_{\mathbf{n},m} = Q_{\mathbf{n},m}$ . From the previous proposition, if  $n_m \ge 1$ ,  $\infty$  is not a zero of any one of these linear forms; thus,  $\infty$  cannot be an accumulation point of zeros of them. Though it is not the case, in principle, some of these linear forms may have an infinite number of zeros which may accumulate on the boundary of the corresponding region of meromorphicity. In that case, for the time being,  $Q_{\mathbf{n},j}$  denotes a formal infinite product.

**Proposition 2** Let  $\mathbf{n} \in \mathbb{Z}^{m+1}_+(\bullet)$ ,  $n_m \ge 1$ , and  $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ . Then, deg  $Q_{\mathbf{n},j} = N_j - 1$ ,  $j = 0, \ldots, m$ , all its zeros are simple and lie in the interior of  $\Delta_j$ . Moreover,

$$\int x^{\nu} L_{\mathbf{n},j}(x) \frac{d\sigma_j(x)}{Q_{\mathbf{n},j-1}(x)} = 0, \quad \nu = 0, \dots, N_j - 2,$$
(7)

 $(Q_{\mathbf{n},-1} \equiv 1).$ 

*Proof* We proceed by induction on *j*. For j = 0, (7) is (6), and this implies that  $L_{\mathbf{n},0}$  has  $N_0 - 1 = |\mathbf{n}| - 1$  simple zeros in the interior of  $\Delta_0$ . Therefore, deg  $Q_{\mathbf{n},0} \ge N_0 - 1$ . If deg  $Q_{\mathbf{n},0} = N_0 - 1$ , we conclude with the initial step. Suppose that deg  $Q_{\mathbf{n},0} \ge N_0$  (including the possible case that deg  $Q_{\mathbf{n},0} = \infty$ ). Choose  $N_0$  zeros of  $Q_{\mathbf{n},0}$ , and denote the monic polynomial with these  $N_0$  zeros by  $Q_{\mathbf{n},0}^*$ .

Notice that

$$\frac{L_{\mathbf{n},0}}{Q_{\mathbf{n},0}^*} \in \mathcal{H}(\overline{\mathbb{C}} \setminus \Delta_1)$$

is analytic in the indicated region and

$$\frac{z^{\nu}L_{\mathbf{n},0}}{Q_{\mathbf{n},0}^{*}} = \mathcal{O}\left(\frac{1}{z^{2}}\right), \quad \nu = 0, \dots, N_{1} - 1.$$

Let  $\Gamma_1$  be a closed, smooth, Jordan curve that surrounds  $\Delta_1$  such that all the zeros of  $Q_{\mathbf{n},0}^*$  lie in the unbounded connected component of the complement of  $\Gamma_1$ . By Cauchy's Theorem, Cauchy's Integral Formula, and Fubini's Theorem, we have

$$0 = \frac{1}{2\pi i} \int_{\Gamma_1} z^{\nu} \frac{L_{\mathbf{n},0}(z)}{Q_{\mathbf{n},0}^*(z)} dz = \frac{1}{2\pi i} \int_{\Gamma_1} z^{\nu} \frac{\sum_{k=1}^m A_{\mathbf{n},k}(z) \widehat{s}_k(z)}{Q_{\mathbf{n},0}^*(z)} dz$$
$$= \frac{1}{2\pi i} \int_{\Gamma_1} z^{\nu} \sum_{k=1}^m A_{\mathbf{n},k}(z) \int \frac{\widehat{s}_{2,k}(x) d\sigma_1(x)}{z-x} \frac{dz}{Q_{\mathbf{n},0}^*(z)} = \int x^{\nu} L_{\mathbf{n},1}(x) \frac{d\sigma_1(x)}{Q_{\mathbf{n},0}^*(x)} dz$$

with  $\nu = 0, ..., N_1 - 1$ . This implies that  $L_{n,1}$  has at least  $N_1$  zeros on  $\Delta_1$ . According to Lemma 2, this linear form can only have  $N_1 - 1$  zeros on this interval. This implies that our initial assumption is false, and deg  $Q_{n,0} = N_0 - 1$ .

Assume that the statement is true for some  $j \in \{0, ..., m-1\}$ , and let us show that it holds for j + 1. Indeed, since deg  $Q_{\mathbf{n},j} = N_j - 1$ , and its zeros are simple and lie in the interior of  $\Delta_j$ , then

$$\frac{L_{\mathbf{n},j}}{Q_{\mathbf{n},j}} \in \mathcal{H}(\overline{\mathbb{C}} \setminus \Delta_{j+1}), \quad \frac{z^{\nu} L_{\mathbf{n},j}}{Q_{\mathbf{n},j}} = \mathcal{O}\left(\frac{1}{z^2}\right), \quad \nu = 0, \dots, N_{j+1} - 2.$$

Let  $\Gamma_{j+1}$  be a closed, smooth, Jordan curve that surrounds  $\Delta_{j+1}$  such that  $\Delta_j$  lies in the unbounded connected component of the complement of  $\Gamma_{j+1}$ . By Cauchy's Theorem, Cauchy's Integral Formula, and Fubini's Theorem, it follows that

$$0 = \frac{1}{2\pi i} \int_{\Gamma_{j+1}} z^{\nu} \frac{L_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j}(z)} dz = \frac{1}{2\pi i} \int_{\Gamma_{j+1}} z^{\nu} \frac{\sum_{k=j+1}^{m} A_{\mathbf{n},k}(z) \widehat{s}_{j+1,k}(z)}{Q_{\mathbf{n},j}(z)} dz$$
$$= \frac{1}{2\pi i} \int_{\Gamma_{j+1}} z^{\nu} \sum_{k=j+1}^{m} A_{\mathbf{n},k}(z) \int \frac{\widehat{s}_{j+2,k}(x) d\sigma_{j+1}(x)}{z-x} \frac{dz}{Q_{\mathbf{n},j}(z)}$$
$$= \int x^{\nu} L_{\mathbf{n},j+1}(x) \frac{d\sigma_{j+1}(x)}{Q_{\mathbf{n},j}(x)},$$

with  $\nu = 0, \ldots, N_{j+1} - 2$ . We have obtained (7) for j + 1.

Formula (7) for j + 1 implies that  $Q_{\mathbf{n},j+1}$  has at least  $N_{j+1} - 1$  simple zeros in the interior of  $\Delta_{j+1}$ . If deg  $Q_{\mathbf{n},j+1} = N_{j+1} - 1$ , we have finished the proof (for example, this is the case when j + 1 = m because  $L_{\mathbf{n},m} \equiv A_{\mathbf{n},m}$ ). Let us suppose that deg  $Q_{\mathbf{n},j+1} \ge N_{j+1}$  (including the possible case that deg  $Q_{\mathbf{n},j+1} = \infty$ , and of course  $j \le m - 2$ ). Choose  $N_{j+1}$  zeros of  $Q_{\mathbf{n},j+1}$ , and denote the monic polynomial with these  $N_{j+1}$  zeros by  $Q_{\mathbf{n},j+1}^*$ . Then

$$\frac{L_{\mathbf{n},j+1}}{Q_{\mathbf{n},j+1}^*} \in \mathcal{H}(\overline{\mathbb{C}} \setminus \Delta_{j+2}), \quad \frac{z^{\nu} L_{\mathbf{n},j+1}}{Q_{\mathbf{n},j+1}^*} = \mathcal{O}\left(\frac{1}{z^2}\right), \quad \nu = 0, \dots, N_{j+2} - 1.$$

Let  $\Gamma_{j+2}$  be a closed, smooth, Jordan curve that surrounds  $\Delta_{j+2}$  such that  $\Delta_{j+1}$  and all the zeros of  $Q^*_{\mathbf{n},j+1}$  lie in the unbounded connected component of the complement of  $\Gamma_{j+2}$ . By Cauchy's Theorem, Cauchy's Integral Formula, and Fubini's Theorem, we have

$$\begin{split} 0 &= \frac{1}{2\pi i} \int_{\Gamma_{j+2}} z^{\nu} \frac{L_{\mathbf{n},j+1}(z)}{Q_{\mathbf{n},j+1}^{*}(z)} dz = \frac{1}{2\pi i} \int_{\Gamma_{j+2}} z^{\nu} \frac{\sum_{k=j+2}^{m} A_{\mathbf{n},k}(z) \widehat{s}_{j+2,k}(z)}{Q_{\mathbf{n},j+1}^{*}(z)} dz \\ &= \frac{1}{2\pi i} \int_{\Gamma_{j+2}} z^{\nu} \sum_{k=j+2}^{m} A_{\mathbf{n},k}(z) \int \frac{\widehat{s}_{j+3,k}(x) d\sigma_{j+2}(x)}{z-x} \frac{dz}{Q_{\mathbf{n},j+1}^{*}(z)} \\ &= \int x^{\nu} L_{\mathbf{n},j+2}(x) \frac{d\sigma_{j+2}(x)}{Q_{\mathbf{n},j+1}^{*}(x)}, \end{split}$$

with  $\nu = 0, ..., N_{j+2} - 1$ . This implies that  $L_{\mathbf{n}, j+2}$  has at least  $N_{j+2}$  zeros on  $\Delta_{j+2}$ . According to Lemma 2, this linear form can only have  $N_{j+2} - 1$  zeros on this interval. This implies that our initial assumption is false; therefore, deg  $Q_{\mathbf{n}, j+1} = N_{j+1} - 1$ , as we needed to prove.

**Proposition 3** Let  $\mathbf{n} \in \mathbb{Z}^{m+1}_+(\bullet)$ ,  $n_m \ge 1$ , and  $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ . Then for each  $j = 0, \ldots, m-1$  and each polynomial q, deg  $q \le N_{j+1} - 1$ ,

$$\frac{q(z)L_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j}(z)} = \int \frac{q(x)L_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)} \frac{d\sigma_{j+1}(x)}{z-x}.$$
(8)

*Proof* From Proposition 2, for any q, deg  $q \le N_{j+1} - 1$ ,

$$\frac{qL_{\mathbf{n},j}}{Q_{\mathbf{n},j}} \in \mathcal{H}(\overline{\mathbb{C}} \setminus \Delta_{j+1}), \quad \frac{qL_{\mathbf{n},j}}{Q_{\mathbf{n},j}} = \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty$$

Let  $\Gamma_{j+1}$  be a closed, smooth, Jordan curve that surrounds  $\Delta_{j+1}$  such that  $\Delta_j$  and z lie in the unbounded connected component of the complement of  $\Gamma_{j+1}$ . By Cauchy's Integral Formula, Cauchy's Theorem, and Fubini's Theorem, it follows that

$$\begin{split} \frac{q(z)L_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j}(z)} &= \frac{1}{2\pi i} \int_{\Gamma_{j+1}} \frac{q(\zeta)L_{\mathbf{n},j}(\zeta)}{Q_{\mathbf{n},j}(\zeta)} \frac{d\zeta}{z-\zeta} \\ &= \frac{1}{2\pi i} \int_{\Gamma_{j+1}} \frac{q(\zeta)\sum_{k=j+1}^{m} A_{\mathbf{n},k}(\zeta)\widehat{s}_{j+1,k}(\zeta)}{Q_{\mathbf{n},j}(\zeta)} \frac{d\zeta}{z-\zeta} \\ &= \int \sum_{k=j+1}^{m} \frac{1}{2\pi i} \int_{\Gamma_{j+1}} \frac{q(\zeta)A_{\mathbf{n},k}(\zeta)}{Q_{\mathbf{n},j}(\zeta)(z-\zeta)} \frac{d\zeta}{\zeta-x} \widehat{s}_{j+2,k}(x) d\sigma_{j+1}(x) \\ &= \int \frac{q(x)L_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)} \frac{d\sigma_{j+1}(x)}{z-x}, \end{split}$$

and we have obtained (8).

## **3** Extremal Problems

Let  $\{\mu_l\} \subset \mathcal{M}(\mathcal{K})$  be a sequence of measures, where  $\mathcal{K}$  is a compact subset of the complex plane and  $\mu \in \mathcal{M}(\mathcal{K})$ . We write

$$*\lim_{l}\mu_{l}=\mu, \quad \mu\in\mathcal{M}(\mathcal{K}),$$

if for every continuous function  $f \in C(\mathcal{K})$ 

$$\lim_{l} \int f \, d\mu_l = \int f \, d\mu,$$

that is, when the sequence of measures converges to  $\mu$  in the weak star topology. Given a polynomial  $q_l$  of degree  $l \ge 1$ , we denote the associated normalized zero counting measure by

$$\nu_{q_l} = \frac{1}{l} \sum_{q_l(x)=0} \delta_x,$$

where  $\delta_x$  is the Dirac measure with mass 1 at x (in the sum, the zeros are repeated according to their multiplicity).

In order to prove our main result, we need Theorem 3.3.3 of [8]. We present it in the form stated in [3], which is more adequate for our purpose. In [3], it was proved under stronger assumptions on the measure.

**Lemma 3** Let  $\{\phi_l\}, l \in \Lambda \subset \mathbb{Z}_+$ , be a sequence of positive continuous functions on a bounded closed interval  $\Delta \subset \mathbb{R}, \sigma \in \operatorname{Reg} \cap \mathcal{M}(\Delta)$ , and let  $\{q_l\}, l \in \Lambda$ , be a sequence of monic polynomials such that deg  $q_l = l$  and

$$\int x^k q_l(x)\phi_l(x)\,d\sigma(x) = 0, \quad k = 0, \dots, l-1.$$

Assume that

$$\lim_{l \in \Lambda} \frac{1}{2l} \log \frac{1}{|\phi_l(x)|} = v(x),$$

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uniformly on  $\Delta$ . Then

$$*\lim_{l\in\Lambda}v_{q_l}=\overline{v}$$

and

$$\lim_{l\to\infty} \left(\int |q_l(x)|^2 \phi_l(x) \, d\mu(x)\right)^{1/2l} = e^{-\omega},$$

where  $\overline{v} \in \mathcal{M}_1(\Delta)$  is the unique solution of the extremal problem

$$V^{\overline{\nu}}(x) + v(x) \begin{cases} = \omega, & x \in supp(\overline{\nu}) \\ \ge \omega, & x \in \Delta, \end{cases}$$

in the presence of the external field v.

Using this result, we can obtain the asymptotic limit distribution of the zeros of the polynomials  $Q_{n,j}$ , j = 0, ..., m.

**Theorem 2** Let  $(\sigma_0; \sigma_1, ..., \sigma_m) \in \text{Reg}, (s_1, ..., s_m) = \mathcal{N}(\sigma_1, ..., \sigma_m)$ , and consider the sequence of multi-indices  $\Lambda = \Lambda(p_0, ..., p_m)$ . Then

\* 
$$\lim_{\mathbf{n}\in\Lambda}\nu_{\mathcal{Q}_{\mathbf{n},j}}=\overline{\mu}_j, \quad j=0,\ldots,m,$$
(9)

where  $\overline{\mu} = \overline{\mu}(C) \in \mathcal{M}_1$  is the vector equilibrium measure determined by the matrix C in (4) on the system of intervals  $F_j = \Delta_j, j = 0, ..., m$ .

*Proof* The unit ball in the cone of positive Borel measures is weakly compact; therefore, it is sufficient to show that each one of the sequences of measures  $\{v_{Q_{\mathbf{n},j}}\}, \mathbf{n} \in \Lambda$ , j = 0, ..., m, has only one accumulation point which coincides with the corresponding component of the vector measure  $\overline{\mu}(\mathcal{C})$ . Let  $\Lambda' \subset \Lambda$  be a subsequence of multiindices such that for each j = 0, ..., m

$$*\lim_{\mathbf{n}\in\Lambda'}\nu_{Q_{\mathbf{n},j}}=\nu_j.$$

Notice that  $v_j \in \mathcal{M}_1(\Delta_j), j = 0, \dots, m$ . Therefore,

$$\lim_{\mathbf{n}\in\Lambda'} \left| \mathcal{Q}_{\mathbf{n},j}(z) \right|^{1/|\mathbf{n}|} = \exp\left(-P_j V^{\nu_j}(z)\right),\tag{10}$$

uniformly on compact subsets of  $\mathbb{C} \setminus \Delta_j$ , where  $P_j = p_j + \cdots + p_m$ .

Because of the normalization adopted on  $A_{\mathbf{n},m}$ ,  $L_{\mathbf{n},m} = Q_{\mathbf{n},m}$ ; consequently, when j = m, (7) takes the form

$$\int x^{\nu} Q_{\mathbf{n},m}(x) \frac{d|\sigma_m|(x)}{|Q_{\mathbf{n},m-1}(x)|} = 0, \quad \nu = 0, \dots, N_m - 2.$$

(By  $|\sigma|$  we denote the total variation of the measure  $\sigma$ .) According to (10),

$$\lim_{\mathbf{n}\in\Lambda'}\frac{1}{2N_m}\log|Q_{\mathbf{n},m-1}(x)| = -\frac{P_{m-1}}{2P_m}V^{\nu_{m-1}}(x),$$

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uniformly on  $\Delta_m$ . Using Lemma 3, it follows that  $\nu_m$  is the unique solution of the extremal problem

$$V^{\nu_m}(x) - \frac{P_{m-1}}{2P_m} V^{\nu_{m-1}}(x) \begin{cases} = \omega_m, & x \in \operatorname{supp}(\nu_m), \\ \ge \omega_m, & x \in \Delta_m, \end{cases}$$
(11)

and

$$\lim_{\mathbf{n}\in\Lambda'} \left( \int \frac{Q_{\mathbf{n},m}^2(x)}{|Q_{\mathbf{n},m-1}(x)|} \, d|\sigma_m|(x) \right)^{1/2N_m} = e^{-\omega_m}. \tag{12}$$

Let us show by induction on decreasing values of *j* that for all  $j \in \{0, ..., m\}$ 

$$V^{\nu_{j}}(x) - \frac{P_{j-1}}{2P_{j}} V^{\nu_{j-1}}(x) - \frac{P_{j+1}}{2P_{j}} V^{\nu_{j+1}}(x) + \frac{P_{j+1}}{P_{j}} \omega_{j+1} \begin{cases} = \omega_{j}, & x \in \text{supp}(\nu_{j}), \\ \ge \omega_{j}, & x \in \Delta_{j}, \end{cases}$$
(13)

where  $P_{-1} = P_{m+1} = 0$ , and

$$\lim_{\mathbf{n}\in\Lambda'} \left( \int \frac{Q_{\mathbf{n},j}^2(x)}{|Q_{\mathbf{n},j-1}(x)|} \frac{|L_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|} d|\sigma_j|(x) \right)^{1/2N_j} = e^{-\omega_j},$$
(14)

where  $Q_{n,-1} \equiv 1$ . For j = m, these relations are none other than (11)–(12), and the initial induction step is settled. Let us assume that the statement is true for  $j + 1 \in \{1, ..., m\}$ , and let us prove it for j.

It is easy to see that the orthogonality relations (7) can be expressed as

$$\int x^{\nu} Q_{\mathbf{n},j}(x) \frac{|Q_{\mathbf{n},j+1}(x)L_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|} \frac{d|\sigma_j|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|} = 0, \quad \nu = 0, \dots, N_j - 2.$$

According to (8) with  $q = Q_{\mathbf{n}, j+1}$ , this can be further transformed into

$$\int x^{\nu} Q_{\mathbf{n},j}(x) \left( \int \frac{Q_{\mathbf{n},j+1}^{2}(t)}{|Q_{\mathbf{n},j}(t)|} \frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|} \frac{d|\sigma_{j+1}|(t)}{|x-t|} \right) \frac{d|\sigma_{j}|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|} = 0,$$

for  $\nu = 0, ..., N_j - 2$ .

Relation (10) implies that

$$\lim_{\mathbf{n}\in\Lambda'}\frac{1}{2N_{j}}\log\left|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)\right| = -\frac{P_{j-1}}{2P_{j}}V^{\nu_{j-1}}(x) - \frac{P_{j+1}}{2P_{j}}V^{\nu_{j+1}}(x),\quad(15)$$

uniformly on  $\Delta_j$ . (Since  $Q_{\mathbf{n},-1} \equiv 1$ , when j = 0 we only get the second term on the right-hand side of this limit.)

Set

$$\rho_{\mathbf{n},j+1} = \int \frac{Q_{\mathbf{n},j+1}^2(t)}{|Q_{\mathbf{n},j}(t)|} \frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|} d|\sigma_{j+1}|(t).$$

It follows that for all  $x \in \Delta_j$ 

$$\frac{\rho_{\mathbf{n},j+1}}{\delta_{j+1}^*} \le \int \frac{Q_{\mathbf{n},j+1}^2(t)}{|Q_{\mathbf{n},j}(t)|} \frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|} \frac{d|\sigma_{j+1}|(t)}{|x-t|} \le \frac{\rho_{\mathbf{n},j+1}}{\delta_{j+1}}$$

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where  $0 < \delta_{j+1} = \inf\{|x - t| : t \in \Delta_{j+1}, x \in \Delta_j\} \le \max\{|x - t| : t \in \Delta_{j+1}, x \in \Delta_j\} \le \delta_{j+1}^* < \infty$ . Taking into consideration these inequalities, from the induction hypothesis we obtain that

$$\lim_{\mathbf{n}\in\Lambda'} \left( \int \frac{Q_{\mathbf{n},j+1}^2(t)}{|Q_{\mathbf{n},j}(t)|} \frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|} \frac{d|\sigma_{j+1}|(t)}{|x-t|} \right)^{1/2N_j} = e^{-P_{j+1}\omega_{j+1}/P_j}.$$
 (16)

Taking (15) and (16) into account, Lemma 3 yields that  $v_j$  is the unique solution of the extremal problem (13) and

$$\lim_{\mathbf{n}\in\Lambda'} \left( \iint \frac{Q_{\mathbf{n},j+1}^2(t)}{|Q_{\mathbf{n},j}(t)|} \frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|} \frac{d|\sigma_{j+1}|(t)}{|x-t|} \frac{Q_{\mathbf{n},j}^2(x)d|\sigma_j|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|} \right)^{1/2N_j} = e^{-\omega_j}.$$

According to (8) with  $q = Q_{\mathbf{n}, j+1}$ 

$$\frac{1}{|Q_{\mathbf{n},j+1}(x)|} \int \frac{Q_{\mathbf{n},j+1}^2(t)}{|Q_{\mathbf{n},j}(t)|} \frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|} \frac{d|\sigma_{j+1}(t)}{|x-t|} = \frac{|L_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|}, \quad x \in \Delta_j,$$

which allows us to reduce the previous formula to (14), thus concluding the induction proof.

Now we can rewrite (13), multiplying through by  $P_j^2$  and taking the constant term on the left to the right to obtain the system of boundary value equations

$$P_{j}^{2}V^{\nu_{j}}(x) - \frac{P_{j-1}P_{j}}{2}V^{\nu_{j-1}}(x) - \frac{P_{j}P_{j+1}}{2}V^{\nu_{j+1}}(x) \begin{cases} = \omega_{j}', & x \in \operatorname{supp}(\nu_{j}), \\ \ge \omega_{j}', & x \in \Delta_{j}, \end{cases}$$
(17)

for j = 0, ..., m, where  $\omega'_j = P_j^2 \omega_j - P_j P_{j+1} \omega_{j+1}$ . (The terms with  $P_{-1}$  and  $P_{m+1}$  do not appear when j = 0 and j = m, respectively.) By Lemma 1,  $(\nu_0, ..., \nu_m) = (\overline{\mu}_0, ..., \overline{\mu}_m)$  and  $(\omega'_0, ..., \omega'_m) = (\omega_0^{\overline{\mu}}, ..., \omega_m^{\overline{\mu}})$  for any convergent subsequence showing the existence of the limits in (9) as stated.

### 4 Proof of Theorem 1

Theorem 1 is a consequence of the following more general result:

**Theorem 3** Let  $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \text{Reg}, (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ , and consider a sequence of multi-indices  $\Lambda = \Lambda(p_0, \ldots, p_m)$ . Let  $(A_{\mathbf{n},0}, A_{\mathbf{n},1}, \ldots, A_{\mathbf{n},m})$ ,  $\mathbf{n} \in \Lambda$ , be the associated sequence of type I Fourier–Padé approximants for the Nikishin system of Markov functions  $(\widehat{s}_1, \ldots, \widehat{s}_m)$  normalized so that for all  $\mathbf{n}, A_{\mathbf{n},m}$  is monic. Then, for  $j = 0, \ldots, m$ 

$$\lim_{\mathbf{n}\in\Lambda} \left| L_{\mathbf{n},j}(z) \right|^{1/|\mathbf{n}|} = G_j(z), \tag{18}$$

uniformly on each compact subset of  $\mathbb{C} \setminus (\Delta_j \cup \Delta_{j+1})$ , where

$$G_{j}(z) = \exp\left(P_{j+1}V^{\overline{\mu}_{j+1}}(z) - P_{j}V^{\overline{\mu}_{j}}(z) - 2\sum_{k=j+1}^{m} \frac{\omega_{k}^{\overline{\mu}}}{P_{k}}\right), \quad j = 0, \dots, m-1,$$

and

$$G_m(z) = \exp(-P_m V^{\mu_m}(z)).$$

Here  $\overline{\mu} = \overline{\mu}(\mathcal{C}) = (\overline{\mu}_0, \dots, \overline{\mu}_m)$  is the equilibrium vector measure, and  $(\omega_0^{\overline{\mu}}, \dots, \omega_m^{\overline{\mu}})$  is the system of equilibrium constants for the vector potential problem determined by the interaction matrix  $\mathcal{C}$  defined in (4) on the system of intervals  $\Delta_j$ ,  $j = 0, \dots, m$ .

*Proof* If j = m,  $L_{\mathbf{n},m} = Q_{\mathbf{n},m}$ , and (9) directly implies that

$$\lim_{\mathbf{n}\in\Lambda} |L_{\mathbf{n},m}(z)|^{1/|\mathbf{n}|} = \exp(-P_m V^{\overline{\mu}_m}(z)),$$

uniformly on compact subsets of  $\mathbb{C} \setminus \Delta_m$ . For  $j \in \{0, ..., m-1\}$ , using (8) with  $q = Q_{\mathbf{n}, j+1}$ , we obtain

$$L_{\mathbf{n},j}(z) = \frac{Q_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j+1}(z)} \int \frac{Q_{\mathbf{n},j+1}^2(x)}{Q_{\mathbf{n},j}(x)} \frac{L_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j+1}(x)} \frac{d\sigma_{j+1}(x)}{z-x}.$$
 (19)

From (9), it follows that

$$\lim_{\mathbf{n}\in\mathcal{A}} \left| \frac{Q_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j+1}(z)} \right|^{1/|\mathbf{n}|} = \exp\left( P_{j+1} V^{\overline{\mu}_{j+1}}(z) - P_j V^{\overline{\mu}_j}(z) \right), \tag{20}$$

uniformly on compact subsets of  $\mathbb{C} \setminus \Delta_j \cup \Delta_{j+1}$  (we also use that the zeros of  $Q_{\mathbf{n},j}$ and  $Q_{\mathbf{n},j+1}$  lie in  $\Delta_j$  and  $\Delta_{j+1}$ , respectively). It remains to find the  $|\mathbf{n}|$ -th root asymptotic behavior of the integral.

Fix a compact set  $\mathcal{K} \subset \mathbb{C} \setminus \Delta_{j+1}$ . It is easy to verify that (for the definition of  $\rho_{\mathbf{n}, j+1}$ , see proof of Theorem 2 above)

$$C_1 \rho_{\mathbf{n},j+1} \le \left| \int \frac{Q_{\mathbf{n},j+1}^2(x)}{Q_{\mathbf{n},j}(x)} \frac{L_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j+1}(x)} \frac{d\sigma_{j+1}(x)}{z-x} \right| \le C_2 \rho_{\mathbf{n},j+1},$$

where

$$C_1 = \frac{\min\{\max\{|u - x|, |v| : z = u + iv\} : z \in \mathcal{K}, x \in \Delta_{j+1}\}}{\max\{|z - x|^2 : z \in \mathcal{K}, x \in \Delta_{j+1}\}} > 0$$

and

$$C_2 = \frac{1}{\min\{|z-x| : z \in \mathcal{K}, x \in \Delta_{j+1}\}} < \infty.$$

Taking into account (14),

$$\lim_{\mathbf{n}\in\Lambda} \left| \int \frac{Q_{\mathbf{n},j+1}^2(x)}{Q_{\mathbf{n},j}(x)} \frac{L_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j+1}(x)} \frac{d\sigma_{j+1}(x)}{z-x} \right|^{1/|\mathbf{n}|} = e^{-2P_{j+1}\omega_{j+1}}.$$
 (21)

From (19)–(21), we obtain

$$\lim_{\mathbf{n}\in\Lambda} |L_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp(P_{j+1}V^{\overline{\mu}_{j+1}}(z) - P_jV^{\overline{\mu}_j}(z) - P_{j+1}\omega_{j+1}).$$

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It remains to show that for all j = 0, ..., m,  $P_j \omega_j = \sum_{k=j}^m \frac{\omega_k^{\mu}}{P_k}$ . At the end of the proof of Theorem 2, we saw that

$$\omega_m^{\overline{\mu}} = P_m^2 \omega_m, \qquad \omega_j^{\overline{\mu}} = P_j^2 \omega_j - P_j P_{j+1} \omega_{j+1}, \quad j = 0, \dots, m-1.$$

From the first relation, it follows that  $P_m \omega_m = \omega_m^{\overline{\mu}} / P_m$ . Let us show that the rest of the relations hold using induction on decreasing values of *j*. Suppose that the formula is true for some  $j + 1 \in \{1, ..., m\}$ . Then, according to the formulas displayed above,

$$P_j\omega_j = \frac{\omega_j^{\mu}}{P_j} + P_{j+1}\omega_{j+1},$$

and using the induction hypothesis the result immediately follows.

Set

$$U_{j}^{\overline{\mu}}(z) = P_{j}V^{\overline{\mu}_{j}}(z) - P_{j+1}V^{\overline{\mu}_{j+1}}(z) + 2\sum_{k=j+1}^{m} \frac{\omega_{k}^{\overline{\mu}}}{P_{k}}, \quad j = 0, \dots, m-1,$$

and

$$U_{-1}^{\overline{\mu}}(z) = -P_0 V^{\overline{\mu}_0}(z), \qquad U_m^{\overline{\mu}}(z) = P_m V^{\overline{\mu}_m}(z)$$

Hence,  $G_j(z) = \exp(-U_j(z)), j = 0, \dots, m$ . We have that for  $j = 0, \dots, m$ 

$$\frac{P_j}{2} \left( U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) \right) = -\frac{P_{j+1}P_j}{2} V^{\overline{\mu}_{j+1}}(z) + P_j^2 V^{\overline{\mu}_j}(z) - \frac{P_j P_{j-1}}{2} V^{\overline{\mu}_{j-1}}(z) - \omega_j^{\overline{\mu}},$$

 $(P_{-1} = P_{m+1} = 0).$ 

From the equilibrium property (see Lemma 1 and (17)), it follows that

$$U_j^{\overline{\mu}}(x) - U_{j-1}^{\overline{\mu}}(x) = 0, \quad x \in \operatorname{supp}(\overline{\mu}_j).$$

On the other hand,

$$U_{j}^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) = \begin{cases} \mathcal{O}((p_{j} - p_{j-1})\log 1/|z|), & z \to \infty, \ p_{j-1} > p_{j}, \\ \mathcal{O}(1), & z \to \infty, \ p_{j-1} = p_{j}. \end{cases}$$
(22)

Let us analyze these two cases separately.

If  $p_j = p_{j-1}$ , the second part of (22) implies that  $U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z)$  is subharmonic in  $\overline{\mathbb{C}} \setminus \operatorname{supp}(\overline{\mu}_j)$ ; consequently,  $U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) \leq 0$  on  $\Delta_j$  and  $U_j^{\overline{\mu}}(z) < U_{j-1}^{\overline{\mu}}(z)$  on  $\overline{\mathbb{C}} \setminus \Delta_j$ .

When  $p_{j-1} > p_j$ , the first part of (22) entails that in a neighborhood of  $z = \infty$ ,  $U_j^{\overline{\mu}}(z) > U_{j-1}^{\overline{\mu}}(z)$ . Let  $\gamma_j = \{z \in \mathbb{C} : U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) = 0\}$ . The equilibrium condition implies that  $\gamma_j \supset \text{supp}(\overline{\mu}_j)$ , and the initial remark of this sentence indicates that  $\gamma_j$  is bounded. Consider any bounded component of the complement of  $\gamma_j$ . On it,  $U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z)$  is subharmonic, and on its boundary,  $U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) = 0$ . Thus,

on any bounded component of the complement of  $\gamma_j$ , we have that  $U_i^{\overline{\mu}}(z) < U_{i-1}^{\overline{\mu}}(z)$ . From the first remark of this sentence it follows that on the unbounded component of the complement of  $\gamma_j$ ,  $U_j^{\overline{\mu}}(z) > U_{j-1}^{\overline{\mu}}(z)$ . Fix  $j \in \{0, ..., m\}$ . For each  $k \in \{j, ..., m\}$  define

$$D_k^j = \left\{ z \in \mathbb{C} \setminus \Delta_j : U_k^{\overline{\mu}}(z) < U_i^{\overline{\mu}}(z), i = j, \dots, m, i \neq k \right\}.$$

Let

$$\zeta_j(z) = \min \left\{ U_k^{\mu}(z) : k = j, \dots, m \right\}$$

**Corollary 1** Let  $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \operatorname{Reg}_{(s_1, \ldots, s_m)} = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ , and consider a sequence of multi-indices  $\Lambda = \Lambda(p_0, \ldots, p_m)$ . Let  $(A_{\mathbf{n},0}, A_{\mathbf{n},1}, \ldots, A_{\mathbf{n},m})$ ,  $\mathbf{n} \in \Lambda$ , be the associated sequence of type I Fourier–Padé approximants for the Nikishin system of Markov functions  $(\widehat{s}_1, \ldots, \widehat{s}_m)$  normalized so that for all  $\mathbf{n}, A_{\mathbf{n},m}$  is monic. Then, for  $j = 0, \ldots, m$ 

$$\lim_{\mathbf{n}\in\Lambda} |A_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp\left(-\zeta_j(z)\right), \quad z \in \bigcup_{k=j}^m D_k^j,$$
(23)

and

$$\limsup_{\mathbf{n}\in\Lambda} |A_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} \le \exp(-\zeta_j(z)), \qquad z \in \mathbb{C} \setminus \left(\Delta_j \bigcup_{k=j}^m D_k^j\right), \qquad (24)$$

uniformly on each compact subset of the indicated set, where  $\overline{\mu} = \overline{\mu}(\mathcal{C}) = (\overline{\mu}_0, \overline{\mu})$  $\ldots, \overline{\mu}_m$ ) is the equilibrium vector measure and  $(\omega_0^{\overline{\mu}}, \ldots, \omega_m^{\overline{\mu}})$  is the system of equilibrium constants for the vector potential problem determined by the interaction matrix C defined in (4) on the system of intervals  $\Delta_j$ , j = 0, ..., m. In particular, if  $p_0 = \cdots = p_m = 1/(m+1)$ , then

$$\lim_{\mathbf{n}\in\Lambda} |A_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp(-U_m(z)), \qquad z\in\mathbb{C}\setminus\bigcup_{k=j}^m \Delta_k.$$
(25)

*Proof* For j = m,  $L_{\mathbf{n},m} = A_{\mathbf{n},m}$ ,  $D_m^m = \mathbb{C} \setminus \Delta_m$  and  $\zeta_m = U_m$ . Therefore, (23) reduces to (18), whereas (24) is satisfied by exclusion since  $\mathbb{C} \setminus (\Delta_m \cup D_m^m) = \emptyset$ . Let us assume that (23)–(24) hold for some  $j + 1 \in \{1, ..., m\}$ , and let us prove that it is also true for j.

Notice that

$$A_{\mathbf{n},j}(z) = L_{\mathbf{n},j}(z) - \sum_{k=j+1}^{m} A_{\mathbf{n},k}(z)\widehat{s}_{j+1,k}(z).$$

Obviously  $\zeta_j(z) = \min(U_j(z), \zeta_{j+1}(z))$ . Taking (18) and (23) (for j + 1) into consideration, on  $D_k^j$  the term containing  $A_{\mathbf{n},k}$  (or  $L_{\mathbf{n},j}$  if k = j) dominates the sum, and (23) immediately follows (notice that  $\hat{s}_{j+1,k}(z) \neq 0, z \in \mathbb{C} \setminus \Delta_{j+1}$ ). On the complement of  $\bigcup_{k=i}^{m} D_{k}^{j}$ , there is no dominating term, and all we can conclude from the previous equality is (24).

Let  $p_0 = \cdots = p_m = 1/(m+1)$ . In this case, on  $\mathbb{C} \setminus \bigcup_{k=j}^m \Delta_k$  we have that  $U_m(z) < U_{m-1}(z) < \cdots < U_j(z)$ , and (25) follows from (23).

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