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Semilocal convergence of secant-like methods for differentiable and nondifferentiable operator equations

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ABSTRACT

From well-known secant-like methods, we observe that we can construct a new family of secant-like methods that includes the secant method and Kurchatov's method. We analyse the local orders of convergence and the efficiencies of the methods of the family and study the semilocal convergence for differentiable and nondifferentiable operators. Finally, we apply our results to conservative problems.

1. Introduction

In this paper, we present a new uniparametric family of secant-like iterative methods, that do not use derivatives in their algorithms, for approximating a solution x^* of a nonlinear equation $F(x) = 0$, where $F : \Omega \subseteq X \to Y$ is a nonlinear operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y.

In [1], Hernández and Rubio construct a uniparametric family of secant-like iterative methods for solving $F(x) = 0$ which depends on a parameter $\lambda \in [0, 1]$. The family is reduced to the secant method if $\lambda = 0$ and to Newton's method if $\lambda = 1$ and F is differentiable. From a new geometric interpretation, we modify the secant method in this paper and construct a new uniparametric family of secant-like iterative methods which depends on a parameter $\lambda \geq 1$. The new family is reduced to the secant method if $\lambda = 1$ and to the well-known method of Kurchatov ([2–5]) if $\lambda = 2$. Remember that Kurchatov's method has R-order of convergence two, as Newton's method, but without the use of derivatives in its algorithm.

The paper is organized as follows. In Section 2, we construct the new uniparametric family of secant-like iterative methods. In Section 3, we study the local order of convergence of the new methods. In Section 4, we analyse the efficiency of the methods. In Section 5, we study the semilocal convergence of the methods in two different situations, for differentiable operators and for any operators. Finally, in Section 6, we illustrate the above-mentioned with two applications to conservative problems.

Throughout the paper, we consider that there exists a first order divided difference $[z, w; F] \in \mathcal{L}(\Omega, Y)$, for all $z, w \in \Omega$, where $\mathcal{L}(\Omega, Y)$ denotes the set of bounded linear operators from Ω into Y. Moreover, we denote $\overline{B(x, \varrho)} = \{y \in X;\ \|y - x\| \leq \varrho\}$ and $B(x, \varrho) = \{y \in X;\ \|y - x\| < \varrho\}$.

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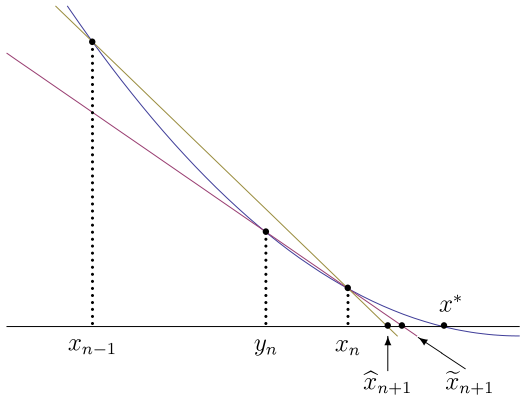

Fig. 1. Geometric interpretation of (1).

2. A new family of secant-like methods

For approximating a solution x^* of the equation $F(x) = 0$, Hernández and Rubio consider in [1] the following uniparametric family of secant-like methods:

$$\begin{cases} x_{-1}, x_0 \text{ given in } \Omega, \quad \lambda \in [0, 1], \\ y_n = \lambda x_n + (1-\lambda)x_{n-1}, \quad n \geq 0, \\ x_{n+1} = x_n - [y_n, x_n; F]^{-1} F(x_n), \end{cases} \tag{1}$$

where $[z, w; F]$, $z, w \in \Omega$, is a first order divided difference, which is a bounded linear operator such that [6]

$$[z, w; F] : \Omega \subset X \longrightarrow Y \quad \text{and} \quad [z, w; F](z - w) = F(z) - F(w).$$

Observe that (1) can be considered as a combination of the secant method and Newton's method, since (1) is reduced to the secant method if $\lambda = 0$ and, provided that F is differentiable, to Newton's method if $\lambda = 1$, since $x_n = y_n$ and $[y_n, x_n; F] = F'(x_n)$. We can also see in [7,8] that the R-order of convergence ([6]) of (1) is at least $\frac{1+\sqrt{5}}{2}$ if $\lambda \in [0, 1)$, the same as that of the secant method. Note that the secant-like methods given by (1) can be considered as a generalization of the secant method.

To obtain family (1), Hernández and Rubio consider the real case and, from the approximations x_{n-1} and x_n, they try to accelerate the secant method by improving the approximation $\widehat{x}_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1}F(x_n)$ to the solution x^*. For this, they fix the approximation x_n, choose an intermediate point y_n between x_{n-1} and x_n, which is defined as $y_n = \lambda x_n + (1-\lambda)x_{n-1}$, with $\lambda \in [0, 1)$, and consider $\widetilde{x}_{n+1} = x_n - [y_n, x_n; F]^{-1} F(x_n)$. From a simple geometric interpretation of the last, see Fig. 1, it is clear that $\widetilde{x}_{n+1}$ is a better approximation to x^* than $\widehat{x}_{n+1}$.

In this paper, we present another improvement of the secant method in the real case that consists of fixing the approximation x_{n-1} and considering the approximation $y_n = \lambda x_n + (1-\lambda)x_{n-1}$, with $\lambda \geq 1$, such that y_n is closer to x^* than x_n, so that the corresponding $\widetilde{x}_{n+1} = x_n - [y_n, x_{n-1}; F]^{-1} F(x_n)$ is also a better approximation to x^* than $\widehat{x}_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1}F(x_n)$. See Fig. 2. So, we extend the previous approximations to Banach spaces and present the following new uniparametric family of secant-like methods:

$$\begin{cases} x_{-1}, x_0 \text{ given in } \Omega, \quad \lambda \geq 1, \\ y_n = \lambda x_n + (1-\lambda)x_{n-1}, \quad n \geq 0, \\ x_{n+1} = x_n - [y_n, x_{n-1}; F]^{-1} F(x_n). \end{cases} \tag{2}$$

In the next sections, we prove that there exists $\lambda_0 \in \mathbb{R}$, $\lambda_0 \geq 2$, such that family (2) has R-order of convergence at least $\frac{1+\sqrt{5}}{2}$ if $\lambda \in [1, \lambda_0]$ and $\lambda \neq 2$, while if $\lambda = 2$, we obtain a second R-order iterative method, the well-known method of Kurchatov ([2–5]). Observe that (2) is reduced to the secant method if $\lambda = 1$.

3. Local order of convergence

In this section, we study the order of convergence of the family of iterations defined in (2), which is established in the next theorem. Firstly, we introduce a development of the first order divide difference of an operator which is based on the ideas presented in [9,10].

If the operator F is differentiable, we have that $[-, -; F] : \Omega \times \Omega \longrightarrow \mathcal{L}(\Omega, Y)$ and

$$[y, x; F](y - x) = F(y) - F(x) = \int_x^y F'(z)\, dz = \int_0^1 F'(x + t(y - x))\, dt\, (y - x),$$

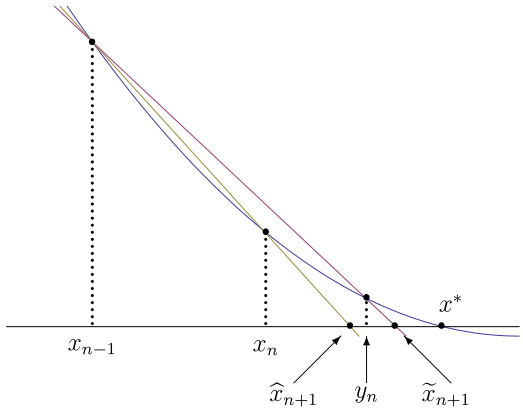

Fig. 2. Geometric interpretation of (2).

for all $x, y \in \Omega$, where $\mathcal{L}(\Omega, Y)$ denotes the set of bounded linear operators from Ω into Y. For F sufficiently differentiable in Ω and $h = y - x$, we obtain:

$$[y, x; F] = \int_0^1 F'(x + th)\, dt = \int_0^1 \left(\sum_{j=1}^{4} \frac{1}{(j-1)!} F^{(j)}(x)(th)^{j-1} + w(x, th) \right) dt$$
$$= \sum_{j=1}^{4} \frac{1}{j!} F^{(j)}(x) h^{j-1} + \int_0^1 w(x, th)\, dt,$$

where $w : \Omega \times \Omega \longrightarrow \mathcal{L}(\Omega, Y)$ with $\|w(x, th)\| = o\left(\|th\|^3\right)$; i.e.: $\lim_{th \to 0} \frac{\|w(x,th)\|}{\|th\|^3} = 0$. In consequence, we have

$$[y, x; F] = F'(x^*) \left(I + \sum_{k=1}^{3} T_k + [F'(\alpha)]^{-1} W(x, e, \tilde{e} - e) \right), \tag{3}$$

where $e = x - x^*$, $\tilde{e} = y - x^*$,

$$T_k = A_{k+1} \sum_{i=0}^{k} e^{k-i}\, \tilde{e}^i \in \mathcal{L}(\Omega, \Omega), \quad k = 1, 2, 3,$$
$$A_k = \frac{1}{k!} [F'(x^*)]^{-1} F^{(k)}(x^*) \in \mathcal{L}\left(\Omega \times \overset{k}{\cdots} \times \Omega, X \right), \quad k = 2, 3, 4,$$
$$W(x, e, \tilde{e} - e) = \sum_{j=0}^{3} w_j(x^*, e)(\tilde{e} - e)^{3-j} + \int_0^1 w(x, t(\tilde{e} - e))\, dt$$

with $W : \Omega \times \Omega \times \Omega \longrightarrow \mathcal{L}(\Omega, \Omega)$ and $\|[F'(x^*)]^{-1} W(x, e, \tilde{e} - e)\| = o(\|e\|^p \|\tilde{e}\|^q)$, $p + q = 3$ and $p, q = 0, 1, 2, 3$.

Before giving the theorem, we introduce the some notations which are used. We denote $w_k(x^*, e)$ when $\|w_k(x^*, e)\| = o(\|e\|^k)$ and $w_{j,k}(x^*, e, \tilde{e})$ when $\|w_{j,k}(x^*, e, \tilde{e})\| = o(\|e\|^j \|\tilde{e}\|^k)$, $j, k \in \mathbb{N}$.

Theorem 1. *The family of iterations defined in (2) has R-order of convergence at least $\frac{1+\sqrt{5}}{2}$ if $\lambda \neq 2$ and at least 2 if $\lambda = 2$. More precisely, if there exists $[F'(x^*)]^{-1}$, then*

$$e_{n+1} = (2 - \lambda) A_2 e_{n-1} e_n + (\lambda - 1) A_2 e_n^2 + w_{2,1}(x^*, e_{n-1}, e_n) \quad \text{if } \lambda \neq 2. \tag{4}$$
$$e_{n+1} = A_2 e_n^2 + A_3 e_{n-1}^2 e_n + w_{2,1}(x^*, e_{n-1}, e_n) \quad \text{if } \lambda = 2. \tag{5}$$

Proof. We do $y = y_n$ and $x = z_n$ in (3) to obtain the expression of $[y_n, z_n; F]$ in terms of $e_n = x_n - x^*$. After that, by expanding in formal power series of e_{n-1} and e_n and taking into account that $[y_n, z_n; F]^{-1}[y_n, z_n; F] = I$, we then see that

$$[y_n, z_n; F]^{-1} = \left(I - (2 - \lambda) A_2 e_{n-1} - \lambda A_2 e_n - (\lambda^2 - 3\lambda + 3) A_3 e_{n-1}^2 + (2 - \lambda)^2 A_2^2\, e_{n-1}^2 + w_2(x^*, e_{n-1}) \right) [F'(x^*)]^{-1}.$$

Observe that the highest local order of convergence for (2) is obtained if we choose $\lambda = 2$, since the term $(2 - \lambda) A_2 e_{n-1}$ disappears, so that

$$[y_n, z_n; F]^{-1} = \left(I - 2 A_2 e_n - A_3 e_{n-1}^2 + w_2(x^*, e_{n-1}) \right) [F'(x^*)]^{-1}.$$

Now, by subtracting the root x^* to both sides of (2) with $\lambda = 2$, we deduce

$$e_{n+1} = e_n - \left(I - 2A_2 e_n - A_3 e_{n-1}^2 + w_2(x^*, e_{n-1})\right) [F'(x^*)]^{-1} F'(x^*) \left(e_n + A_2 e_n^2 + w_2(x^*, e_n)\right)$$

and (5). Taking then norms, we have

$$\|e_{n+1}\| \le \|A_2\| \|e_n\|^2 + \|A_3\| \|e_{n-1}\|^2 \|e_n\|.$$

Consequently the associated equation is $t^2 - t - 2 = 0$ [11,12], whose only real positive root is two. In addition, the R-order of convergence of family (2) with $\lambda = 2$ (Kurchatov's method) is at least two.

On the other case, family (2) with $\lambda \ne 2$, if we follow as above, deduce (4) and

$$\|e_{n+1}\| \le |2 - \lambda| \|A_2\| \|e_{n-1}\| \|e_n\| + (\lambda - 1) \|A_2\| \|e_n\|^2.$$

Therefore the associated equation is now $t^2 - t - 1 = 0$, whose unique positive root is $\frac{1+\sqrt{5}}{2}$. Then, the R-order of convergence is at least $\frac{1+\sqrt{5}}{2}$, the same as that of the secant method. □

Notice that uniparametric family (2) is reduced to the following two well-known methods:

- The secant method if $\lambda = 1$:

$$\begin{cases} x_{-1}, x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n, x_{n-1}; F]^{-1} F(x_n), \quad n \ge 0. \end{cases} \tag{6}$$

- Kurchatov's method if $\lambda = 2$:

$$\begin{cases} x_0, x_{-1} \text{ given in } \Omega, \\ x_{n+1} = x_n - [2x_n - x_{n-1}, x_{n-1}; F]^{-1} F(x_n), \quad n \ge 0. \end{cases} \tag{7}$$

Note that Kurchatov's method has a geometrical interpretation similar to the secant method in the scalar case (see [2]).

4. Efficiency analysis

Once we know the R-order of convergence of methods of family (2), we analyse their efficiency. For this, we consider that equation $F(x) = 0$ represents a nonlinear system of dimension m, namely, $F(x_1, x_2, \ldots, x_m) = 0$, where $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is a nonlinear function and $F \equiv (F_1, F_2, \ldots, F_m)$ with $F_i : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}$, $i = 1, 2, \ldots, m$, and count the evaluations of operators involved and analyse the operational cost needed to apply the methods.

To make the analysis of the efficiency easier, we write (2), (6) and (7) as:

$$\text{method (2):} \quad \begin{cases} x_{-1}, x_0 \in \Omega, \quad \lambda \ge 1, \\ y_n = \lambda x_n + (1 - \lambda) x_{n-1}, \quad n \ge 0, \\ [y_n, x_{n-1}; F]\, b_n = -F(x_n), \\ x_{n+1} = x_n + b_n, \end{cases}$$

$$\text{method (6):} \quad \begin{cases} x_{-1}, x_0 \in \Omega, \\ [x_n, x_{n-1}; F]\, c_n = -F(x_n), \quad n \ge 0, \\ x_{n+1} = x_n + c_n, \end{cases}$$

$$\text{method (7):} \quad \begin{cases} x_{-1}, x_0 \in \Omega, \\ [2x_n - x_{n-1}, x_{n-1}; F]\, d_n = -F(x_n), \quad n \ge 0, \\ x_{n+1} = x_n + d_n. \end{cases}$$

After that, we count the number of functions that are required by the two methods per iteration. We observe that method (6) requires the m functions F_j $(j = 1, 2, \ldots, m)$ and the $m^2 - m$ evaluations of functions included in the divided difference matrix $[u, v; F] = ([u, v; F]_{ij})_{i,j=1}^m \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, where

$$[u, v; F]_{ij} = \frac{1}{u_j - v_j} \left(F_i(u_1, \ldots, u_{j-1}, u_j, v_{j+1}, \ldots, v_m) - F_i(u_1, \ldots, u_{j-1}, v_j, v_{j+1}, \ldots, v_m)\right), \tag{8}$$

$i, j = 1, 2, \ldots, m$, $u = (u_1, u_2, \ldots, u_m)^T$ and $v = (v_1, v_2, \ldots, v_m)^T$ to be evaluated per iteration, that sum $a_1(m) = m^2$ evaluations of functions. Methods (2) and (7) require m evaluations of functions more, so that the total number of evaluations of functions is $a_0(m) = a_2(m) = m^2 + m$.

Next, we count the number of products and divisions that are required by the three methods per iteration. We observe that method (6) requires m^2 divisions to compute $[x_n, x_{n-1}; F]$, $\frac{m}{6}(m-1)(2m-1)$ products and $\frac{m}{2}(m-1)$ divisions for the decomposition LU and $m(m-1)$ products and m divisions for solving two triangular linear systems, that sum

$$p_1(m) = \frac{m}{6}(2m^2 + 3m - 5) + \frac{m}{2}(3m + 1)\,\ell,$$

products, where $\ell \geq 1$ is the ratio between products and quotients. For example, $\ell = 1$ if Maple is used. However, for MPFR with the processor and the precision used in this work, $\ell = 1.7$ approximately. Method (7) requires the same number of products and divisions as method (6), so that $p_2(m) = p_1(m)$. Notice that method (2) presents $2m$ additional products respect to (6) when we compute $y_n = \lambda x_n + (1-\lambda)x_{n-1}$, where $\lambda \notin \mathbb{N}$. Namely, $p_0(m) = p_1(m) + 2m$.

After that, we define the computational efficiency index of three iterative methods (2), (6) and (7) respectively, by

$$CEI_i(\mu, m) = \rho_i^{\frac{1}{\mathcal{C}_i(\mu,m)}}, \quad i = 0, 1, 2, \tag{9}$$

where ρ_i is the R-order of convergence and $\mathcal{C}_i(\mu, m)$ the computational cost of the corresponding method. A similar definition of CEI can be found in [13,14,6].

The computational cost is given by

$$\mathcal{C}_i(\mu, m) = a_i(m)\mu + p_i(m), \quad i = 0, 1, 2, \tag{10}$$

where $a_i(m)$ is the number of scalar functions, $p_i(m)$ is the number of products per iteration and μ is the ratio between products and evaluations of functions that are required to express $\mathcal{C}_i(\mu, m)$ in terms of products. Notice that the definition of CEI given in (9) is a generalization of the scalar case to several variables (see [11,12]). In consequence, the costs are respectively

$$\mathcal{C}_0(\mu, m) = (m^2 + m)\,\mu + \frac{m}{6}\,(2m^2 + 3m + 7) + \frac{m}{2}\,(3m + 1)\,\ell, \qquad \rho_0 = \frac{1+\sqrt{5}}{2};$$
$$\mathcal{C}_1(\mu, m) = m^2\,\mu + \frac{m}{6}\,(2m^2 + 3m - 5) + \frac{m}{2}\,(3m + 1)\,\ell, \qquad \rho_1 = \frac{1+\sqrt{5}}{2};$$
$$\mathcal{C}_2(\mu, m) = (m^2 + m)\,\mu + \frac{m}{6}\,(2m^2 + 3m - 5) + \frac{m}{2}\,(3m + 1)\,\ell, \qquad \rho_2 = 2.$$

The comparison between CEI_0 and the other ones is easy since $\rho_0 = \rho_1$ and $\mathcal{C}_0 > \mathcal{C}_1$, and then $CEI_0 < CEI_1$. In the other case we have $\rho_0 < \rho_2$ and $\mathcal{C}_0 > \mathcal{C}_2$, and, in consequence, $CEI_0 < CEI_2$. Moreover, we compare the CEI of (6) and (7) using the following expression

$$R_{12} = \frac{\log CEI_1}{\log CEI_2} = \frac{\log \rho_1}{\log \rho_2}\,\frac{\mathcal{C}_2}{\mathcal{C}_1},$$

and we have the following theorem.

Theorem 2. *We have $CEI_2 > CEI_1$ $\forall m \geq 3$ and $\forall \ell \geq 1$, and for $m = 2$ when μ is sufficiently small depending on ℓ. In all cases $CEI_1 > CEI_0$ and $CEI_2 > CEI_0$.*

In particular, we have $CEI_2 > CEI_1$ for $m = 2$, when $\ell = 1$ with $\mu < 18.48023$ and if we take $\ell = 2.5$, then $\mu < 37.8845$. Considering the results of CEI in Theorem 2, for the methods analysed by family (2), Kurchatov's method (7) is the most efficient if $m \geq 3$.

5. Semilocal convergence

If we consider, for the first order divided difference, the condition

$$\|[x, y; F] - [u, v; F]\| \leq \omega(\|x - u\|, \|y - v\|), x, y, u, v \in \Omega, \tag{11}$$

it is known that the operator F is differentiable if $\omega(0, 0) = 0$, see [6]. In other case, the operator F may be non-differentiable. So, to analyse the semilocal convergence of (2), we consider two cases: differentiable operators ($\omega(0, 0) = 0$) and any operators. In the case of differentiable operators we consider the usual Lipschitz-condition. On the other hand, notice that if the operator F is not differentiable, we shall consider that there exists a first-order divided difference $[z, w; F] \in \mathcal{L}(X, Y)$, for all $z, w \in \Omega$, with $z \neq w$.

5.1. Differentiable operators

In this section we analyse the semilocal convergence of family (2). For this, we use a technique based on proving first a system of recurrence relations. Firstly, we suppose that there exists a first-order divided difference $[x, y; F] \in \mathcal{L}(X, Y)$, for all $x, y \in \Omega$, where $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y. Secondly, we suppose that

(I) $x_{-1}, x_0 \in \Omega$, such that $\|x_0 - x_{-1}\| \leq \alpha$, and $y_0 = \lambda x_0 + (1-\lambda)x_{-1} \in \Omega$, where $\lambda \geq 1$,
(II) the bounded linear operator $L_0 = [y_0, x_{-1}; F]$ is invertible and such that $\|L_0^{-1}\| \leq \beta$ and $\|L_0^{-1}F(x_0)\| \leq \eta$,
(III) $\|[x, y; F] - [u, v; F]\| \leq K(\|x - u\| + \|y - v\|)$, $K \geq 0$, $x, y, u, v \in \Omega$.

Under conditions (I)–(III), we establish a system of recurrence relations in the next. Before, we denote

$$a_{-1} = \frac{\eta}{\alpha+\eta}, \qquad b_{-1} = \frac{K\beta\alpha^2}{\alpha+\eta}$$

and define the real sequences

$$a_n = f(a_{n-1})g(a_{n-1})b_{n-1}, \qquad b_n = f(a_{n-1})^2 a_{n-1} b_{n-1}, \quad n \geq 0, \tag{12}$$

where

$$f(t) = \frac{1}{1-t} \quad \text{and} \quad g(t) = (2-\lambda) + \lambda f(t)t.$$

Note that $f(t)$ and $g(t)$ are increasing and $f(t) > 1$ provided that $t \in (0, 1)$.

From conditions (I)–(III), it follows that x_1 is well-defined, since the operator L_0^{-1} exists and

$$\|x_1 - x_0\| = \|L_0{}^{-1}F(x_0)\| \leq \eta = f(a_{-1})a_{-1}\|x_0 - x_{-1}\|, \tag{13}$$
$$\|y_1 - x_0\| \leq \|y_1 - x_1\| + \|x_1 - x_0\| \leq \lambda\|x_1 - x_0\| \leq \lambda\eta,$$
$$K\|L_0{}^{-1}\|\,\|x_0 - x_{-1}\| \leq K\beta\alpha = f(a_{-1})b_{-1}. \tag{14}$$

Next, we prove the following recurrence relations for $j \in \mathbb{N}$ by mathematical induction on j:

(i) there exists $L_j{}^{-1} = \left[y_j, x_{j-1}; F\right]^{-1}$ and $\|L_j{}^{-1}\| \leq f(a_{j-1})\|L_{j-1}{}^{-1}\|$,
(ii) $\|x_{j+1} - x_j\| \leq f(a_{j-1})a_{j-1}\|x_j - x_{j-1}\|$,
(iii) $\|y_{j+1} - x_0\| \leq \left(1 + \sum_{k=0}^{j-2}\left(\prod_{i=0}^{k} f(a_i)a_i\right) + \lambda\prod_{i=0}^{j-1} f(a_i)a_i\right)\|x_1 - x_0\|$,
(iv) $\|x_{j+1} - x_0\| \leq \left(1 + f(a_0)a_0 + \cdots + \prod_{i=0}^{j-1} f(a_i)a_i\right)\|x_1 - x_0\|$,
(v) $K\|L_j{}^{-1}\|\,\|x_j - x_{j-1}\| \leq f(a_{j-1})b_{j-1}$.

Suppose now $a_0 < 1$ and $x_1 \in \Omega$. Then from (12), (13) and (14) we have:

$$\begin{aligned}\|I - L_0{}^{-1}L_1\| &\leq \|L_0{}^{-1}\|\,\|L_0 - L_1\| \leq K\|L_0{}^{-1}\|(\|y_1 - y_0\| + \|x_0 - x_{-1}\|)\\ &\leq K\beta\left((2-\lambda) + \lambda f(a_{-1})a_{-1}\right)\|x_0 - x_{-1}\| \leq a_0 < 1\end{aligned}$$

and, by the Banach lemma, the operator $L_1{}^{-1}$ exists and $\|L_1{}^{-1}\| \leq f(a_0)\|L_0{}^{-1}\|$, so that (i) holds for $j = 1$.

From Taylor's series, it follows

$$F(x_1) = (F'(x_0) - L_0)(x_1 - x_0) + \int_0^1 \left(F'(x_0 + \tau(x_1 - x_0)) - F'(x_0)\right)(x_1 - x_0)\,d\tau,$$

and taking into account (III) we obtain

$$\|F(x_1)\| \leq K\left((2-\lambda)\|x_0 - x_{-1}\| + \|x_1 - x_0\|\right)\|x_1 - x_0\| \leq Kg(a_{-1})\|x_0 - x_{-1}\|\,\|x_1 - x_0\|.$$

In addition, since the operator $L_1{}^{-1}$ exists, x_2 is well-defined and

$$\|x_2 - x_1\| \leq \|L_1{}^{-1}\|\,\|F(x_1)\| \leq f(a_0)\|L_0{}^{-1}\|\,\|F(x_1)\| \leq f(a_0)a_0\|x_1 - x_0\|,$$
$$\|y_2 - x_0\| \leq \lambda\|x_2 - x_1\| + \|x_1 - x_0\| \leq (1 + \lambda f(a_0)a_0)\|x_1 - x_0\|,$$
$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq (1 + f(a_0)a_0)\|x_1 - x_0\|,$$

so that (ii) and (iii) hold for $j = 1$.

From (14) and (i) for $j = 1$, we see that

$$K\|L_1{}^{-1}\|\,\|x_1 - x_0\| \leq Kf(a_0)\|L_0{}^{-1}\|\,\|x_1 - x_0\| \leq f(a_0)b_0,$$

so that (iv) holds for $j = 1$.

If we now suppose that $a_n < 1$ and $x_n \in \Omega$ for all $n \in \mathbb{N}$, items (i)–(iv) for $j = n + 1$ follow similarly to (i)–(iv) for $j = 1$ and the mathematical induction is complete.

On the other hand, it is not difficult to prove that the real sequences $\{a_n\}$ and $\{b_n\}$ defined in (12) are decreasing if

$$\lambda \in [1, \lambda_0] \quad \text{with } \lambda_0 \in \left[2, \frac{2\alpha}{\alpha - \eta}\right), \qquad a_{-1} < \frac{3-\sqrt{5}}{2} \quad \text{and} \quad b_{-1} < \frac{a_{-1}(1-a_{-1})^2}{2(1-a_{-1}) - \lambda(1-2a_{-1})}. \tag{15}$$

Now, we are ready to prove the following semilocal convergence theorem for family (2) when conditions (I)–(III) are satisfied.

Theorem 3. *Let X and Y be two Banach spaces and let $F : \Omega \subseteq X \to Y$ be a nonlinear operator defined on a non-empty open convex domain Ω. We suppose that there exists $[x, y; F] \in \mathcal{L}(X, Y)$, for all $x, y \in \Omega$, and conditions (I)–(III) and (15) are satisfied. If $\overline{B(x_0, R)} \subset \Omega$, where $R = \frac{1-a_0}{1-2a_0}\lambda\eta$, then family (2), starting from x_{-1} and x_0, converges to a solution x^* of $F(x) = 0$. Moreover, the solution x^* and the iterates x_n belong to $\overline{B(x_0, R)}$. Furthermore, the solution x^* is unique in $\Omega_0 = B(x_0, \sigma) \cap \Omega$, where $\sigma = \frac{1}{K\beta} - \lambda\alpha - R$, provided that $R < \frac{1}{2}\left(\frac{1}{K\beta} - \lambda\alpha\right)$.*

Proof. Firstly, since $\{a_n\}$ is a decreasing sequence and $a_0 < 1$, then $a_n < 1$ for all $n \in \mathbb{N}$.

Secondly, for $p \in \mathbb{N}$, we have

$$\begin{aligned}\|x_{n+p} - x_n\| &\le \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\le \sum_{j=n}^{n+p-1} f(a_{j-1})a_{j-1}\|x_j - x_{j-1}\| \le (f(a_0)a_0)^n \sum_{j=0}^{p-1} (f(a_0)a_0)^j \\ &= (f(a_0)a_0)^n \, \frac{1-(f(a_0)a_0)^p}{1-f(a_0)a_0}\, \|x_1 - x_0\|,\end{aligned}$$

as $f(a_j)a_j \le f(a_0)a_0$, for $j = 1, 2, \ldots, n+p-1$, then the sequence $\{x_n\}$ defined in (2) is of Cauchy, and consequently $\{x_n\}$ is convergent and $\lim_n x_n = x^* \in \overline{B(x_0, R)}$.

Thirdly, if $n = 0$ in the above, we obtain

$$\|x_p - x_0\| \le \frac{1-(f(a_0)a_0)^p}{1-f(a_0)a_0}\|x_1 - x_0\| < R$$

and $x_p \in B(x_0, R)$ for all $p \in \mathbb{N}$.

Fourthly, from

$$F(x_n) = (F'(x_{n-1}) - L_{n-1})(x_n - x_{n-1}) + \int_0^1 \left(F'(x_{n-1} + \tau(x_n - x_{n-1})) - F'(x_{n-1})\right)(x_n - x_{n-1})\, d\tau$$

it follows that

$$\|F(x_n)\| \le K\left((2-\lambda)\|x_{n-1} - x_{n-2}\| + \|x_n - x_{n-1}\|\right)\|x_n - x_{n-1}\|.$$

Thus, $\lim_{n\to\infty} \|F(x_n)\| = 0$ and, since F is continuous at x^*, $\lim_{n\to\infty} \|F(x_n)\| = \|F(x^*)\| = 0$. Consequently, $\{x_n\}$ converges to a solution x^* of $F(x) = 0$.

Finally, to prove the uniqueness of the solution x^*, we first assume that y^* is another solution of $F(x) = 0$ in $\Omega_0 = B(x_0, \sigma) \cap \Omega$. Next, we consider the operator $T = [y^*, x^*; F]$, so that if T is invertible, we have $x^* = y^*$, since $T(y^* - x^*) = T(y^*) - T(x^*)$. Indeed,

$$\begin{aligned}\|L_0^{-1}T - I\| &\le \|L_0^{-1}\|\|T - L_0\| \le \|L_0^{-1}\|\|[y^*, x^*; F] - [y_0, x_{-1}; F]\| \\ &\le K\beta\left(\|y^* - y_0\| + \|x^* - x_{-1}\|\right) < K\beta(\sigma + \lambda\alpha + R) = 1,\end{aligned}$$

and the operator T^{-1} exists. □

5.2. Any operators

Note that condition (III) guarantees the differentiability of the operator F (see [6]). If we are interested in obtaining a valid result for nondifferentiability operators, condition (III) must be relaxed.

Now, we analyse the semilocal convergence of family (2) when it is applied to any operator. For this, we suppose (I), (II) and

(IIIb) $\|[x, y; F] - [u, v; F]\| \le \omega(\|x-u\|, \|y-v\|)$, $x, y, u, v \in \Omega$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous non-decreasing function in both arguments.

Theorem 4. *Let X and Y be two Banach spaces and let $F : \Omega \subseteq X \to Y$ be a nonlinear operator defined on a non-empty open convex domain Ω. We suppose that there exists $[x, y; F] \in \mathcal{L}(X, Y)$, for all $x, y \in \Omega$, and conditions (I), (II) and (IIIb) are satisfied. We also suppose that the equation*

$$t\left(1 - \frac{m}{1 - \beta\,\omega((2\lambda-1)t + (\lambda-1)\alpha, t+\alpha)}\right) - \lambda\eta = 0, \tag{16}$$

where $m = \max\{\beta\,\omega(\eta + (\lambda-1)\alpha, \alpha), \beta\,\omega(\lambda\eta, \eta)\}$, has at least one positive root and r denotes the smallest positive root of (16). If $\beta\,\omega((2\lambda-1)r + (\lambda-1)\alpha, r+\alpha) < 1$, $M = \frac{m}{1-\beta\,\omega((2\lambda-1)r+(\lambda-1)\alpha, r+\alpha)} < 1$ and $\overline{B(x_0, r)} \subset \Omega$, then family (2), starting from x_{-1} and x_0, is well-defined and converges to a unique solution x^* of $F(x) = 0$. Moreover, the solution x^* and the iterates x_n belong to $\overline{B(x_0, r)}$ and x^* is unique in $B(x_0, r)$.

Proof. Firstly, by mathematical induction, we prove that family (2) is well-defined; namely, iterative method (2) makes sense if the operator $[y_n, x_{n-1}; F]$ is invertible and the approximation $x_{n+1} \in \Omega$.

From the convergence conditions, it follows that x_1 is well-defined and $\|x_1 - x_0\| \le \eta < r$. Consequently, $x_1, y_1 \in B(x_0, r) \subseteq \Omega$.

Next, from (IIIb), we have

$$\begin{aligned}\|I - L_0^{-1}L_1\| &\le \|L_0^{-1}\|\|L_0 - L_1\| \le \beta\omega(\lambda\|x_1 - x_0\| + (\lambda - 1)\|x_0 - x_{-1}\|, \|x_0 - x_{-1}\|)\\ &\le \beta\omega((2\lambda - 1)r + (\lambda - 1)\alpha, r + \alpha) < 1,\end{aligned}$$

and then, by Banach's lemma, the operator L_1^{-1} exists and

$$\|L_1^{-1}\| \le \frac{\beta}{1 - \beta\omega((2\lambda - 1)r + (\lambda - 1)\alpha, r + \alpha)}.$$

After that, we observe

$$F(x_1) = F(x_0) + [x_1, x_0; F](x_1 - x_0) = ([x_1, x_0; F] - L_0)\,(x_1 - x_0)$$

and taking again (IIIb) it follows

$$\|F(x_1)\| \le \|[x_1, x_0; F] - L_0\|\,\|x_1 - x_0\| \le \omega(\|x_1 - y_0\|, \|x_0 - x_{-1}\|)\|x_1 - x_0\| \le \omega(\eta + (\lambda - 1)\alpha, \alpha)\|x_1 - x_0\|.$$

In addition, the approximation x_2 is well-defined and

$$\begin{aligned}&\|x_2 - x_1\| \le \|L_1^{-1}\|\|F(x_1)\| \le M\|x_1 - x_0\| < \eta,\\ &\|y_2 - x_0\| \le (1 + \lambda M)\|x_1 - x_0\| < r,\\ &\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\| \le (1 + M)\|x_1 - x_0\| < r,\end{aligned}$$

since r is a positive root of Eq. (16), so that $x_2 \in B(x_0, \rho)$.

If we now suppose that $L_j = \left[y_j, x_{j-1}; F\right]$ is invertible and $x_{j+1} \in B(x_0, r) \subseteq \Omega$, for all $j = 1, 2, \ldots, n - 1$, then items

- the operator L_n^{-1} exists and $\|L_n^{-1}\| \le \frac{\beta}{1-\beta\omega((2\lambda-1)r+(\lambda-1)\alpha,r+\alpha)}$,
- $\|x_{n+1} - x_n\| \le M\|x_n - x_{n-1}\| \le M^n\|x_1 - x_0\| < \eta$,

follow similarly to the above and the mathematical induction concludes that family (2) is well-defined.

Secondly, we prove that $\{x_n\}$ is a Cauchy sequence. As,

$$\begin{aligned}\|x_{n+p} - x_n\| &\le \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \cdots + \|x_{n+1} - x_n\|\\ &\le \left(M^{p-1} + M^{p-2} + \cdots + 1\right)\|x_{n+1} - x_n\| = \frac{1 - M^p}{1 - M}\|x_{n+1} - x_n\| < \frac{M^n}{1 - M}\|x_1 - x_0\|,\end{aligned}$$

it is clear that $\{x_n\}$ is a Cauchy sequence, so that $\{x_n\}$ is convergent.

Thirdly, if $\lim_n = x^*$, we see that x^* is a solution of $F(x) = 0$. Indeed, as

$$\|F(x_n)\| \le \omega(\lambda\eta, \eta)\|x_n - x_{n-1}\|,$$

by letting $n \to \infty$, we obtain $\|x_n - x_{n-1}\| \to 0$, and consequently $F(x^*) = 0$.

Finally, we see the uniqueness of x^*. For this, we suppose that there exists another solution y^* of $F(x) = 0$ in $B(x_0, r)$, and consider the operator $T = [y^*, x^*; F]$. If T is invertible, we have that $x^* = y^*$, since $T(y^* - x^*) = T(y^*) - T(x^*)$. Indeed,

$$\begin{aligned}\|L_0^{-1}T - I\| &\le \|L_0^{-1}\|\|T - L_0\| \le \|L_0^{-1}\|\|[y^*, x^*; F] - [y_0, x_{-1}; F]\|\\ &\le \beta\omega(\|y^* - y_0\|, \|x^* - x_{-1}\|) \le \beta\omega((2\lambda - 1)r + (\lambda - 1)\alpha, r + \alpha) < 1,\end{aligned}$$

and the operator T^{-1} exists. □

Note that if $\omega(0, 0) = 0$, then we can prove that F is differentiable (see [6]). However, if F is nondifferentiable, we obtain that $\omega(0, 0) \ne 0$, as we can see in Section 6.3.

6. Application to a special case of conservative problems

It is well known that energy is dissipated in the action of any real dynamical system, usually through some form of friction. However, in certain situations this dissipation is so slow that it can be neglected over relatively short periods of time. In such cases we assume the law of conservation of energy, namely, that the sum of the kinetic energy and the potential energy is constant. A system of this kind is said to be conservative.

If φ and ψ are arbitrary functions with the property that $\varphi(0) = 0$ and $\psi(0) = 0$, the general equation

$$\mu\,\frac{d^2x(t)}{dt^2} + \psi\left(\frac{dx(t)}{dt}\right) + \varphi(x) = 0, \tag{17}$$

can be interpreted as the equation of motion of a mass μ under the action of a restoring force $-\varphi(x)$ and a damping force $-\psi(dx/dt)$. In general these forces are nonlinear, and Eq. (17) can be regarded as the basic equation of nonlinear mechanics. In this paper we shall consider the special case of a nonlinear conservative system described by the equation

$$\mu \frac{d^2x(t)}{dt^2} + \varphi(x(t)) = 0,$$

in which the damping force is zero and there is consequently no dissipation of energy. Extensive discussions of (17), with applications to a variety of physical problems, can be found in classical references [15,16].

In this paper, we study the existence of a unique solution for a special case of a nonlinear conservative system described by the equation

$$\frac{d^2x(t)}{dt^2} + \phi(x(t)) = 0 \tag{18}$$

with the boundary conditions

$$x(0) = x(1) = 0. \tag{19}$$

In order to study the application of (2) for the numerical solution of differential equation problems, we illustrate the theory for the case of particular second-order ordinary differential Eq. (18) subject to the boundary conditions (19).

It is required to find a solution of problem (18) and (19) in the interval $t \in [0, 1]$. Under suitable restrictions on the function ϕ, we will see that a unique solution of (18) and (19) exists. Moreover the method of discretization is used to project the boundary value problem of second order into a finite-dimensional space. The new family of secant-like methods are applied to this problem to approximate the solution of the corresponding system of equations.

For this, we consider the operator

$$[\mathcal{F}(x)](t) = \frac{d^2x(t)}{dt^2} + \phi(x(t)), \tag{20}$$

which is an operator from $C^{(2)}[0, 1]$ into $C[0, 1]$. Then, solving problem (18) is equivalent to solving the equation $\mathcal{F}(x) = 0$, where $\mathcal{F}$ is defined in (20).

Initially, we transform problem (18) and (19) into a finite dimensional problem. For this, we approximate the second derivative by a standard numerical formula. Moreover, from now on, we use the infinity norm.

6.1. Setting up a finite difference scheme

For the direct numerical solution of problem (18) and (19), we introduce the points $t_j = jh, j = 0, 1, \ldots, m + 1$, where $h = \frac{1}{m+1}$ and m is an appropriate integer. A scheme is then designed for the determination of numbers x_j, it is hoped, approximate the values $x(t_j)$ of the true solution at the points t_j. A standard approximation for the second derivative at these points is

$$x_j'' \approx \frac{x_{j-1} - 2x_j + x_{j+1}}{h^2}, \quad j = 1, 2, \ldots, m.$$

A natural way to obtain such a scheme is to demand that the x_j satisfy at each interior mesh point t_j the difference equation

$$x_{j-1} - 2x_j + x_{j+1} + h^2\phi(x_j) = 0. \tag{21}$$

Since x_0 and x_{m+1} are determined by the boundary conditions, the unknowns are $x_1, x_2, \ldots, x_m$.

A further discussion is simplified by the use of matrix and vector notation. Introducing the vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \qquad v_{\mathbf{x}} = \begin{pmatrix} \phi(x_1) \\ \phi(x_2) \\ \vdots \\ \phi(x_m) \end{pmatrix}$$

and the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & \ldots & 0 \\ 0 & 1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -2 \end{pmatrix},$$

the system of equations, arising from demanding that (21) holds for $j = 1, 2, \ldots, m$, can be written compactly in the form

$$F(\mathbf{x}) \equiv A\mathbf{x} + h^2 v_{\mathbf{x}} = 0, \tag{22}$$

which is a function from $\mathbb{R}^m$ into $\mathbb{R}^m$.

If the function $\phi(x)$ is not linear in x, we cannot hope to solve system (22) by algebraic methods. Some iterative procedure must be resorted to. Next, we analyse (2) for this purpose.

6.2. A differentiable case

In this section we consider a particular case of (18). The steady temperature distribution is known in a homogeneous rod of length 1 in which, as a consequence of a chemical reaction or some such heat-producing process, heat is generated at a rate $\phi(x(t))$ per unit time per unit length, $\phi(x(t))$ being a given function of the excess temperature x of the rod over the temperature of the surroundings. If the ends of the rod, $t = 0$ and $t = 1$, are kept at given temperatures, we are to solve the boundary value problem given by (18) and (19), measured along the axis of the rod. For an example we choose an exponential law $\phi(x(t)) = \exp(x(t))$ for the heat generation.

Taking into account that the solution of (18) and (19) with $\phi(x(t)) = \exp(x(t))$ is of the form

$$x(t) = \int_0^1 G(t,\xi)\exp(x(\xi))\,d\xi,$$

where $G(t,\xi)$ is the Green function in $[0,1]$, we can locate the solution $x^*(t)$ in some domain. So, we have

$$\|x^*(t)\| - \frac{1}{8}\exp(\|x^*(t)\|) \le 0,$$

so that $\|x^*(t)\| \in [0,\varrho_1] \cup [\varrho_2,+\infty]$, where $\varrho_1 = 0.1444$ and $\varrho_2 = 3.2616$ are the two positive real roots of the scalar equation $8t - \exp(t) = 0$.

By Theorem 3, we could only guarantee the semilocal convergence to a solution $x^*(t)$ such that $\|x^*(t)\| \in [0,\varrho_1]$. For this, we can consider the domain

$$\Omega = \{x(t) \in C^2[0,1]\,;\, \|x(t)\| \le \ln 6,\ t \in [0,1]\},$$

since $\varrho_1 < \ln 6 < \varrho_2$.

In view of what the domain Ω is for Eq. (18), we then consider (22) with $F : \widetilde{\Omega} \subset \mathbb{R}^m \to \mathbb{R}^m$ and

$$\widetilde{\Omega} = \{(x_1,x_2,\ldots,x_m) \in \mathbb{R}^m;\ |x_i| \le \ln 6, \text{for} i = 1,2,\ldots,m\}.$$

6.2.1. Approximating the solution

According to Section 6.1, $v_{\mathbf{x}} = (\exp(x_1),\exp(x_2),\ldots,\exp(x_m))^t$ if $\phi(x(t)) = \exp(x(t))$. Consequently, the first derivative of the function F defined in (22) is given by

$$F'(\mathbf{x}) = A + h^2\mathrm{diag}(v'_{\mathbf{x}}),$$

where $v'_{\mathbf{x}} = (\exp(x_1),\exp(x_2),\ldots,\exp(x_m))$. Moreover,

$$F'(\mathbf{x}) - F'(\mathbf{y}) = -h^2\mathrm{diag}(\mathbf{z}),$$

where $\mathbf{y} = (y_1,y_2,\ldots,y_m)^t$ and $\mathbf{z} = (\exp(x_1)-\exp(y_1),\exp(x_2)-\exp(y_2),\ldots,\exp(x_m)-\exp(y_m))$. In addition,

$$\|F'(\mathbf{x}) - F'(\mathbf{y})\| \le h^2 \max_{1\le i\le m} |\exp(c_i)|\,\|\mathbf{x}-\mathbf{y}\|,$$

where $\mathbf{c} = (c_1,c_2,\ldots,c_m)^t \in \widetilde{\Omega}$, so that

$$\|F'(\mathbf{x}) - F'(\mathbf{y})\| \le 6h^2\|\mathbf{x}-\mathbf{y}\|. \tag{23}$$

Considering (see [6])

$$[\mathbf{x},\mathbf{y};F] = \int_0^1 F'(\tau\mathbf{x} + (1-\tau)\mathbf{y})\,d\tau,$$

taking into account

$$\int_0^1 \|\tau(\mathbf{x}-\mathbf{u}) + (1-\tau)(\mathbf{y}-\mathbf{v})\|d\tau \le \frac{1}{2}\left(\|\mathbf{x}-\mathbf{u}\| + \|\mathbf{y}-\mathbf{v}\|\right),$$

and (23), we have

$$\begin{aligned}\|[\mathbf{x},\mathbf{y};F] - [\mathbf{u},\mathbf{v};F]\| &\le \int_0^1 \|F'(\tau\mathbf{x} + (1-\tau)\mathbf{y}) - F'(\tau\mathbf{u} + (1-\tau)\mathbf{v})\|\,d\tau \\ &\le 6h^2\int_0^1 (\tau\|\mathbf{x}-\mathbf{u}\| + (1-\tau)\|\mathbf{y}-\mathbf{v}\|)\,d\tau \\ &= 3h^2\left(\|\mathbf{x}-\mathbf{u}\| + \|\mathbf{y}-\mathbf{v}\|\right).\end{aligned}$$

From the preceding result we have $K = 3h^2$. If we take $m = 19$, then $h = \frac{1}{20}$ and $K = \frac{3}{400} = 0.0075$. Discretizing the parabolas given by $x_{-1}(t) = \frac{5}{2}t(1-t)$ and $x_0(t) = \frac{1}{2}t(1-t)$ in 21 points in the interval $[0,1]$, we have $\alpha = 0.5$.

Table 1
Numerical solution $\mathbf{x}^*$ of (22) with $\phi(x) = \exp(x)$.

i	x_i^*	i	x_i^*
1	2.62125617742571819416e-02	11	1.39136149148458656192e-01
2	4.98587257172283280025e-02	12	1.34824278757197113339e-01
3	7.08770831869588552737e-02	13	1.27651569159449126900e-01
4	8.92118174747010710997e-02	14	1.17638466907795998138e-01
5	1.04813271226358889813e-01	15	1.04813271226358889813e-01
6	1.17638466907795998138e-01	16	8.92118174747010710997e-02
7	1.27651569159449126900e-01	17	7.08770831869588552737e-02
8	1.34824278757197113339e-01	18	4.98587257172283280025e-02
9	1.39136149148458656192e-01	19	2.62125617742571819416e-02
10	1.40574818132414702110e-01		

Table 2
Numerical results for nonlinear system (22) with $\phi(x) = \exp(x)$.

Method	I	$\Delta\widetilde{\rho}_i$	CEI	TF	q	Time (ms)
(6)	15	3.6E-6	1.000015431	64805.1	3926	9625
(7)	10	1.1E-3	1.000021231	47101.4	2682	6567

We then consider the secant method ($\lambda = 1$) and Kurchatov's method ($\lambda = 2$). For $\lambda = 1$, we have $\beta = 42$, $\eta = 0.22$. In this case $y_0 = x_0 \in \widetilde{\Omega}$. The conditions (15) are accomplished and we get $R = 0.3132$. For $\lambda = 2$, we obtain $\beta = 44$, $\eta = 0.232$. We have now $y_0 = 2x_0 - x_{-1} \in \widetilde{\Omega}$. The conditions (15) are also accomplished and, in this case, $R = 0.5664$. In both cases the conditions of Theorem 3 are accomplished with $\sigma = 2.3633$ for $\lambda = 1$ and $\sigma = 1.4639$ for $\lambda = 2$, so that the secant method and Kurchatov's method converge to the solution $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_{19}^*)^T$ given in Table 1. Note that 4096 significative figures are used in the computations.

6.2.2. Numerical considerations

The numerical computations were performed on MPFR library of C++ multi-precision arithmetics [17,18] with 4096 digits of mantissa. All programs were compiled by g++(4.2.1) for i686-apple-darwin1 with `libgmp (v.4.2.4)` and `libmpfr (v.2.4.0)` libraries in a processor Intel® Xeon E5620, 2.4 GHz (64-bit machine). Within each example the starting point is the same for all methods tested. The classical stopping criteria $\|e_{I+1}\| = \|x_{I+1} - \alpha\| < \varepsilon$ and $\|e_I\| > \varepsilon$, with $\varepsilon = 10^{-\nu}$, where $\nu = 4096$, is replaced by $\|\tilde{e}_{I+1}\| = \|x_{I+1} - \tilde{\alpha}_{I+1}\| < 10^{-\eta}$ and $\|\tilde{e}_I\| > 10^{-\eta}$, where $\eta = \left[\nu(2\rho - 1)/\rho^2\right]$ and $\tilde{\alpha}_n$ is obtained by the δ^2-Aitken procedure, that is

$$\tilde{e}_n = \left(\frac{\left(\delta x_{n-1}^{(r)}\right)^2}{\delta^2 x_{n-2}^{(r)}} \right)_{r=1\div m} \tag{24}$$

where $\delta x_{n-1} = x_n - x_{n-1}$. Note that this criterion is independent of the knowledge of the root (see [19,20]).

In order to compare easily the efficiency of the iterative methods we introduce a new measure, called time factor (TF), defined as

$$TF = 1/\log(CEI).$$

Table 2 shows the results obtained for iterative methods (6) and (7). In each row we can read the necessary iteration number I, the error of the local order of convergence $\Delta\widetilde{\rho}_i$, the computational efficiency index CEI, the time factor TF, the estimation q of the corrected decimal number of last iterated x_I and the time necessary to achieve the I iteration. That is, we compute $q = \left[\rho^2/(2\rho - 1)\eta^\star\right]$, where $\eta^\star = -\log_{10}\|\tilde{e}_I\|$.

In this case the concrete values of parameters for methods (6) and (7) are $(m, \mu) = (19, 77)$ since the evaluation of exponential function is equivalent to 76 products. Note that in Table 2 we have $CEI_2 > CEI_1$ as Theorem 2 asserts and the execution time of method (6) is greater than the execution time of method (7).

In the computations we substitute the computational order of convergence (COC) [21] by an extrapolation ($ECLOC$) denoted by $\tilde{\rho}$ and defined as follows ([19])

$$\tilde{\rho} = \frac{\ln\|\tilde{e}_I\|}{\ln\|\tilde{e}_{I-1}\|},$$

where $\tilde{e}_I$ is given in (24). If $\rho = \tilde{\rho} \pm \Delta\tilde{\rho}$, where ρ is the local order of convergence and $\Delta\tilde{\rho}$ is the error of $ECLOC$, then we get $\Delta\tilde{\rho} < 10^{-3}$. This fact means that in all computations of $ECLOC$ we obtain at least 3 significant digits and this results is a good check of the local convergence orders of the family of iterative methods presented in this paper.

Table 3
Numerical solution $\mathbf{x}^*$ of (22) with $\phi(x) = \frac{1}{3}x^2 + |x| + 1$.

i	x_i^*	i	x_i^*
1	2.61571297987090693801e-02	11	1.38759070015023621721e-01
2	4.97482966100552773252e-02	12	1.34462697508586908805e-01
3	7.07130302690300120457e-02	13	1.27315101410860990497e-01
4	8.89968144084573144406e-02	14	1.17335709947068538197e-01
5	1.04551506151051095720e-01	15	1.04551506151051095720e-01
6	1.17335709947068538197e-01	16	8.89968144084573144406e-02
7	1.27315101410860990497e-01	17	7.07130302690300120457e-02
8	1.34462697508586908805e-01	18	4.97482966100552773252e-02
9	1.38759070015023621721e-01	19	2.61571297987090693801e-02
10	1.40192499780163247060e-01		

Table 4
Numerical results for nonlinear system (22) with $\phi(x) = \frac{1}{3}x(t)^2 + |x(t)| + 1$.

Method	I	$\widetilde{\rho}_i$	$\widetilde{CEI}$	$\widetilde{TF}$	q	Time (ms)
(6)	14	1.61803518875	1.000101883	9815.7	2809	334
(7)	10	2.00000000007	1.000144606	6915.8	2339	247

6.3. A non-differentiable case

In this section, we consider a nondifferentiable operator of type (22), so that Theorem 4 is illustrated. Observe that $\omega(0, 0) \neq 0$ is obtained from condition (IIIb).

Setting $\phi(x(t)) = \frac{1}{3}x(t)^2 + |x(t)| + 1$ in (18), we obtain (22) with $v_{\mathbf{x}} = (v_1, v_2, \ldots, v_m)^t$ and $v_i = \frac{1}{3}x_i^2 + |x_i| + 1$ for $i = 1, 2, \ldots, m$.

6.3.1. Approximating the solution

Taking the definition of divided difference operator (8) we have

$$[\mathbf{x}, \mathbf{y}; F] = A + h^2 \mathrm{diag}(\mathbf{z}),$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_m)$ and $z_i = \frac{x_i+y_i}{3} + \frac{|x_i|-|y_i|}{x_i-y_i}$, for $i = 1, 2, \ldots, m$.

In consequence,

$$\|[\mathbf{x}, \mathbf{y}; F] - [\mathbf{u}, \mathbf{v}; F]\| \leq \frac{h^2}{3}\left(\|\mathbf{x} - \mathbf{u}\| + \|\mathbf{y} - \mathbf{v}\|\right) + 2h^2,$$

so that we consider $\omega(s, t) = h^2\left(\frac{s+t}{3} + 2\right)$ for condition (IIIb) in Theorem 4.

If we take $m = 19$, then $h = \frac{1}{20}$ and $\omega(s, t) = \frac{1}{400}\left(\frac{s+t}{3} + 2\right)$. Furthermore, if we discretize the parabolas given by $x_{-1}(t) = \frac{3}{2}t(1-t)$ and $x_0(t) = \frac{1}{2}t(1-t)$, we have $\alpha = 0.25$. We then consider the secant method ($\lambda = 1$) and Kurchatov's method ($\lambda = 2$). For $\lambda = 1$, we have $\beta = 44.642$, $\eta = 0.235$, and for $\lambda = 2$, we obtain $\beta = 45.904$, $\eta = 0.242$. In both cases the conditions of Theorem 4 are accomplished and the solution with these initial approximations is unique, see Table 3. For $\lambda = 1$ we get $r = 0.3483$, while for $\lambda = 2$ we have $r = 0.8237$.

6.3.2. Numerical considerations

Table 4 shows the results obtained for iterative methods (6) and (7). Observe that the columns of Table 4 are the necessary iteration number I, $ECLOC$, an approximation of the CEI, say $\widetilde{CEI} = \widetilde{\rho}^{1/c}$, and a new time factor defined by $\widetilde{TF} = 1/\log(\widetilde{CEI})$.

In this case the concrete values of parameters for methods (6) and (7) are $(m, \mu) = (19, 3.7)$. Note that not only from a theoretical point of view Theorem 2 is accomplished, but the practical results (execution time) does also.

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