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Attracting cycles for the relaxed Newton's method

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ABSTRACT

We study the relaxed Newton's method applied to polynomials. In particular, we give a technique such that for any $n \geq 2$, we may construct a polynomial so that when the method is applied to a polynomial, the resulting rational function has an attracting cycle of period n. We show that when we use the method to extract radicals, the set consisting of the points at which the method fails to converge to the roots of the polynomial $p(z) = z^m - c$ (this set includes the Julia set) has zero Lebesgue measure. Consequently, iterate sequences under the relaxed Newton's method converge to the roots of the preceding polynomial with probability one.

1. Introduction

As is well known, for some $h \in D_1(1) = \{h \in \mathbb{C} : |h-1| < 1\} \cup \{1\}$, there are polynomials f for which the relaxed Newton's method $N_{f,h}(z) = z - h\frac{f(z)}{f'(z)}$, has attracting cycles. For example, if we consider the one-parameter family of cubic polynomials $f_\lambda(z) = z^3 + (\lambda-1)z - \lambda$ where $\lambda \in \mathbb{C}$, we denote the relaxed Newton's method applied to f_λ by $N_{\lambda,h}$. Then there are open regions in the parameter space (λ-space) such that, for λ in these regions, there are open sets of points in the complex plane such that $N_{\lambda,h}$ fails to converge to the roots of f_λ due to the existence of attracting cycles. For example, when $h = 1.5$, Fig. 1a shows parameter regions (shaded black) for which the iterates of the critical point $z = 0$ of $N_{\lambda,h}$ accumulate on an attracting cycle. Fig. 1b shows a zoom of one of these regions (shaded black).

For $\lambda = 0.945 + 1.01i$, black shaded regions in Fig. 2a, are those where $N_{\lambda,h}$ fails to converge to any root of f_λ. Fig. 2b shows a zoom of one of these regions (shaded black). Note that these black shaded regions are similar to those of the filled Julia set for polynomials (see [1]).

A first study of Newton's method for the family of polynomials f_λ was carried out in [2]. In [3], Saupe makes several numerical studies of the relaxed Newton's method $N_{\lambda,h}$. In [4], Kriete gives a description for $N_{\lambda,h}$ similar to that of [2]. Similar phenomena for the same family of cubic polynomials are observed for both the Schröder and the König iteration functions (see [5–7]). In fact, these authors show the patterns of non-convergent iterated sequences in the complex domain.

Barna seems to be the first who established the existence of a (super)attracting cycle for the Newton's method and demonstrate how badly Newton's method fails (see [8]). As an example, $p(x) = 3x^5 - 10x^3 + 23x$ is a polynomial for which $\Omega = \{-1, 1\}$ is a super-attracting 2-cycle for Newton's iterative function. A general procedure for constructing polynomials whose Newton's maps have a prescribed set of distinct complex numbers $z_1, \ldots, z_n$ as a super-attracting n-cycle is given in [9].

It is important to observe that the iterates' under an iterative method, possible, failure to converge to roots of a nonlinear equation may be due to the existence of a (super)attracting cycle which may trap the sequence of iterates of a given point.

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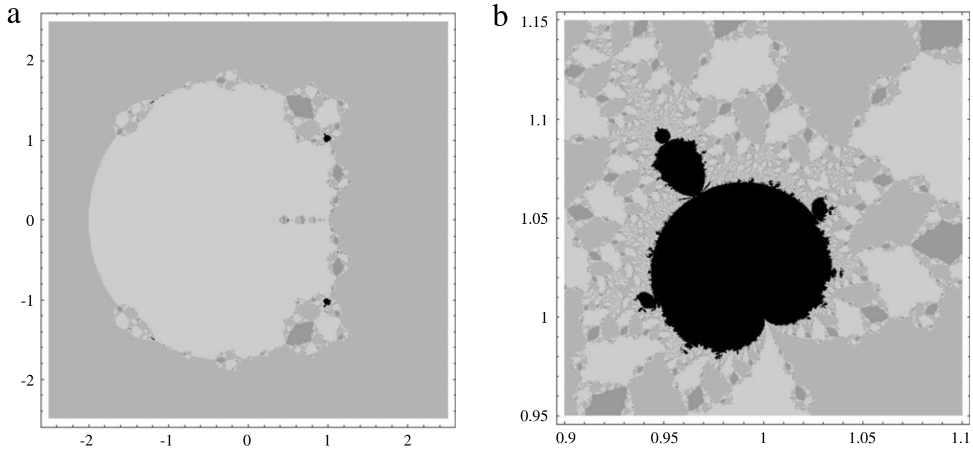
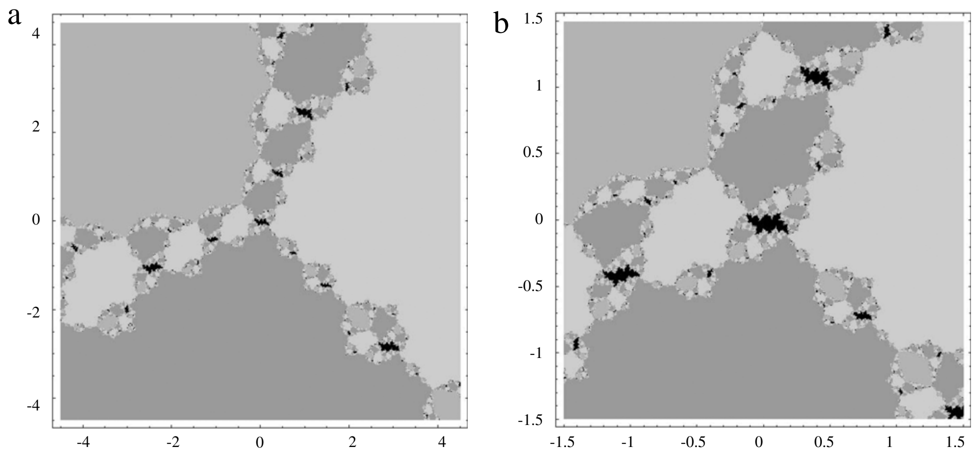


λ-plane

Zoom of a black region in the λ-plane

Fig. 1.



Basins of attraction for N_λ, for
$\lambda = 0.945 + 1.01i$,

Zoom of a black region in Figure 2a

Fig. 2.

The existence of either cycles or fixed points other than the roots of the polynomial to which we are applying our iteration method, whether (super)attracting, or repelling, or indifferent, interferes with the search for roots and alters the convergence regions of roots that are (super)attracting fixed points for the method being used. For a review of basic concepts of Complex Dynamics, see [1,10]; and for a study of the dynamics of Newton's method, see [11].

2. The relaxed Newton's method

Let f be a complex polynomial, and let h be a complex number. The relaxed Newton's method associated to f is the rational map $N_{f,h} : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ given by

$$N_{f,h}(z) = z - h\frac{f(z)}{f'(z)}. \tag{1}$$

The iteration of $N_{f,h}$ gives an algorithm for finding the roots of f.

We let $G(f, h)$ denote the set of "good" initial values, that is, the set of values z_0 for which the sequence of iterates under $N_{f,h}$ converges to a root of f, or in other words $G(f, h) = \{z_0 \in \overline{\mathbb{C}} : N_{f,h}^n(z_0) \rightarrow \text{ a root of } f, \text{ as } n \rightarrow \infty\}$. This set is the union of the convergence regions of the roots of f. As is well known the Julia set of $N_{f,h}$ is closed, invariant, and has an empty interior. Also, the Julia set of $N_{f,h}$ is contained in the set $B(f, h) = \overline{\mathbb{C}} - G(f, h)$, which is called the set of "bad" initial conditions. In addition, there may exist (super)attracting k-cycles, with $k \geq 2$, so that there is an open set U of initial value z_0 such that the iterates $N_{f,h}^n(z_0)$ approach the cycle as $n \rightarrow \infty$. Hence, $N_{f,h}$ fails to converge to a root of f. In other words, $B(f, h)$ may have a non-empty interior. Consequently, $B(f, h)$ may have large Lebesgue measure.

It is well known that for every polynomial f, the (spherical) Lebesgue measure of $B(f, h)$ tends to zero as h decreases to zero (see [12,13]). In addition, for generic polynomial f, there exist some $h^* \in]0, 1]$ such that, for any $0 < h < h^*$, the set of bad initial values $B(f, h)$ has Lebesgue measure zero.

In [14], Smale shows that Newton's method is not generally convergent. For this, he constructs a polynomial f such that the associated iteration function N_f has a 2-cycle that is super-attracting. More precisely, the author obtains the degree three polynomial $f(z) = z^3 - 2z + 2$ for which N_f has $\Omega = \{0, 1\}$ as a super-attracting 2-cycle. Following Smale's argument for each h, D. Saupe constructs a polynomial $f_h : \mathbb{C} \longrightarrow \mathbb{C}$ such that the relaxed Newton's method $N_{f_h,h}$ has the attracting 2-cycle $\Omega = \{0, 1\}$. In this case, the polynomial found is given by $f_h(z) = (2 - h)z^3 - (1 - h)z^2 - 2z + \frac{2}{h}$, with $0 < h \leq 1$, which coincides with Smale's polynomial for $h = 1$. (See [3].) Note that these techniques work only for constructing 2-cycles. In addition, in [14] Smale poses the following question: *"Find more systematically the set of f whose Newton's map N_f has attracting n-cycles* $(n \geq 2)$ *(and thus fails to generally converge)*".

Given a set Ω consisting of n $(n \geq 2)$ distinct complex numbers, the main result of this paper, Theorem 4, gives a technique for constructing, for any $h \in D_1(1)$, a polynomial $\tilde{f}_h$ such that the relaxed Newton's method $N_{\tilde{f}_h,h}$ has Ω as a super-attracting cycle. This can be considered as a contribution to solving the problem posed by Smale.

3. Basic features of the relaxed Newton's method

Note that the relaxed Newton's method satisfies the following properties:

1. $N_{f,h}(z) = z$ if and only if $f(z) = 0$, or in other words the finite fixed points of $N_{f,h}$ are the roots of f.
2. We have $N'_{f,h}(z) = \frac{(1-h)f'(z)^2 + hf(z)f''(z)}{f'(z)^2} = 1 - h(1 - L_f(z))$ where $L_f(z) = f(z)f''(z)/f'(z)^2$ is the degree of the logarithmic convexity function (see [15]).
3. If r is a simple root of f, or in other words $f(r) = 0$ and $f'(r) \neq 0$, then $N'_{f,h}(r) = 1 - h$. Therefore, r is an attracting fixed point if $|1 - h| < 1$, and it is super-attracting if $h = 1$, that is, for the classical Newton's method.

   In order to ensure that the roots of f are (super)attracting fixed points for $N_{f,h}$, we consider only values of h in $D_1(1)$.
4. We will write $f(z) = (z - r)^m g(z)$, with $g(r) \neq 0$, when r is a root of multiplicity $m \geq 2$ of f. We have $N_{f,h}(z) = z - h\frac{(z-r)g(z)}{mg(z)+(z-r)g'(z)}$, $N_{f,h}(r) = r$, and $N'_{f,h}(z) = 1 - \frac{h}{m}$. Thus r is an attracting fixed point of $N_{f,h}$ since $|1 - h/m| < 1$.
5. The critical points of $N_{f,h}$ are solutions of $N'_{f,h}(z) = 0$, or in other words they are solutions of the equation $L_f(z) = (h - 1)/h$. A solution of the preceding equation, other than a root of f, if any, is called a *free critical point*. Note that the free critical points of $N_{f,h}$ are solutions of the equation $hf(z)f''(z) + (1 - h)f'(z)^2 = 0$.
6. Conjugating with the Möbius transformation $z \to 1/z$, we have that $z = \infty$ is a fixed point of $N_{f,h}$ if and only if $z = 0$ is a fixed point of the map $F(z) = \frac{1}{N_{f,h}(1/z)} = \frac{zf'(1/z)}{f'(1/z) - hzf(1/z)}$.

   If f is a polynomial of degree n, then a straightforward calculation yields $F(0) = 0$ and $|F'(0)| = \left|\frac{n}{n-h}\right| > 1$ for any $h \in D_1(1)$. In other words, the point $z = \infty$ is a repelling fixed point of $N_{f,h}$.

Let $R_1, R_2 : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ be two rational maps. We say that R_1 and R_2 are conjugated if there is a Möbius transformation $\psi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ such that $R_2 \circ \psi(z) = \psi \circ R_1(z)$, for all z.

An important feature of conjugation of rational maps is as follows. Let R_1 and R_2 be two rational maps, and let ψ be a Möbius transformation conjugating R_1 and R_2. Then $\mathcal{F}(R_2) = \psi(\mathcal{F}(R_1))$ and $\mathcal{J}(R_2) = \psi(\mathcal{J}(R_1))$. From a global point of view of the iteration of rational maps, this means that conjugacy plays a central role in understanding the behavior of classes of maps in the following sense. Suppose we wish to describe both the quantitative and the qualitative behaviors of the map $z \longrightarrow T_f(z)$ where T_f is some iterative root-finding map. Since a conjugacy preserves both fixed and periodic points, as well as their types, and since the attraction basins and the dynamical data concerning f are carried by the fixed points of T_f, for polynomials of degree greater than or equal to two, it is worthwhile constructing parameterized families, consisting of polynomials p_μ that are *as simple as possible*, so that there exists a conjugacy between T_p and T_{p_μ} for a suitable choice of the complex parameter μ. In particular, when ψ is a Möbius transformation, we say that the rational maps R_1 and R_2 are affine conjugated. We have the following.

Theorem 1 (*Scaling Theorem for the Relaxed Newton's Method*)**.** *Let f be an analytic function, and let $T(x) = \alpha x + \beta$, with $\alpha \neq 0$, be an affine map. Let $g(x) = (f \circ T)(x)$. Then $T \circ N_{g,h} \circ T^{-1}(x) = N_{f,h}(x)$, that is, T is a conjugacy between $N_{f,h}$ and $N_{g,h}$.*

The proof follows from a straightforward calculation.

4. Results

In this section, we give a method for constructing polynomials whose relaxed Newton's method has an attracting periodic orbit of period greater than or equal to two. Thus, the set of starting points for which the iterate under the relaxed Newton's method does not converge to a root of the polynomial contains an open set.

Theorem 2. *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a complex analytic function. Let $\Omega = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct complex numbers. Then Ω is an n-cycle of $N_{f,h}$ if and only if $f'(x_{[i]}) = h\frac{f(x_{[i]})}{x_{[i]}-x_{[i+1]}}$, for $i = 1, 2, \ldots, n$. Here $[i] = [i']$ if and only if $i - i' = n$. Furthermore, if $f''(x_{[i]}) = h(h-1)\frac{f(x_{[i]})}{(x_{[i]}-x_{[i+1]})^2}$ for some $i \in \{1, \ldots, n\}$, then Ω is a super-attracting cycle for $N_{f,h}$.*

Proof. For the first part, note that $x_{[i+1]} = N_{f,h}(x_{[i]}) = x_{[i]} - h\frac{f(x_{[i]})}{f'(x_{[i]})}$ is equivalent to $f'(x_{[i]}) = h\frac{f(x_{[i]})}{x_{[i]}-x_{[i+1]}}$.

For the second, let $f(x_i) = y_i$ and $f'(x_i) = w_i$. Then the set $\Omega = \{x_1, \ldots, x_n\}$ is an n-cycle for $N_{f,h}$ if and only if $y_i = \frac{1}{h}(x_{[i]} - x_{[i+1]})w_i$. Since $(N^n_{f,h})'(x_j) = N'_{f,h}(x_1)N'_{f,h}(x_2)\cdots N'_{f,h}(x_n)$, for any $j \in \{1, \ldots, n\}$, without loss of generality, we may assume $N'_{f,h}(x_1) = 0$. It is not difficult to see that $N'_{f,h}(x_1) = 0$ if and only if $f''(x_{[1]}) = h(h-1)\frac{f(x_{[1]})}{(x_{[1]}-x_{[2]})^2}$. The proof is now complete. □

Theorem 3. *For any $h \in D_1(1)$ and any positive integer $n \geq 2$, a polynomial f of degree less than or equal to $2n-1$ can be constructed so that $N_{f,h}$ has an n-cycle.*

Proof. Let $x_1, x_2, \ldots, x_n$ be n distinct complex numbers, and let $y_1, y_2, \ldots, y_n$ be given complex numbers. Suppose there is a polynomial f such that

$$\begin{cases} f(x_i) = y_i, \quad \text{for } i = 1, \ldots, n, \\ f'(x_i) = \dfrac{hy_i}{x_i - x_{i+1}}, \quad \text{for } i = 1, \ldots, n-1 \text{ and} \\ f'(x_n) = \dfrac{hy_n}{x_n - x_1}. \end{cases} \tag{2}$$

According to Theorem 2, the set $\Omega = \{x_1, x_2, \ldots, x_n\}$ is an n-cycle for $N_{f,h}$. We claim that such a polynomial f can be constructed. Indeed, using the Hermite interpolation method, we may construct a polynomial of degree $2n-1$ which satisfies conditions (2) as asserted. We begin the construction by writing f as

$$f(x) = a_1f_1(x) + a_2f_2(x) + \cdots + a_{2n}f_{2n}(x)$$

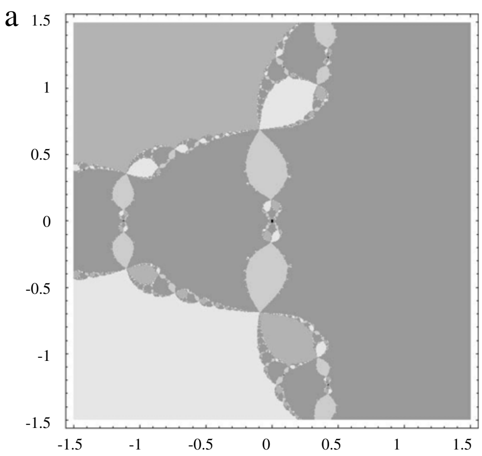
where, for each $i = 1, \ldots, 2n$, the functions f_i are polynomials of degree $i-1$ defined inductively as follows:

$$\begin{aligned}
&f_1(x) = 1\\
&f_2(x) = f_1(x)\cdot(x-x_1) = x - x_1\\
&f_3(x) = f_2(x)\cdot(x-x_1) = (x-x_1)^2\\
&f_4(x) = f_3(x)\cdot(x-x_2) = (x-x_1)^2\cdot(x-x_2)\\
&f_5(x) = f_4(x)\cdot(x-x_2) = (x-x_1)^2\cdot(x-x_2)^2\\
&f_6(x) = f_5(x)\cdot(x-x_3) = (x-x_1)^2\cdot(x-x_2)^2\cdot(x-x_3)\\
&\vdots\\
&f_{2i-1}(x) = f_{2i-2}(x)\cdot(x-x_{i-1})\\
&f_{2i}(x) = f_{2i-1}(x)\cdot(x-x_i)\\
&\vdots\\
&f_{2n-1}(x) = f_{2n-2}(x)\cdot(x-x_{n-1})\\
&f_{2n}(x) = f_{2n-1}(x)\cdot(x-x_n).
\end{aligned}$$

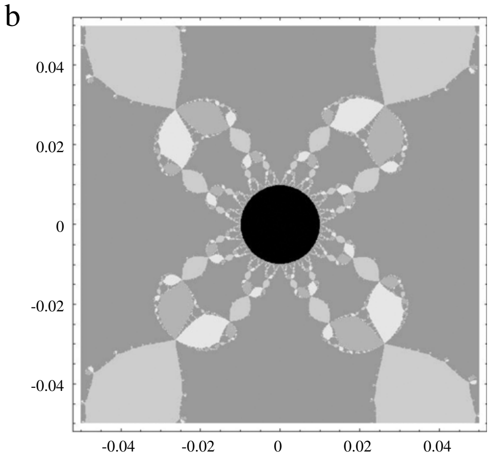
Note that $f_{2i-1}(x_i) = f_{2i-2}(x_i)\cdot(x_i - x_{i-1}) \neq 0$ and $f_{2i}(x_i) = f_{2i-1}(x_i)\cdot(x_i - x_i) = 0$. In other words, $f_j(x_i) \neq 0$, for $j \leq 2i-1$, and $f_j(x_i) = 0$, for $j \geq 2\mathrm{i}$.

On the other hand, we have $f'_{2i}(x) = f'_{2i-1}(x)\cdot(x-x_i) + f_{2i-1}(x)$. Thus $f'_{2i}(x_i) = f_{2i-1}(x_i) \neq 0$ and $f'_{2i+1}(x) = f'_{2i}(x)\cdot(x-x_i) + f_{2i}(x)$. Therefore, $f'_{2i+1}(x_i) = f_{2i}(x_i) = 0$, and hence $f'_j(x_i) \neq 0$, for $j \leq 2\mathrm{i}$, and $f'_j(x_i) = 0$, for $j \geq 2i+1$. To determine the polynomial f, we must find suitable coefficients a_i, for $i = 1, \ldots, 2n$. For this, we must solve a linear system of $2n$ equations with $2n$ unknowns. The associated matrix is a lower triangular matrix whose rows are

$$\begin{array}{rccccccccc}
A_{2i-1} = f_1(x_i) & f_2(x_i) & f_3(x_i) & \cdots & f_{2i-1}(x_i) & 0 & 0 & 0 & \cdots & 0\\
A_{2i} = f'_1(x_i) & f'_2(x_i) & f'_3(x_i) & \cdots & f'_{2i-1}(x_i) & f'_{2i}(x_i) & 0 & 0 & \cdots & 0\\
A_{2i+1} = f_1(x_{i+1}) & f_2(x_{i+1}) & f_3(x_{i+1}) & \cdots & f_{2i-1}(x_{i+1}) & f_{2i}(x_{i+1}) & f_{2i+1}(x_{i+1}) & 0 & \cdots & 0
\end{array}$$



Basin of attraction of the attracting cycle $\mathcal{O} = \{0, 3\}$ for $N^2_{\tilde{f},0.8}$



Zoom of the basin of attraction of the point $z = 0$ as a fixed point of $N^2_{\tilde{f},0.8}$

Fig. 3.

for $i = 1, \ldots, n$. Thus the system of equations may be written in the form $Ax = b$ where

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_{2n} \end{pmatrix}_{2n\times 2n}, \quad x = \begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix}_{2n\times 1}, \quad \text{and} \quad b = \begin{pmatrix} y_1 \\ \dfrac{hy_1}{x_1 - x_2} \\ \vdots \\ y_n \\ \dfrac{hy_n}{x_n - x_1} \end{pmatrix}_{2n\times 1}.$$

Note that the system of linear equations $Ax = b$ has a solution, since the determinant of the matrix A is non-zero. In fact, note that the components of the diagonal of A are $f_{2i-1}(x_i)$ and $f'_{2i}(x_i)$, which are non-zero for $i = 1, \ldots, n$. The proof is now complete. □

Theorem 4. *Let $n \geq 2$ be an integer. Then, for any $h \in D_1(1)$, a polynomial f_h of degree less than or equal to $2n$ can be constructed so that $N_{f_h,h}$ has an attracting n-cycle.*

Proof. Fix $h \in D_1(1)$. Given $\Omega = \{x_1, \ldots, x_n\}$ with $x_i \neq x_j$ for all $i, j \in \{i, \ldots, j\}$, let $y_1, \ldots, y_n$ be any n complex numbers, Theorem 3 allows the construction of a polynomial f_h such that Ω is an n-cycle for $N_{f,h}$. Recall that

$$f(x) = a_1 f(x) + \cdots + a_{2n} f_{2n}(x)$$

for suitable values of $a_1, \ldots, a_{2n}$. We define a new polynomial $\tilde{f}$ by

$$\tilde{f}(x) = f(x) + a_{2n+1} f_{2n+1}(x)$$

where the coefficient a_{2n+1} is a parameter to be determined. The new condition does not alter the n-cycle $\Omega = \{x_1, \ldots, x_n\}$, since $f_{2n+1}(x_i) = 0$ for each $i = 1, \ldots, n$. Finally, to determine a_{2n+1} we apply the condition $\tilde{f}''(x_1) = h(h-1)\frac{f(x_1)}{(x_1-x_2)^2}$ and solve the equation for a_{2n+1}. By Theorem 3, the set Ω is a super-attracting n-cycle, which completes the proof. □

5. Examples

Given any $n \geq 2$ and $h \in D_1(1)$, we construct a polynomial f_h so that $N_{f,h}$ has a super-attracting n-cycle. As an example, for $n = 2$ and $h = 0.8$, we will construct such a polynomial.

Consider the set $\mathcal{O} = \{x_1, x_2\}$ where $x_1 = 0$ and $x_2 = 3$. Let $y_1 = 2$ and $y_2 = -2$. We construct a polynomial $f(x)$ so that $f(x_i) = y_i, f'(x_1) = \frac{hy_1}{x_1-x_2}$, and $f'(x_2) = \frac{hy_2}{x_2-x_1}$. Using Theorems 3 and 4, we obtain the polynomial $\tilde{f}(x) = 2 - \frac{8}{15}x - \frac{4}{225}x^2 - \frac{232}{675}x^3 + \frac{176}{2025}x^4$ which is such that the relaxed Newton's method $N_{\tilde{f},h}(x)$ has the super-attracting 2-cycle $\mathcal{O} = \{0, 3\}$. Fig. 3a shows the basin of attraction of the attracting cycle $\mathcal{O} = \{0, 3\}$ for $N_{\tilde{f},0.8}$ (shaded black). Fig. 3b shows a zoom of the basin of attraction of the point $z = 0$ as a fixed point of $N^2_{\tilde{f},0.8}$ (shaded black).

6. General convergency for extracting radicals

In this section, we study general convergency of the relaxed Newton's method, at least for the polynomials $f(z) = z^n - c$ where c is a non-zero complex number. For references on the subject in this section, see [14,16].

Definition 6.1. Given a polynomial f of degree d, we say that a rational function T_f whose coefficients are functions of the coefficients of f is an *iteration function for the polynomial* if each root of f is an (super)attracting fixed point of T_f. If this property holds for all polynomial of degree d, we say that $T_f : f \longrightarrow T_f$ is a *purely iterative algorithm*. Finally, we say that an iteration function T_f is *generally convergent* if for almost all (in the sense of Lebesgue's measure) $z \in \mathbb{C}$, the iterates $T_f^n(z)$ converge to a root of f.

As mentioned above, Newton's method is a purely iterative algorithm. However, there exist polynomials f for which N_f is not generally convergent and, consequently, there exist open sets of polynomials $\mathcal{U}$ containing f such that, for any g in $\mathcal{U}$, Newton's method N_g fails to converge in an open set of starting points (see [14]). According to the Scaling Theorem, for the polynomials $g(z) = z^n + c$ and $f(z) = z^n - 1$ we have that $N_{g,h}$ and $N_{f,h}$ are conjugate. Thus to describe the convergence regions of $N_{g,h}$, it suffices to describe the convergence regions of $N_{f,h}$.

Recall that a point $w \in \mathbb{C}$ is *preperiodic* for a rational map R if for some $n \geq 1$, $R^n(w)$ belongs to some k-cycle.

It is well known that for a polynomial f, an iteration function T_f is generally convergent, provided that the critical points of T_f are either pre-periodic or converge to a root of f under iteration by the map T_f. This is equivalent to the following assertions: The Lebesgue measure of the Julia set $\mathcal{J}(T_f)$ is zero and, for all z in the Fatou set $\mathcal{F}(T_f)$, the iterates $T_f^n(z)$ converge to a root of f as n tends to infinity. In other words, for a polynomial f, an iteration function T_f is generally convergent if and only if the Lebesgue measure of the Julia set $\mathcal{J}(T_f)$ is zero, and there are neither (super)attracting n-cycles with $n \geq 2$, nor cycles of parabolic domains, nor cycles of Siegel disks, nor cycles of Herman rings. In addition, we have the following. Let R be a rational map which has an attracting cycle. If all critical points are either pre-periodic or converge to attracting cycles, then the Lebesgue measure of $\mathcal{J}(R)$ is zero. Moreover, for any $z \notin \mathcal{J}(R)$, the iterates $R^n(z)$ converge to an attracting cycle as n tends to infinity.

We have the following.

Theorem 5. *For the polynomial $f(z) = z^n - 1$, the relaxed Newton's method is generally convergent. Moreover, the Lebesgue measure of the Julia set $\mathcal{J}(N_{f,h})$ is zero and, for all $z \in \mathcal{F}(N_{f,h})$, the iterates $N_{f,h}^n(z)$ converge to a root of f as n tends to infinity.*

Proof. We have $N_{p,h}(z) = z - h\frac{p(z)}{p'(z)}$. Then $N'_{p,h}(z) = \frac{(1-h)p'(z)^2 + hp(z)p''(z)}{p'(z)^2}$. Thus the critical points of $N_{p,h}$ are the solutions of the equation $(1-h)p'(z)^2 + hp(z)p''(z) = 0$. It follows from this equation that either $z = 0$ or $z^n = \frac{h(n-1)}{n-h}$. Let ξ_j be nth roots of unity, with $j = 1, \ldots, n$. Then the solutions of the preceding equation are given by $z_j = \sqrt[n]{\frac{h(n-1)}{n-h}}\,\xi_j$. A straightforward calculation shows that $N_{p,h}^k(z_1) \to 1$ as $k \to \infty$. It follows that $N_{p,h}^k(z_j) \to \xi_j$ for $j = 1, \ldots, n$. Finally since $N_{p,h}(0) = \infty$ and ∞ is a repelling fixed point for $N_{p,h}$, the proof is complete. $\square$

Conclusions

In this work we develop a technique which allows one to construct attracting cycles of prescribed period $n \geq 2$ for the relaxed Newton's method. We show that this method is generally convergent when used to extract radicals.

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