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On the Stability of Equilibria in Two-Degrees-of-Freedom Hamiltonian Systems Under Resonances

A. Elipe,[1] V. Lanchares,[2] and A. I. Pascual[2]

[1] Grupo de Mecánica Espacial, Universidad de Zaragoza, 50009 Zaragoza, Spain

[2] Dept. Matemáticas y Computación, Universidad de La Rioja, 26004 Logroño, Spain
E-mail: vlancha@dmc.unirioja.es

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Summary. We consider the problem of stability of equilibrium points in Hamiltonian systems of two degrees of freedom under resonances. Determining the stability or instability is based on a geometrical criterion based on how two surfaces, related with the normal form, intersect one another. The equivalence of this criterion with a result of Cabral and Meyer is proved. With this geometrical procedure, the hypothesis may be extended to more general cases.

Key words. Nonlinear stability, Normal forms.

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1. Introduction

In the analysis of dynamical systems, the search for equilibria and their stability is one of the first tasks to be carried out, since phase flow depends on them to a large extent. A precise knowledge of the phase portrait is very helpful when pursuing other tasks, like the existence of periodic orbits, limit cycles, or the numerical integration of some specific orbits.

Determining the stability of the equilibria of fixed points, in general, is not an easy task, since it usually requires finding a Lyapunov function. However, the theory for linear stability is well established and easy to apply. This is why the first step in analyzing the stability of a nonlinear system lies in linearizing it and finding the linear stability. If at least one eigenvalue has nonzero real part, the equilibrium is unstable for the linearized system and is also unstable for nonlinear system. Unfortunately, the converse is not true; since the rest of the terms may destroy the stability of the linear part, and a different method must be used.

In this paper, we deal with two-degrees-of-freedom autonomous Hamiltonians, defined by an analytical function $\mathcal{H} = \mathcal{H}(x_1, x_2, X_1, X_2)$. Without loss of generality we will assume that the origin is an isolated equilibrium of the system. A power series expansion of the analytical function $\mathcal{H}$ in a neighborhood of the origin gives

$$\mathcal{H}(\boldsymbol{x}) = \mathcal{H}_2(\boldsymbol{x}) + \mathcal{H}_3(\boldsymbol{x}) + \cdots,$$

where $\mathcal{H}_k(\boldsymbol{x})$ is a homogeneous polynomial of degree k in $\boldsymbol{x}$.

The linear stability analysis of the equilibrium gives much information, since on the one hand, if the equilibrium is hyperbolic, it is nonlinear unstable, and on the other hand, inasmuch as the eigenvalues of the linearized Hamiltonian $\mathcal{H}_2$ appear in pairs $\pm\lambda$, to have linear stability it is necessary that all the eigenvalues of the linearized system be purely imaginary numbers.

Thus, let us assume that there exist $\omega_1, \omega_2 \in \mathbb{R}^+$, such that the eigenvalues of the corresponding linear system around the origin are $\pm\omega_1 i$, $\pm\omega_2 i$. In this way, after a suitable symplectic linear transformation, the Hamiltonian $\mathcal{H}$ is expressed as

$$\mathcal{H} = H_2 + F(q_1, q_2, p_1, p_2), \tag{1}$$

where

$$H_2 = \tfrac{1}{2}\omega_1(q_1^2 + p_1^2) \pm \tfrac{1}{2}\omega_2(q_2^2 + p_2^2), \tag{2}$$

and F is an analytic function verifying

$$\lim_{\|(q,p)\| \longrightarrow 0} \frac{\|F\|}{\|(q, p)\|^2} = 0. \tag{3}$$

Two situations must be considered. On the one hand, for the plus sign in (2), according to the classical Lyapunov theory, a result of Dirichlet ensures the stability of the origin for the whole system defined by (1), because H_2 is positive-defined (see e.g. [25], [5]). On the other hand, for the minus sign in (2), H_2 is not sign-defined and Dirichlet's theorem of stability cannot be applied. However, for this interesting case, an Arnold's theorem [3] gives sufficient conditions to determine the stability character of the origin if the fundamental frequencies ω_1 and ω_2 satisfy a general condition of irrationality and the Hamiltonian is in Birkhoff normal form (Theorem 1).

Arnold's theorem has a wide field of application in Celestial Mechanics; thus, Leontovich [17] determined that the Lagrangian points in the planar restricted three-body problem (RTBP) are stable for almost every value of the mass ratio of the primaries below the critical Routh value. Later on, Deprit and Deprit-Bartholomé [8] provided a procedure to apply the theorem. By so doing, they established the stability of the triangular points except for three values, the resonant cases, excluded from the hypothesis of Arnold's theorem. This result is applied to a wide range of problems, as we can see, for example, in [11], [15], [21], [24] and the references therein.

The resonances among the eigenvalues were tackled for particular cases [18], [19], [20], [26], [27], [28], [1], [2], and each resonance needed an ad hoc criterion. Recently, Cabral and Meyer [4] gave an analytic method (hereafter, the CM method) that generalizes Arnold theorem and is valid for the resonant cases. Independently, Elipe et al. [14] showed that the stability for the resonances may be geometrically determined depending on how two surfaces cut each other. In this paper, we analyze the relation between these two criteria and prove that, although both are almost equivalent, the geometrical one is a bit more general, since it is valid for cases (related to the normal forms) that the CM method does not consider. Indeed, in the CM method, the Hamiltonian function is brought to its Birkhoff normal form up to terms of the order of the resonance, and the stability character of the origin is determined, precisely, by these terms. However, the geometric criterion allows us to consider other cases where stability is decided by terms of higher order than that of the resonance. This is the case if all the previous terms vanish when restricted to a certain manifold.

The main idea of the geometric criterion implies studying the orbits in the reduced space after normalization. In fact, the reduced flow lies on a fibered three-dimensional space [23]. Each fiber is a two-dimensional space labeled by (2), which turns out to be a formal integral constructed by the normalization transformation. The origin belongs to the fiber labeled by $H_2 = 0$ and the orbits on this fiber determine the stability of the origin: If they are closed around it, the origin is Lyapunov stable; otherwise, if there are asymptotic orbits, it is unstable. This simple idea is proved to be equivalent to the classical isoenergetic reduction in the more general hypothesis mentioned above.

2. The Theorems of Arnold and Cabral and Meyer

Let us introduce the set of Poincaré action and angle variables:

$$q_k = \sqrt{2\Phi_k}\cos\phi_k, \qquad p_k = \sqrt{2\Phi_k}\sin\phi_k, \quad k = 1, 2.$$

With this symplectic transformation, the Hamiltonian (2) is converted into

$$H_2 = \omega_1\Phi_1 - \omega_2\Phi_2.$$

Arnold's stability theorem [3] as given in a more readable form in [22] is as follows:

Theorem 1 (Arnold). *Let us consider a two-degrees-of-freedom Hamiltonian system $\mathcal{H}$ expressed, in the real canonical coordinates $(\Phi_1, \Phi_2, \phi_1, \phi_2)$, as*

$$\mathcal{H} = H_2 + H_4 + \cdots + H_{2n} + \tilde{H},$$

where

1. *$\mathcal{H}$ is real analytic in a neighborhood of the origin in $\mathbb{R}^4$.*
2. *H_{2k}, $1 \le k \le n$, is a homogeneous polynomial of degree k in Φ_i, with real coefficients. In particular,*

$$H_2 = \omega_1\Phi_1 - \omega_2\Phi_2, \quad 0 < \omega_1,\ 0 < \omega_2;$$
$$H_4 = \tfrac{1}{2}\left(A\Phi_1^2 - 2B\Phi_1\Phi_2 + C\Phi_2^2\right).$$

$\tilde{\mathcal{H}}$ has a power expansion in Φ_i which starts with terms at least of order $2n + 1$.

Under these assumptions, the origin is stable *provided that for some k, $2 \le k < n$, H_2 does not divide H_{2k} or, likewise, provided that $D_{2k} = H_{2k}(\omega_2, \omega_1) \ne 0$.*

There are several implicit assumptions in stating that $\mathcal{H}$ has this form. On the one hand, since $H_2, \ldots, H_{2n}$ depend only on the actions Φ_1 and Φ_2, $\mathcal{H}$ is in its Birkhoff's normal form up to degree $2n$. On the other hand, the frequencies ω_i must satisfy a nonresonance condition. This assumption is what Deprit [8] called the *general condition of irrationality*, which implies assuming that for any pair (k_1, k_2) of rational integers, $k_1\omega_1 + k_2\omega_2 \neq 0$. However, it is enough that the frequencies satisfy a weak resonant condition [8], [22], [25], namely, $k_1\omega_1 + k_2\omega_2 \neq 0$ for any pair of rational numbers (k_1, k_2) such that $|k_1| + |k_2| \leq 2n$.

For the resonant cases, Cabral and Meyer [4] extended Arnold's theorem based on two fundamental lemmas, the first one already established by Sokolsky [27] in the study of stability of equilibrium positions under first-order resonances.

Lemma 1. *Let $K(s, \phi, t) = \Psi(\phi)s^n + O(s^{n+\frac{1}{2}})$, where $n = m/2$ with $m \geq 3$, an integer. Let us assume that K is an analytic function of $\sqrt{s}, \phi, t$, τ-periodic in ϕ and T-periodic in t. If $\Psi(\phi) \neq 0$, for all ϕ, then the origin $s = 0$ is a stable equilibrium for the Hamiltonian system*

$$\dot{s} = \frac{\partial K}{\partial \phi}, \qquad \dot{\phi} = -\frac{\partial K}{\partial s},$$

in the sense that given $\epsilon > 0$, there exists $\delta > 0$ such that if $s(0) < \delta$, then the solution is defined for all t and $s(t) < \epsilon$. If $\Psi(\phi)$ has a simple zero, i.e., if there exists ϕ^* such that $\Psi(\phi^*) = 0$ and $\Psi'(\phi^*) \neq 0$, then the equilibrium $s = 0$ is unstable.

The proof of the lemma is based on Chetaev's theorem [6] for instability and on Moser's invariant curve theorem [25] for stability.

Lemma 2. *Let $K(s, \phi, t) = \epsilon^m \Psi(\phi)s^n + O(\epsilon^{m+1})$, where m and $2n$ are positive integers. Let us assume that K is an analytic function of s, ϕ, t, τ-periodic in ϕ, T-periodic in t for all $\frac{1}{2} \leq s \leq 3$ and all $0 \leq \epsilon \leq \epsilon_0$. If $\Psi(\phi) \neq 0$, for all ϕ, then if ϵ_0 is sufficiently small, any solution of*

$$\dot{s} = \frac{\partial K}{\partial \phi}, \qquad \dot{\phi} = -\frac{\partial K}{\partial s},$$

which starts with $|s(0)| \leq 1$ for $0 \leq \epsilon \leq \epsilon_0$, satisfies $|s(t)| \leq 2$ for all t.

This proof is based on the fact that there are invariant curves for the section map which separate $s = 1$ from $s = 2$.

Let us assume that the frequencies ω_1, ω_2 of the quadratic Hamiltonian (2) are in resonance, i.e., there are two relatively prime integers n and m verifying

$$n\omega_1 - m\omega_2 = 0,$$

or $n = m = 1$. If $n = m = 1$, we also assume that the matrix of the linearized system is diagonalizable.

Let us write the Hamiltonian in the action-angle variables $(\Phi_1, \Phi_2, \phi_1, \phi_2)$, and let us assume that after some symplectic transformations the Hamiltonian $\mathcal{H}$ is the normal form through terms of order r where $r = 2l - 1$ or $r = 2l$, that is,

$$\begin{aligned}\mathcal{H} \;=\; & H_2(\Phi_1, \Phi_2) + H_4(\Phi_1, \Phi_2) + \cdots \\ & + H_{2l-2}(\Phi_1, \Phi_2) + H_r(\Phi_1, \Phi_2, n\phi_1 + m\phi_2) + \cdots .\end{aligned} \tag{4}$$

Moreover,

1. $H_2 = \omega_1 \Phi_1 - \omega_2 \Phi_2$.
2. H_{2k} is a homogeneous polynomial of degree k in Φ_1, Φ_2.
3. $H_r(\Phi_1, \Phi_2, n\phi_1 + m\phi_2)$ is a homogeneous polynomial of degree r in $\sqrt{\Phi_1}$, $\sqrt{\Phi_2}$, with coefficients that are finite Fourier series in the angle $n\phi_1 + m\phi_2$.
4. $\mathcal{H}$ is an analytic function of the variables $\sqrt{\Phi_1}$, $\sqrt{\Phi_2}$, ϕ_1, ϕ_2, and 2π periodic in $n\phi_1 + m\phi_2$.

Let

$$\Psi(\phi) = H_r(\omega_2, \omega_1, n\phi),$$

where

$$\phi = \phi_1 + \frac{m}{n}\phi_2.$$

Let $D_{2k} = H_{2k}(\omega_2, \omega_1)$. If $D_{2k} \neq 0$ for some $k = 2, \ldots, l-1$, then Arnold's theorem ensures the stability of the origin. Therefore, we assume that $D_{2k} = 0$ for all $2 \leq k \leq l-1$. Under these hypotheses, we have the following theorem.

Theorem 2 (Cabral and Meyer). *If $\Psi(\phi) \neq 0$ for all ϕ, then the origin is stable. If Ψ has a simple zero, that is, if there exists ϕ^* such that $\Psi(\phi^*) = 0$ and $\Psi'(\phi^*) \neq 0$, then the origin is unstable.*

This proof comes down to establishing the stability properties in the surface defined by $\mathcal{H} = 0$, where the Hamiltonian function can be reduced to that of Lemma 1.

3. A Geometric Counterpart

An interesting geometric approach to the stability problem can be given if we consider the structure of the phase space after normalization and study the shape of the orbits on the reduced phase space. Indeed, after normalization, a new formal integral is introduced (namely, $H_2 = \omega_1 \Phi_1 - \omega_2 \Phi_2$), and the phase space can be considered as a foliation of two-dimensional surfaces in terms of the value of H_2. Taking this into account, we do not follow the standard procedure of the isoenergetic reduction that considers the motion at the energy level $\mathcal{H} = 0$ (this is the basis of the proof given by Cabral and Meyer). On the contrary, we consider the motion at the surface $H_2 = 0$ where the origin lies. This idea was introduced in [14] and we will see that it is a generalization of Theorem 2.

To begin with, it is advisable to introduce suitable sets of variables with a twofold objective: reveal the structure of the reduced phase space and give a compact expression for each order in the normal form.

As in the hypothesis of Theorem 2, we suppose that the frequencies ω_1 and ω_2 satisfy the resonant condition

$$n\omega_1 - m\omega_2 = 0, \qquad \text{or} \qquad \frac{\omega_1}{m} = \frac{\omega_2}{n} = \omega,$$

with n and m relatively prime integers, such that $n + m = r$, the order of the resonance, as was implicitly supposed in Theorem 2.

To find the structure of the reduced phase space, we follow the steps in [10], [13] to obtain a set of variables that remains invariant under the normalization procedure. This task is easy when performed with complex variables like those introduced by Gustavson [16]. This set of coordinates is related to q_k, p_k variables of (2) through

$$q_k = \frac{1}{\sqrt{2}}(u_k + i v_k), \qquad p_k = \frac{i}{\sqrt{2}}(u_k - i v_k), \quad k = 1, 2.$$

In complex variables, the quadratic part of the Hamiltonian, H_2, becomes

$$H_2 = i\omega_1 u_1 v_1 - i\omega_2 u_2 v_2,$$

and the operator $\mathcal{L}_2$, associated with H_2 is diagonal and of the form

$$\mathcal{L}_2 = i\omega_1 \left(u_1 \frac{\partial}{\partial u_1} - v_1 \frac{\partial}{\partial v_1} \right) - i\omega_2 \left(u_2 \frac{\partial}{\partial u_2} - v_2 \frac{\partial}{\partial v_2} \right),$$

whereas every term H_j in the Hamiltonian is a homogeneous polynomial of degree j in u_k, v_k.

Let us now define the following four monomials,

$$I_1 = u_1 v_1, \qquad I_2 = u_2 v_2, \qquad I_3 = u_1^n u_2^m, \qquad I_4 = v_1^n v_2^m,$$

that do not belong to the image of the Lie operator $\mathcal{L}_2$. As it is in diagonal form, they belong to $\ker \mathcal{L}_2$ and are invariants with respect to H_2. Then,

Theorem 3. *The invariants I_1, I_2, I_3, I_4 generate the normal form that, up to order N, can be expressed as*

$$\mathcal{H} = i\omega_1 I_1 - i\omega_2 I_2 + \sum_{\substack{2(\alpha_1+\alpha_2)+r\alpha_3=k \\ 3\le k\le N}} (a_{\alpha_1\alpha_2\alpha_3} I_3^{\alpha_3} + b_{\alpha_1\alpha_2\alpha_3} I_4^{\alpha_3}) I_1^{\alpha_1} I_2^{\alpha_2}.$$

Proof. Let $M = u_1^{\beta_1} u_2^{\beta_2} v_1^{\gamma_1} v_2^{\gamma_2}$ be a monomial of order j, with $\beta_1 + \beta_2 + \gamma_1 + \gamma_2 = j$. It is in normal form if $M \in \ker \mathcal{L}_2$.

The action of the Lie derivative over M is

$$\mathcal{L}_2 M = i M \left[\omega_1(\beta_1 - \gamma_1) - \omega_2(\beta_2 - \gamma_2) \right],$$

thus,

$$M \in \ker \mathcal{L}_2 \Longleftrightarrow \omega_1(\beta_1 - \gamma_1) - \omega_2(\beta_2 - \gamma_2) = 0.$$

Taking into account the resonant condition, $M \in \ker \mathcal{L}_2$ if and only if the following Diophantine equation is satisfied:

$$m(\beta_1 - \gamma_1) - n(\beta_2 - \gamma_2) = 0. \tag{5}$$

The solutions of (5) are

$$\beta_1 - \gamma_1 = kn, \qquad \beta_2 - \gamma_2 = km, \quad k \in \mathbb{Z}.$$

If $k = 0$, we obtain the trivial solution $\beta_1 = \gamma_1$, $\beta_2 = \gamma_2$, which is satisfied for every ω_1, ω_2 even if they are not resonant. In this case, the monomial M is of the form

$$M = (u_1 v_1)^{\beta_1} (u_2 v_2)^{\beta_2} = I_1^{\beta_1} I_2^{\beta_2}, \quad 2(\beta_1 + \beta_2) = j,$$

and j must be an even number.

If $k \neq 0$, we obtain the *resonant terms* of the normal form, and they can be of two different classes depending on the sign of k. If $k > 0$, then

$$M = (u_1 v_1)^{\gamma_1} (u_2 v_2)^{\gamma_2} (u_1^n u_2^m)^k = I_1^{\gamma_1} I_2^{\gamma_2} I_3^k, \quad 2(\gamma_1 + \gamma_2) + rk = j,$$

and j must be at least r.

Finally, if $k < 0$, then

$$M = (u_1 v_1)^{\beta_1} (u_2 v_2)^{\beta_2} (v_1^n v_2^m)^{-k} = I_1^{\beta_1} I_2^{\beta_2} I_4^{-k}, \quad 2(\beta_1 + \beta_2) - rk = j,$$

and j, as in the previous case, must be at least r. □

It is worth noting that the four invariants I_k are not independent, but related through

$$I_1^n I_2^m = I_3 I_4. \tag{6}$$

Besides, every linear combination of them is also invariant by the Lie operator. We take advantage of this fact in order to introduce a similar set of variables. Indeed, by defining M_1, M_2, S, and C, as

$$\begin{aligned} M_1 &= \frac{i}{2}(mI_1 + nI_2), \qquad S = \frac{i}{2} n^{m/2} m^{n/2} (I_4 - i^{n+m} I_3), \\ M_2 &= \frac{i}{2}(mI_1 - nI_2), \qquad C = \frac{1}{2} n^{m/2} m^{n/2} (I_4 + i^{n+m} I_3), \end{aligned} \tag{7}$$

the normal form, up to order N, is written as

$$\mathcal{H} = 2\omega M_2 + \sum_{\substack{2(\alpha_1+\alpha_2)+r(\alpha_3+\alpha_4)=k \\ 3 \le k \le N}} a_{\alpha_1\alpha_2\alpha_3\alpha_4} M_1^{\alpha_1} M_2^{\alpha_2} C^{\alpha_3} S^{\alpha_4}. \tag{8}$$

Moreover, the relation (6) becomes

$$C^2 + S^2 = (M_1 + M_2)^n (M_1 - M_2)^m, \tag{9}$$

where $M_1 \ge |M_2|$.

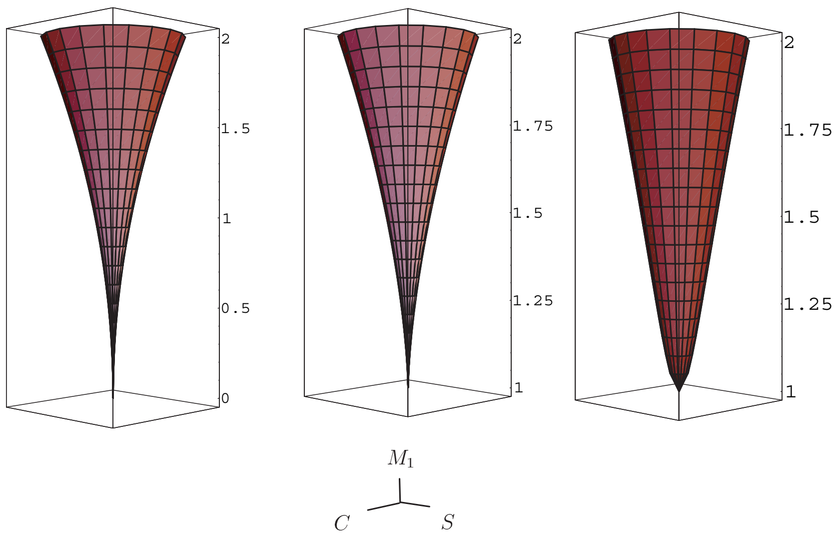

Fig. 1. Reduced phase space for values of M_2 less than, equal to, and greater than zero.

N.B. In spite of their appearance, the variables M_1, M_2, S, and C defined in (7) are real ones.

Equation (9) defines the reduced phase space as a revolution surface (as a matter of fact, a semialgebraic variety) for each constant value of M_2; see Figure 1. In particular, the origin is the vertex of the surface corresponding to $M_2 = 0$. Taking into account (7), we find the Lie algebra underlying the new variables M_1, S, and C by means of Poisson brackets,

$$\begin{aligned} \{M_1, C\} &= mnS, \\ \{S, M_1\} &= mnC, \\ \{S, C\} &= \tfrac{1}{2}mn(M_1+M_2)^{n-1}(M_1-M_2)^{m-1}\left[(m+n)M_1+(m-n)M_2\right]. \end{aligned} \tag{10}$$

In this way, the orbits, for a fixed value of M_2, are the solutions of the system of differential equations

$$\dot{M}_1 = \{M_1, \mathcal{H}\}, \qquad \dot{S} = \{S, \mathcal{H}\}, \qquad \dot{C} = \{C, \mathcal{H}\}, \tag{11}$$

together with the constraint (9). In fact, the orbits can be obtained as the intersection of the surface defined by (9) and that defined by the truncated normal form of the Hamiltonian function up to order N, given by Equation (8) as can be seen in Figure 2. Two interesting consequences are worth noting:

Proposition 1. *The system* (11) *is invariant under the action of the group* $SO(2)$ *of rotations around the axis* M_1.

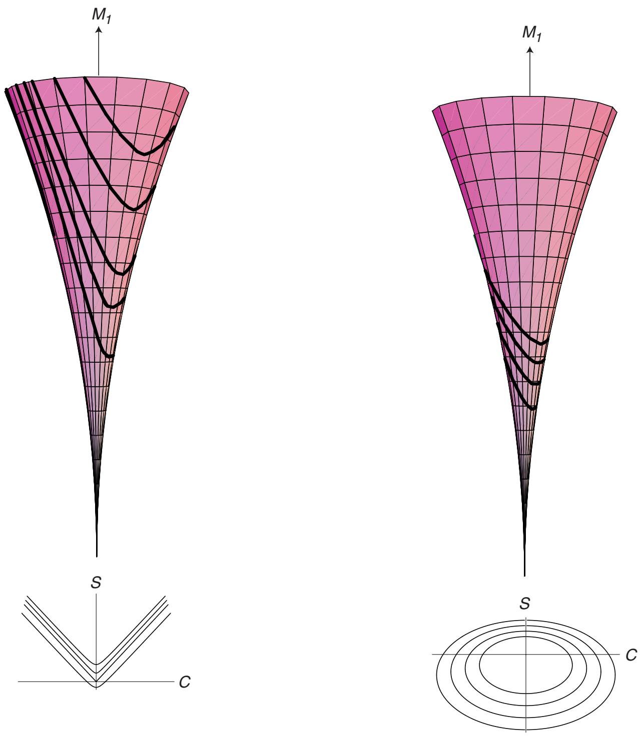

Fig. 2. Phase flow on the reduced phase space.

Proof. This is a consequence of the axial symmetry of the Casimir function (9). Indeed, let us consider the rotation around the M_1 axis:

$$\bar{S} = S\cos\sigma - C\sin\sigma, \qquad \bar{C} = S\sin\sigma + C\cos\sigma.$$

A straightforward calculation yields

$$\begin{aligned}
\{M_1, \bar{C}\} &= mn\bar{S},\\
\{\bar{S}, M_1\} &= mn\bar{C},\\
\{\bar{S}, \bar{C}\} &= \tfrac{1}{2}mn(M_1 + M_2)^{n-1}(M_1 - M_2)^{m-1}\left[(m+n)M_1 + (m-n)M_2\right],
\end{aligned}$$

which proves the proposition. □

Proposition 2. *If the vertex of the surface (9) is nonregular, then it is an equilibrium point of the system (11). If $M_2 = 0$, then the origin (the vertex) is always an equilibrium point.*

Proof. The (C, S, M_1) coordinates of the vertex of the surface (9) are $(0, 0, |M_2|)$. It is not a regular point if the three partial derivatives of the function

$$f(M_1, S, C) = S^2 + C^2 - (M_1 + M_2)^n(M_1 - M_2)^m$$

vanish at the same time.

This is the situation if either $M_2 > 0$ and $m > 1$, or $M_2 < 0$ and $n > 1$, or $M_2 = 0$. In these three cases, Poisson brackets (10), when evaluated at the vertex coordinates, are zero. Therefore, the system of differential equations (11) vanishes at the vertex, and then it is an equilibrium. □

Note that we have implicitly used a singular reduction, as we have obtained reduced spaces which are not smooth manifolds [7]. In this case, the aim is to reduce the S^1 symmetry induced by considering H_2 as an integral. The algebra of S^1-invariant polynomials is generated by (7), subject to the relation (9). This relation defines the reduced orbit space as $\mathbb{R}^4/S^1$, and it is a fibered three-dimensional space. Each fiber is a two-dimensional semialgebraic variety, labeled by H_2, with at most one singular point. The appearance of singular points depends on the isotropy group associated with each point of the phase space by the natural action of the S^1 symmetry. If every isotropy group is trivial, then the reduced space is regular, and has singularities otherwise. This situation depends on the resonance and the value of H_2. A detailed discussion on this topic from the point of view of singular reduction is given in [23].

Now, we are in position to propose a geometric criterion based on how the phase portrait looks in a neighborhood of the origin (let us recall again that we are only interested in the manifold $M_2 = 0$). In this sense, if the orbits are closed trajectories, then the origin will be stable, whereas if there are asymptotic orbits crossing the origin, it will be unstable [14]. In fact, this comes down to determining the zero-level energy curves on the surface (9) for $M_2 = 0$, that is, the intersection of the surfaces (9) and $H = 0$ (for $M_2 = 0$).

Let us assume that the Hamiltonian is normalized up to a certain order $N \geq r$, the first term that does not vanish for $M_2 = 0$. Under these conditions, we get the following result.

Theorem 4 (Geometric criterion). *Let us consider the two surfaces $\mathcal{G}_1 \equiv C^2 + S^2 = M_1^r$ and $\mathcal{G}_2 \equiv H(C, S, M_1; M_2 = 0) = 0$. If their only common point is the origin, then it is stable. If they intersect each other transversally, then the origin is unstable.*

Proof. First, note that taking $M_2 = 0$ is equivalent to making $\Phi_1 = \omega_2$ and $\Phi_2 = \omega_1$, as Arnold does in his theorem. Indeed,

$$\Phi_1 = \omega_2 = \frac{M_1 + M_2}{m}, \qquad \Phi_2 = \omega_1 = \frac{M_1 - M_2}{n},$$

and hence, $2M_2 = m\omega_2 - n\omega_1 = 0$, due to the resonance condition.

On the manifold $M_2 = 0$, the Hamiltonian (8) becomes

$$H = H_N(M_2 = 0) = \sum_{2\alpha_1 + r(\alpha_3+\alpha_4) = N} a_{\alpha_1\alpha_2\alpha_3\alpha_4} M_1^{\alpha_1} C^{\alpha_3} S^{\alpha_4},$$

and the surface (9) is now

$$\mathcal{G}_1 \equiv C^2 + S^2 = M_1^r.$$

Now, let us parameterize the surface $\mathcal{G}_1$. Although it is not the simplest one, for convenience reasons we will use Lissajous variables, specially designed to handle oscillators in resonances [12], [13]. They are related to Poincaré variables through the formulae

$$\begin{aligned} 2m\Phi_1 &= \Psi_1 + \Psi_2, \qquad n\phi_1 + m\phi_2 = 2mn\psi_1, \\ 2n\Phi_2 &= \Psi_1 - \Psi_2, \qquad n\phi_1 - m\phi_2 = 2mn\psi_2. \end{aligned}$$

In this set of variables, $H_2 = \omega\Psi_2$, and the invariants in this set of variables are

$$\begin{aligned} M_1 &= \tfrac{1}{2}\Psi_1, \\ M_2 &= \tfrac{1}{2}\Psi_2, \\ S &= 2^{-(m+n)/2}(\Psi_1 - \Psi_2)^{m/2}(\Psi_1 + \Psi_2)^{n/2} \sin 2mn\psi_1, \\ C &= 2^{-(m+n)/2}(\Psi_1 - \Psi_2)^{m/2}(\Psi_1 + \Psi_2)^{n/2} \cos 2mn\psi_1. \end{aligned} \tag{12}$$

By using Lissajous variables, we have that

$$\mathcal{G}_2 \equiv H_N = 0 \equiv \Psi_1^{N/2}\, G(\psi_1) = 0,$$

where $G(\psi_1)$ is, except for a constant factor, the function appearing in Theorem 2 if N is the order of the resonance. The surface $\mathcal{G}_1$ (let us recall that $\mathcal{G}_1$ is evaluated for $M_2 = \Psi_2 = 0$) may be parameterized by

$$S = 2^{-r/2}\Psi_1^{r/2} \sin 2nm\psi_1, \qquad C = 2^{-r/2}\Psi_1^{r/2} \cos 2nm\psi_1, \qquad M_1 = \tfrac{1}{2}\Psi_1.$$

It is clear that if the function $G(\psi_1)$ does not vanish for every ψ_1, the surface $\mathcal{G}_2$ is defined by $\Psi_1 = 0$ and, therefore, the intersection with $\mathcal{G}_1$ is the point $(0, 0, 0)$, that is to say, the origin. On the other hand, if $G(\psi_1)$ has a simple zero at ψ_1^*, then the two surfaces intersect along the curve defined by

$$S = 2^{-r/2}\Psi_1^{r/2} \sin 2nm\psi_1^*, \qquad C = 2^{-r/2}\Psi_1^{r/2} \cos 2nm\psi_1^*, \qquad M_1 = \tfrac{1}{2}\Psi_1.$$

Note that if ψ_1^* is a multiple root of $G(\psi_1)$, then the surfaces are tangent and the origin is not an isolated equilibrium point.

As can be seen, the intersection of the two surfaces comes down to determining the zeroes of the function $G(\psi_1)$, the same function that appears in Cabral and Meyer's theorem. Thus, in this sense, the two theorems are equivalent. However, in the hypothesis of the geometric criterion, N is not supposed to be the order of the resonance, but the first order in the normal form that does not have M_2 as a common factor. This fact allows us to extend the results given in [4].

To complete the proof it is necessary to verify that the properties of stability on the surface $M_2 = 0$ ensure the nonlinear stability of the equilibrium position. In this way, we follow the line given in [4], although we work with Lissajous variables.

Let us introduce ψ_2 as a new independent variable and consider the tail of the normal form. In fact, ψ_2 is a timelike variable because

$$\dot{\psi}_2 = \frac{\partial \mathcal{H}}{\partial \Psi_2} = \omega + O(\Psi_1^{1/2}),$$

and it is an increasing function of time. Moreover, if we introduce the scaled time τ by means of

$$\omega dt = d\tau,$$

then

$$\frac{d\psi_2}{d\tau} = 1 + O(\Psi_1^{1/2}).$$

Taking all of this into account, we have, on the surface $M_2 = 0$,

$$\frac{d\psi_1}{d\psi_2} = \frac{\partial \Psi_1^{N/2} G(\psi_1)}{\partial \Psi_1} + O(\Psi_1^{(N-1)/2}), \qquad \frac{d\Psi_1}{d\psi_2} = -\frac{\partial \Psi_1^{N/2} G(\psi_1)}{\partial \psi_1} + O(\Psi_1^{(N+1)/2}),$$

and the stability of the origin on this surface follows from the fundamental Lemma 1.

To prove full stability, we scale the action variables $\Psi_k = \epsilon^2 J_k$, where ϵ is a small-scale parameter. Now, the Hamiltonian is

$$\mathcal{H} = H_2(J_2) + \epsilon^2 H_4(J_1, J_2) + \cdots + \epsilon^{N-2} H_N(J_1, J_2, \psi_1) + O(\epsilon^{N-1}).$$

If we consider the flow on the surface $J_2 = h\epsilon^{N-1}$, for $h \in [-1, 1]$, it is the same as the flow of

$$\mathcal{H} = \epsilon^{N-2} J_1^{N/2} G(\psi_1) + O(\epsilon^{N-1})$$

provided that Ψ_2 is a common factor for all the terms in the Hamiltonian function with degree less than N. Then, the full stability of the origin follows by applying Lemma 2. □

Let us turn to Theorem 2 and consider the normal form up to the order r, with $r = n + m$ the order of the resonance as is implicitly assumed in the hypothesis. Then, the truncated normal form is written as

$$\mathcal{H} = 2\omega M_2 + \sum_{j=2}^{\lfloor \frac{r}{2} \rfloor} H_{2j}(M_1, M_2) + \alpha S + \beta C, \tag{13}$$

where $H_{2j}(M_1, M_2)$ is a homogeneous polynomial of degree j in M_1 and M_2. It is worth noting that, if r is an odd number, the highest-order terms in (13) are

$$\alpha S + \beta C.$$

On the other hand, if r is an even number, the highest-order terms are given by

$$H_r(M_1, M_2) + \alpha S + \beta C.$$

Under the hypothesis that the stability of the origin is determined by the terms of order r, we have $H_{2j}(M_1, 0) = 0$ for $j = 1, \ldots, \lfloor \frac{r-1}{2} \rfloor$, and then on the manifold $M_2 = 0$, the normal form reduces to

$$H = \alpha S + \beta C \quad r \text{ odd}, \qquad H = a_r M_1^{r/2} + \alpha S + \beta C \quad r \text{ even}.$$

A simpler expression can be obtained if we apply Proposition 1. Indeed, a rotation of angle σ defined by

$$\tan\sigma = -\frac{\beta}{\alpha}, \quad \text{if } \alpha^2+\beta^2 \neq 0,$$

and $\sigma = 0$ otherwise, transforms the normal form into

$$H = S\sqrt{\alpha^2+\beta^2} \quad \text{if } r \text{ odd}, \qquad \text{and} \quad H = a_r M_1^{r/2} + S\sqrt{\alpha^2+\beta^2} \quad \text{if } r \text{ even}.$$

Then, the next result follows:

Theorem 5. *If r is an odd number and $\alpha^2+\beta^2 \neq 0$, then the origin is unstable. If r is an even number and $a_r^2 > \alpha^2+\beta^2$, then the origin is stable, whereas it is unstable if $a_r^2 < \alpha^2+\beta^2$.*

This theorem includes the well-known results of Alfriend and Markeev for third- and fourth-order resonances. Furthermore, a simple geometric interpretation can be given, provided that $\mathcal{G}_2$ is a cylindrical surface. A projection onto the plane $C = 0$ shows that stability depends on the relative position of the curve $S^2 = M_1^r$, and

$$S = 0 \quad \text{if } r \text{ is odd}, \quad \text{or} \quad a_r M_1^{r/2} + S\sqrt{\alpha^2+\beta^2} = 0 \quad \text{if } r \text{ is even}.$$

If the last curve is inside $S^2 = M_1^r$, there are asymptotic orbits and the origin is unstable. On the other hand, if the curve is outside $S^2 = M_1^r$, the origin is stable (see Figure 3).

Note that in the case of stability, the function $H_r(M_2 = 0)$ is sign-defined and the energy can be either positive or negative. On the other hand, if the origin is unstable, the function $H_r(M_2 = 0)$ is no longer sign-defined and the energy takes positive and negative values. In this way, the nontrivial level set $H = 0$ acts as a separatrix and, since the origin is an equilibrium point, asymptotic orbits must appear. Note, also, the following result:

Corollary 1. *If the origin is stable, all the orbits are bounded for all $M_2 \in \mathbb{R}$. If the origin is unstable, there are always unbounded orbits for all $M_2 \in \mathbb{R}$.*

In this way, the stability properties of the origin can be deduced from the phase flow on any manifold defined by M_2. It is enough that there exists an unbounded orbit to ensure that the origin is unstable. On the other hand, if every orbit is bounded, then the origin is stable.

4. Conclusions

We have studied the stability properties of an equilibrium position of a two-degrees-of-freedom Hamiltonian system from a geometric point of view. To this end, different sets of variables have been introduced. In particular, the set of Lissajous variables has been revealed to be very convenient as they act as parametric coordinates of the reduced phase space. On the other hand, they suggest a different proof of Cabral and Meyer's theorem

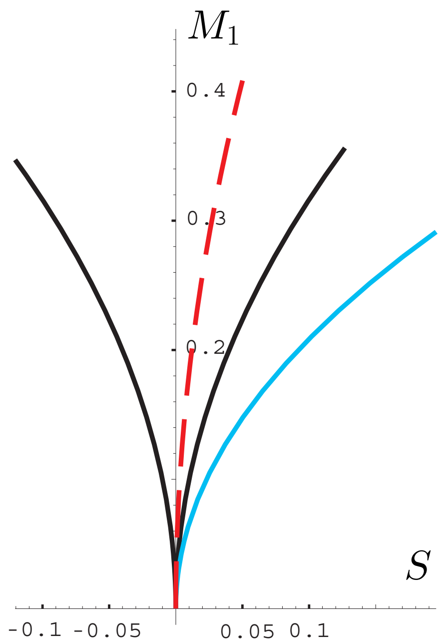

Fig. 3. Geometric view of Theorem 5 by projection onto the plane $C = 0$. The black solid line is the revolution surface $\mathcal{G}_1(M_2 = 0)$. The other two lines correspond to the surface $H_r(M_2 = 0)$ for two cases: the blue solid line represents stability, since the only common point is the origin; on the contrary, the dashed red line corresponds to instability, since both surfaces cut each other transversally.

of stability. Instead of considering the stability properties of the equilibrium position on the zero isoenergetic manifold, we have considered the motion on the surface defined by the new formal integral where the equilibrium lies. Besides, by using appropriate invariants, the normal form of the Hamiltonian function can be given in a compact form. In this way, it is possible to generalize the hypothesis of Cabral and Meyer's theorem and give a general stability criterion for odd and even resonances.

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References

[1] Alfriend, T. K. "The stability of the triangular Lagrangian points for commensurability of order two." *Celest. Mech.* **1** (1970) 351–359.
[2] Alfriend, T. K. "The stability of the triangular Lagrangian points for commensurability of order three." *Celest. Mech.* **4** (1971) 60–77.
[3] Arnold, V. I. "The stability of the equilibrium position of a Hamiltonian system of ordinary differential equations in the general elliptic case." *Soviet Math. Dokl.* **2** (1964) 247–249.
[4] Cabral, H. E. and Meyer, K. R. "Stability of equilibria and fixed points of conservative systems." *Nonlinearity* **12** (1999) 1351–1362.
[5] Cabral, H. E. "Normal forms of Hamiltonian systems and stability of equilibria." In *Classical and Celestial Mechanics* (Cabral, H. E., and Diacu, F., eds.), 117–169. Princeton Univ. Press., Princeton (2002).
[6] Chetaev, N. *The Stability of Motion*. Pergamon, Oxford (1961).
[7] Cushman, R. H. and Bates, L. M. *Global Aspects of Classical Integrable Systems.* Birkhäuser, Berlin (1997).
[8] Deprit, A. and Deprit-Bartholomé, A. "Stability of the triangular Lagrangian points." *Astron. J.* **72** (1967) 173–179.
[9] Deprit, A. "Delaunay normalisations." *Celestial Mech.* **26** (1982) 9–21 .
[10] Deprit, A. "The Lissajous transformation I. Basics." *Celest. Mech.* **51** (1991) 201–225.
[11] Deprit, A. and López-Moratalla, T. "Estabilidad orbital de satélites estacionarios." *Rev. Mat. Complutense* **9** (1996) 311–333.
[12] Elipe, A. and Deprit, A. "Oscillators in resonance." *Mech. Res. Commun.* **26** (1999) 635–640.
[13] Elipe, A. "Complete reduction of oscillators in resonance $p : q$." *Phys. Rev. E* **61** (2000) 6477–6484.
[14] Elipe, A., Lanchares, V., López-Moratalla, T. and Riaguas, A. "Nonlinear stability in resonant cases: A geometrical approach." *J. Nonlinear Sci.* **11** (2001) 211–222.
[15] Goździewski, K. "Stability of the triangular points in the unrestricted planar problem of a symmetric rigid body and a point mass." *Celest. Mech. & Dynam. Astron.* **85** (2003) 79–103.
[16] Gustavson, F. G. "On constructing formal integrals of a Hamiltonian system near an equilibrium point." *Astron. J.* **71** (1966) 670–686.
[17] Leontovich, A. M. "On the stability of the Lagrangian periodic solutions for the reduced problem of three bodies." *Soviet Math. Dokl.* **3** (1962) 425–430.
[18] Markeev, A. P. "On the stability of the triangular libration points in the circular bounded three-body problem." *Appl. Math. Mech.* **33** (1966) 105–110.
[19] Markeev, A. P. *Libration Points in Celestial Mechanics and Astrodynamics.* Nauka, Moscow (1978).
[20] Markeev, A. P. "The problem of the stability of the equilibrium position of a Hamiltonian system at 3:1 resonance." *J. Appl. Math. Mech.* **65** (2001) 639–645.
[21] Markeev, A. P. and Bardin, B. "On the stability of planar oscillations and rotations of a satellite in a circular orbit." *Celest. Mech. & Dynam. Astron.* **85** (2003) 51–66.
[22] Meyer, K. R. and Schmidt, D. S. "The stability of the Lagrange triangular point and a theorem of Arnold." *J. Diff. Eq.* **62** (1986) 222–236.
[23] Palacián, J. F. and Yanguas, P. "Reduction of polynomial planar Hamiltonians with quadratic unperturbed part." *SIAM Rev.* **42** (2000) 671–691.
[24] Riaguas, A., Elipe, A. and López-Moratalla, T. "Nonlinear stability of the equilibria in the gravity field of a finite straight segment." *Celest. Mech. Dynam. Astron.* **81** (2001) 235–248.
[25] Siegel, C. L. and Moser, J. K. *Lectures on Celestial Mechanics.* Springer-Verlag, Berlin (1971).
[26] Sokolsky, A. G. "On the stability of a Hamiltonian system with two degrees of freedom in the case of equal frequencies." *Prikh. Mat. Mech.* **38** (1974) 791–799.
[27] Sokolsky, A. G. "On the stability of an autonomous Hamiltonian system with two degrees of freedom at the resonance of the first order." *Prikh. Mat. Mech.* **41** (1977) 24–33.
[28] Sokolsky, A. G. "On the stability of a Hamiltonian system with two degrees of freedom in the case of null characteristic exponents." *Prikh. Mat. Mech.* **45** (1981) 441–449.