

Fundamental pro-groupoids and covering projections

by

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Abstract. We introduce a new notion of covering projection $E \rightarrow X$ of a topological space X which reduces to the usual notion if X is locally connected. We use locally constant presheaves and covering reduced sieves to find a pro-groupoid $\pi \text{crs}(X)$ and an induced category $\text{pro}(\pi \text{crs}(X), \text{Sets})$ such that for any topological space X the category of covering projections and transformations of X is equivalent to the category $\text{pro}(\pi \text{crs}(X), \text{Sets})$. We also prove that the latter category is equivalent to $\text{pro}(\pi CX, \text{Sets})$, where πCX is the Čech fundamental pro-groupoid of X . If X is locally path-connected and semilocally 1-connected, we show that $\pi \text{crs}(X)$ is weakly equivalent to πX , the standard fundamental groupoid of X , and in this case $\text{pro}(\pi \text{crs}(X), \text{Sets})$ is equivalent to the functor category $\text{Sets}^{\pi X}$. If $(X, *)$ is a pointed connected compact metrisable space and if $(X, *)$ is 1-movable, then the category of covering projections of X is equivalent to the category of continuous $\tilde{\pi}_1(X, *)$ -sets, where $\tilde{\pi}_1(X, *)$ is the Čech fundamental group provided with the inverse limit topology.

Introduction. It is well known that if X is a locally path-connected and semilocally 1-connected space then the category $\text{Cov proj } X$ of covering projections and transformations of X is equivalent to the category of πX -sets, that is, to the functor category $\text{Sets}^{\pi X}$. The aim of this work is to study the category $\text{Cov proj } X$ for any space X , without local conditions of connectedness.

In 1972–73, Fox [F1, F2] introduced the notion of overlay of a metrisable space. The fundamental theorem of Fox’s overlay theory establishes the existence of a bi-unique correspondence between the d -fold overlayings of a connected metrisable space X and the representations of the fundamental trope of X in the symmetric group Σ_d of degree d .

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On the other hand, for a locally connected distributive category \mathcal{C} , using the filtered small category of hypercoverings, Artin and Mazur [A-M] constructed a pro-simplicial set HC . In particular, for the category \mathcal{C} induced by a locally connected space, this pro-simplicial set is the Čech pro-simplicial set defined by the Čech nerve of all open coverings \mathcal{U} of the space X . As a consequence of this construction they classify the covering projections of X which are trivial over an open covering \mathcal{U} . The construction given by Artin and Mazur cannot be applied to non-locally connected spaces.

The objective of this paper is to solve the classification problem of “covering projections” for a general space. We want to remove the condition of local connectedness considered by Artin and Mazur and the conditions of metrisable space and finite fibre of the d -fold overlays analysed by Fox. Of course we want to have a classification up to isomorphism of covering projections but we also want to have a classification of morphisms between “covering projections”.

For this aim, we consider a new notion of covering projection $E \rightarrow X$ using atlases and an equivalence relation between atlases. If X is a connected metrisable space and all the fibres of $E \rightarrow X$ have a finite cardinal d , we have Fox’s d -fold overlay and if X is a locally connected space we have the usual notion of covering projection given, for instance, in Spanier’s book [S] and analysed by Artin and Mazur. To generalise the Fox fundamental trope or the Artin–Mazur fundamental pro-group of a space we consider a fundamental pro-groupoid $\pi \text{crs}(X)$ and a category $\text{pro}(\pi \text{crs}(X), \text{Sets})$ which is equivalent to the category of covering projections of the space X . This kind of category is also related to the notion of Galois category characterized by Grothendieck [Gro] and to the notion of Galois topos considered by Moerdijk [M].

If G is a pro-finite group, we can consider the category $G\text{-FinSets}$ of continuous finite G -sets. A category \mathcal{C} equivalent to $G\text{-FinSets}$ is said to be a *Galois category*. Grothendieck [Gro] gave an axiomatic description of these categories and proved that the pro-finite group G is unique up to isomorphism. The fundamental group of a pointed connected Grothendieck topos \mathcal{E} can be defined as the group determined by the Galois category \mathcal{E}_{lcf} of locally constant finite objects in \mathcal{E} . For instance, if X is a connected CW -complex, then the category of finite covering projections of X is equivalent to the category of continuous finite $\widehat{\pi_1 X}$ -sets, where $\widehat{\pi_1 X}$ is the pro-finite completion of $\pi_1 X$.

We also note that Moerdijk [M] gave a characterization of the toposes of the form BG for G a pro-discrete localic group. He also proved that the category of surjective pro-groups is equivalent to the category of pro-discrete localic groups. For a connected locally connected space this equivalence of categories carries the Artin–Mazur fundamental surjective pro-group to the

fundamental localic group considered by Moerdijk. This implies that the category of covering projections of a connected locally connected space is determined either by the Artin–Mazur fundamental pro-group or by the corresponding localic group. Nevertheless, this construction does not characterise the category of covering projections of a non-locally connected space X . In this case the fundamental pro-group(oid) $\pi \text{crs}(X)$ that we consider need not be a surjective pro-group(oid). At present, we do not know if for every pro-group G the category $\text{pro}(G, \text{Sets})$ is a Galois topos in the sense of the definition given in [M]. One interesting property of a Galois topos is that the pro-discrete localic group is determined up to isomorphism. However, the category of the form $\text{pro}(G, \text{Sets})$ does not determine the pro-groupoid G up to isomorphism. In §1, we give an example of two non-isomorphic pro-groups G and G' such that $\text{pro}(G, \text{Sets})$ is equivalent to $\text{pro}(G', \text{Sets})$. We do not know if the existence of an equivalence of categories implies that G and G' are weakly equivalent in some sense.

We also analyse the category of covering projections of a compact metrisable space X . In this case, the fundamental pro-groupoid is isomorphic to a tower of groupoids. If we assume that X is connected, then the tower of groupoids reduces to a tower of groups G and if for a given point $x \in X$, (X, x) is 1-movable, then the category $\text{pro}(G, \text{Sets})$ is equivalent to the category of continuous $\lim G$ -sets, where $\lim G$ is provided with the inverse limit topology. We note that the condition of pointed 1-movability implies that the Čech fundamental pro-group is isomorphic to a tower $G = \{G(n)\}$ of groups with surjective bonding homomorphisms. Since we are working with a tower we see that $\lim G$ is not trivial and the maps $\lim G \rightarrow G(n)$ are surjections. For the more general case of surjective pro-groups, Moerdijk [M] has noted that $\lim G$ can be trivial; he has solved this pathology by considering the inverse limit in the category of localic groups. I suppose that for some notion of “surjective pro-groupoid” G the category $\text{pro}(G, \text{Sets})$ will be equivalent to a category of LG -sets, where LG will be an associated pro-discrete localic groupoid.

This paper illustrates some nice relationships between the étale homotopy developed by Artin and Mazur [A-M], the theory of classifying toposes of localic groupoids of Moerdijk [M] and the methods used by Fox [F1, F2], by Edwards and Hastings [E-H] and by Porter [P] in shape theory and strong shape theory.

We finish this introduction by giving a summary of the main results of the paper. In §1, for a pro-groupoid G we define the category $\text{pro}(G, \text{Sets})$ and we show that a map $f : G \rightarrow G'$ in pro Gpd induces an equivalence of categories $f^* : \text{pro}(G', \text{Sets}) \rightarrow \text{pro}(G, \text{Sets})$ if f is an isomorphism in pro Gps , or if f is a (level) weak equivalence, or if G, G' are towers and f is an isomorphism in the category $\text{tow } \pi_0 \text{Gpd}$. The main results of §2 are the right definition

of covering projection, the determination of the pro-groupoid $\pi \text{crs}(X)$, and Theorem 2.2, which shows that for any space X the category of covering projections of X is equivalent to the category $\text{pro}(\pi \text{crs}(X), \mathbf{Sets})$. In §3, we establish a connection with the Artin–Mazur fundamental pro-groupoid which is isomorphic to the Čech fundamental pro-groupoid, and we find a weak equivalence $\pi \text{Sd} CX \rightarrow \pi \text{crs}(X)$ from the fundamental pro-groupoid of the subdivision of the Čech pro-simplicial set to the pro-groupoid $\pi \text{crs}(X)$ of reduced covering sieves of X . Therefore the category $\text{pro}(\pi \text{crs}(X), \mathbf{Sets})$ is equivalent to $\text{pro}(\pi CX, \mathbf{Sets})$. In §4, we give an easy proof of the standard classification of covering projections of a locally connected and semilocally 1-connected space. Finally, in §5, we show that under some shape conditions, we can obtain surjective towers of groups and in this case the category of covering projections reduces to a category of continuous $\check{\pi}_1(X, *)$ -sets, where $\check{\pi}_1(X, *)$ is the Čech fundamental topological group. We also prove as a corollary a version of the fundamental theorem of Fox’s overlay theory.

0. Preliminaries. In this section, we introduce some notation and terminology which is frequently used in this paper.

Let C be a small category and C^{op} its opposite. As usual, we denote by $\mathbf{Sets}^{C^{\text{op}}}$ the category whose objects are all functors $P : C^{\text{op}} \rightarrow \mathbf{Sets}$ and morphisms $P \rightarrow P'$ are all the natural transformations $\theta : P \rightarrow P'$ between such functors. A functor $P : C^{\text{op}} \rightarrow \mathbf{Sets}$ is also called a *presheaf* on C . A presheaf P on C is said to be *locally constant* if for every arrow $f : A \rightarrow B$ in C , $Pf : PB \rightarrow PA$ is an isomorphism. We denote by $(\mathbf{Sets}^{C^{\text{op}}})_{\text{lc}}$ the category of locally constant presheaves on C .

For the category C we have the Yoneda embedding $y : C \rightarrow \mathbf{Sets}^{C^{\text{op}}}$ defined on objects by $yA(B) = \text{Hom}_C(B, A)$. The following result will be used; for more details we refer the reader to [M-M, Theorem I.5.2].

THEOREM 0.1. *Let $l : C \rightarrow \mathcal{D}$ be a functor from a small category C to a cocomplete category \mathcal{D} . Then the functor $R : \mathcal{D} \rightarrow \mathbf{Sets}^{C^{\text{op}}}$ defined by*

$$RX(C) = \text{Hom}_{\mathcal{D}}(lC, X)$$

has a left adjoint $L : \mathbf{Sets}^{C^{\text{op}}} \rightarrow \mathcal{D}$.

Let Δ denote the small category whose objects are finite ordered sets $[n] = \{0 < \dots < n\}$ and whose morphisms are those $\phi : [n] \rightarrow [m]$ which preserve the order. We shall consider the category of simplicial sets as the functor category $\mathbf{SS} = \mathbf{Sets}^{\Delta^{\text{op}}}$. By Theorem 0.1 the functor $y : \Delta \rightarrow \mathbf{Sets}^{\Delta^{\text{op}}}$ and the functor $\Delta \rightarrow \mathbf{Top}$, $[n] \rightarrow \Delta_n$, where Δ_n is the standard n -simplex, induce the *singular functor* $\text{Sing} : \mathbf{Top} \rightarrow \mathbf{SS}$ and its left adjoint, the *realization functor* $|-| : \mathbf{SS} \rightarrow \mathbf{Top}$, $X \rightarrow |X|$. We recall that a map $f : X \rightarrow Y$ in \mathbf{SS} is a *weak equivalence* if for every $x \in X_0$ and $q \geq 0$ the induced

map $\pi_q(f) : \pi_q(X, x) \rightarrow \pi_q(Y, fx)$ is an isomorphism, where π_q denotes the homotopy group functor.

Let G be a group. We can view G as a small category with one object $*$ and arrows given by the elements of G . The composition is given by the product of G . In this case \mathbf{Sets}^G is the category of left G -sets and $\mathbf{Sets}^{G^{\text{op}}}$ is the category of right G -sets. An object in \mathbf{Sets}^G is determined by a homomorphism $\eta : G \rightarrow \text{Aut } X$, where $\text{Aut } X$ is the group of automorphisms of a set X . For a given map $\widehat{f} : X \rightarrow X'$ in \mathbf{Sets} , we denote by $\text{Aut } \widehat{f}$ the group of automorphisms of \widehat{f} in the category $\mathbf{Maps}(\mathbf{Sets})$ of maps in \mathbf{Sets} . If $\eta : G \rightarrow \text{Aut } X$, $\eta' : G \rightarrow \text{Aut } X'$ determine two objects in \mathbf{Sets}^G , a morphism $f : \eta \rightarrow \eta'$ is given by a pair $f = (\theta_f, \widehat{f})$ where $\widehat{f} : X \rightarrow X'$ is a map and $\theta_f : G \rightarrow \text{Aut } \widehat{f}$ is a group homomorphism such that the following diagram is commutative:

$$\begin{array}{ccc}
 & & \text{Aut } X \\
 & \nearrow \eta & \uparrow \text{pr}_1 \\
 G & \xrightarrow{\theta_f} & \text{Aut } \widehat{f} \\
 & \searrow \eta' & \downarrow \text{pr}_2 \\
 & & \text{Aut } X'
 \end{array}$$

For G a topological group, we have an analogous category of continuous G -sets, which is denoted by $G\text{-Sets}$. In this case, $\text{Aut } X$ and $\text{Aut } \widehat{f}$ are provided with the discrete topology and we consider continuous homomorphisms η and θ_f .

Given a topological space X , we denote by $\mathcal{O}(X)$ the small category whose objects are all open subsets U of X , and arrows $V \rightarrow U$ are inclusions $V \subset U$. In this paper, we will consider the category $\mathbf{Sets}^{\mathcal{O}(X)^{\text{op}}}$ of presheaves on X , and the full subcategory $\text{Sh}(X)$ of sheaves on X . For more properties of sheaves we refer the reader to [M-M] and [J]. We also consider the category $\text{Etale } X$ whose objects are étale maps $p : E \rightarrow X$; that is, p is a local homeomorphism in the following sense: For each $e \in E$, there is an open set V such that pV is open in X and $p|_V : V \rightarrow pV$ is a homeomorphism. A morphism $f : p \rightarrow p'$ is given by a continuous map $f : E \rightarrow E'$ such that $f p' = p$.

If $F : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Sets}$ is a presheaf on X , we can consider $F_x = \text{colim}_{x \in U} F(U)$ and the map $\text{germ}_x : F(U) \rightarrow F_x$. For each $\sigma \in F(U)$, $\text{germ}_x \sigma = \{ \text{germ}_x \sigma \mid x \in U \}$ is a subset of $E = \bigsqcup_{x \in E} F_x$. All the subsets $\text{germ}_x \sigma$ form the base of a topology on E such that the map $p : E \rightarrow X$, $p(\sigma) = x$ if $\sigma \in F_x$, is an étale map, which is also called the *bundle of germs* of F . This construction gives a functor $\Lambda : \mathbf{Sets}^{\mathcal{O}(X)^{\text{op}}} \rightarrow \text{Etale } X$ which is left adjoint to the functor $\Gamma : \text{Etale } X \rightarrow \mathbf{Sets}^{\mathcal{O}(X)^{\text{op}}}$ defined by

$$\Gamma p(U) = \{s \mid s \text{ is a continuous section of } p \text{ on } U\}.$$

The restriction $\Lambda : \text{Sh}(X) \rightarrow \text{Etale } X$ is an equivalence of categories with quasi-inverse Γ .

We shall often use categories of fractions and categories of right fractions (see [G-Z]). Let \mathcal{C} be a category and let Σ be a class of morphisms in \mathcal{C} . The category of fractions induced by Σ will be denoted by $\mathcal{C}[\Sigma^{-1}]$, and by $\mathcal{C}\Sigma^{-1}$ if Σ admits a calculus of right fractions. In the last case the hom-set can be defined by

$$\text{Hom}_{\mathcal{C}\Sigma^{-1}}(X, Y) = \text{colim}_{s \in \Sigma, \text{codomain}(s)=X} \text{Hom}_{\mathcal{C}}(\text{domain}(s), Y).$$

A category I is said to be *left filtering* if it satisfies the following conditions:

- (a) given two objects i, i' in I , there is an object j in I and morphisms $j \rightarrow i, j \rightarrow i'$,
- (b) if $u, v : j \rightarrow i$ are morphisms in I , there is k in I and a morphism $w : k \rightarrow j$ such that $uw = vw$.

A *pro-object* in C is a functor $X : I \rightarrow C$, where I is a left filtering small category. An arrow $u : j \rightarrow i$ is carried by X to a morphism $X(u) : X(j) \rightarrow X(i)$, which is called a *bonding morphism*. In some cases the hom-set $\text{Hom}_I(j, i)$ only has one arrow and we then use the notation $X_i^j : X(j) \rightarrow X(i)$. We also use this notation when no confusion is possible.

We are going to consider the category $\text{pro } C$ whose objects are pro-objects in C . Given pro-objects $X : I \rightarrow C$ and $Y : J \rightarrow C$ in C , the morphism set from X to Y is defined by

$$\text{Hom}_{\text{pro } C}(X, Y) = \lim_j \text{colim}_i \text{Hom}_C(X(i), Y(j)).$$

An alternative description of morphisms in $\text{pro } C$ can be given as follows: A morphism $u : X \rightarrow Y$ is represented by a pair $(\varphi, f(j))$, where $\varphi : |J| \rightarrow |I|$ is a map from the object set of J to the object set of I and $f(j) : X(\varphi(j)) \rightarrow Y(j)$ is a morphism in C , $j \in |J|$, such that if $j \rightarrow j'$ is a morphism in J , then there are $i \in |I|$, $i \rightarrow \varphi(j)$ and $i \rightarrow \varphi(j')$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X(\varphi(j)) & \longrightarrow & Y(j) \\ & \nearrow & & \downarrow \\ X(i) & & & \\ & \searrow & & \\ & X(\varphi(j')) & \longrightarrow & Y(j') \end{array}$$

Two pairs $(\varphi, f(j)), (\psi, g(j))$ represent the same morphism u if for each

$j \in |J|$, there are $i \in |I|$, $i \rightarrow \varphi(j)$ and $i \rightarrow \psi(j)$ such that the following diagram commutes:

$$\begin{array}{ccc} & X(\varphi(j)) & \\ \nearrow & & \searrow \\ X(i) & & Y(j) \\ \searrow & & \nearrow \\ & X(\psi(j)) & \end{array}$$

One of the more interesting properties of the category $\text{pro } C$ is that if $Y : J \rightarrow C$ is a pro-object and $\phi : I \rightarrow J$ is a cofinal functor, then $Y\phi : I \rightarrow C$ is isomorphic to $Y : J \rightarrow C$ in the category $\text{pro } C$.

For each left filtering small category I , we denote by C^I the category whose objects are functors $X : I \rightarrow C$ and morphisms are natural transformations; that is, a morphism $f : X \rightarrow Y$ is given by a coherent family of morphisms $f(i) : X(i) \rightarrow Y(i)$, $i \in |I|$. There is a canonical functor $\gamma : C^I \rightarrow \text{pro } C$, and a morphism of the form $\gamma f : \gamma X \rightarrow \gamma Y$ is said to be a *level morphism*.

Of particular interest is the full subcategory $\text{tow } C$ of $\text{pro } C$ determined by objects whose indexing category is \mathbb{N} , where \mathbb{N} is the category whose objects are non-negative integer numbers and $\text{Hom}_{\mathbb{N}}(n, m)$ has either one element if $n \geq m$ or is the empty set if $n < m$.

1. The category $\text{pro}(G, \text{Sets})$. In this section, we define and study the category $\text{pro}(G, \text{Sets})$, where G is a pro-groupoid. Later in §2 we shall prove that the category of covering projections of a space is equivalent to a category of the form $\text{pro}(G, \text{Sets})$.

Recall that a *groupoid* G is a small category where any morphism in G is an isomorphism. Given two groupoids G and G' , a groupoid homomorphism is just a functor $f : G \rightarrow G'$. Let \mathbf{Gpd} denote the category of groupoids.

We denote by $[0, 1]$ the groupoid with two objects $0, 1$ and whose morphisms are the identities and two mutually inverse maps $u : 0 \rightarrow 1$ and $u^{-1} : 1 \rightarrow 0$. If G is a groupoid, we can consider the product groupoid $G \times [0, 1]$ and the groupoid homomorphisms $\partial_0, \partial_1 : G \rightarrow G \times [0, 1]$, where for example ∂_0 carries an arrow $\alpha : U \rightarrow U'$ in G to the arrow $\partial_0 \alpha = (\alpha, \text{id}_0) : (U, 0) \rightarrow (U', 0)$. Using this cylinder, we can consider homotopies making commutative diagrams of the form

$$\begin{array}{ccc} G + G & \xrightarrow{f+g} & G' \\ \partial_0 + \partial_1 \downarrow & \nearrow F & \\ G \times [0, 1] & & \end{array}$$

where $G + G$ is the sum groupoid, and F is a groupoid homomorphism.

We note that a homotopy F determines a natural transformation η_F from f to g by $\eta_F(U) = F(\text{id}_U, u)$. Conversely, a natural transformation $\eta : f \rightarrow g$ determines a homotopy F_η from f to g by $F_\eta(\text{id}_U, u) = \eta(U)$.

If G and G' are two groupoids, we can consider a groupoid $\text{HOM}_{\text{Gpd}}(G, G')$ whose objects are given by the elements of the set $\text{Hom}_{\text{Gpd}}(G, G')$ and if $f, g : G \rightarrow G'$ are objects in $\text{HOM}_{\text{Gpd}}(G, G')$ a morphism $\eta : f \rightarrow g$ is a natural transformation from f to g . We denote by $\pi_0 \text{HOM}_{\text{Gpd}}(G, G')$ the set of isomorphism classes of the groupoid $\text{HOM}_{\text{Gpd}}(G, G')$. This set is also the set of homotopy classes of groupoid homomorphisms from G to G' . We also consider the category $\pi_0 \text{Gpd}$ which has the same objects as Gpd and the hom-set is defined by $\text{Hom}_{\pi_0 \text{Gpd}}(G, G') = \pi_0 \text{HOM}_{\text{Gpd}}(G, G')$. Denote by $\gamma : \text{Gpd} \rightarrow \pi_0 \text{Gpd}$ the projection functor which carries an arrow $f : G \rightarrow G'$ to the homotopy class $\gamma f : \gamma G \rightarrow \gamma G'$. We note that f is an equivalence (of categories) if and only if f is a homotopy equivalence; that is, if γf is an isomorphism in $\pi_0 \text{Gpd}$.

For a given pro-groupoid $G : I \rightarrow \text{Gpd}$, we consider the category (G, Sets) . An object of (G, Sets) is given by a pair $(G(i), F)$ where i is an object in I and $F : G(i) \rightarrow \text{Sets}$ is a functor. A morphism α from $(G(i), F)$ to $(G(j), H)$ is a pair $\alpha = (i \rightarrow j, \theta_\alpha : F \rightarrow HG_j^i)$ where $i \rightarrow j$ is a morphism in I and $\theta_\alpha : F \rightarrow HG_j^i$ is a natural transformation (G_j^i is the corresponding bonding map).

Consider the class

$$\Sigma = \{\alpha \mid \alpha \text{ is a morphism in } (G, \text{Sets}) \text{ and } \theta_\alpha \text{ is an equivalence}\}.$$

It is easy to check that the class Σ admits a calculus of right fractions (see §1 and [G-Z]). Therefore we can consider the category of right fractions $(G, \text{Sets})\Sigma^{-1}$ that will be denoted by $\text{pro}(G, \text{Sets})$.

If I is the indexing category of the pro-groupoid G , and i, j are two objects in I , we consider the category $I \downarrow \{i, j\}$ whose objects are pairs (u, v) of maps $u : k \rightarrow i, v : k \rightarrow j$, and a morphism from (u, v) to (u^1, v^1) is given by a map $w : k \rightarrow k^1$ such that $u^1 w = u, v^1 w = v$. If $(G(i), F)$ and $(G(j), H)$ are two objects in $\text{pro}(G, \text{Sets})$, we can consider the category $I \downarrow \{i, j\}$; for an object (u, v) in $I \downarrow \{i, j\}$, we write $k = \text{domain}(u) = \text{domain}(v)$. From the definition of the hom-set in a category of right fractions, one has

$$\text{Hom}_{\text{pro}(G, \text{Sets})}((G(i), F), (G(j), H)) \cong \text{colim}_{I \downarrow \{i, j\}} \text{Hom}_{\text{Sets}^{G(k)}}(FG_i^k, HG_j^k).$$

Now assume that $f : G \rightarrow G'$ is a morphism in pro Gpd represented by a pair $(\varphi, f(i'))$. We are going to see how the pair $(\varphi, f(i'))$ induces a functor $(\varphi, f(i'))^* : \text{pro}(G', \text{Sets}) \rightarrow \text{pro}(G, \text{Sets})$. First we define a functor from (G', Sets) to $\text{pro}(G, \text{Sets})$. Let $\alpha' = (i' \rightarrow j', \theta_{\alpha'} : F' \rightarrow H'G_{j'}^{i'})$ be a morphism in (G', Sets) from $(G'(i'), F')$ to $(G'(j'), H')$. Then $(\varphi, f(i'))^*$ carries these objects to $(G(\varphi i'), F' f(i'))$ and $(G(\varphi j'), H' f(j'))$, respectively.

In order to get $(\varphi, f(i'))^*(\alpha')$, we choose k in I and arrows $k \rightarrow \varphi i'$ and $k \rightarrow \varphi j'$ such that the diagram

$$\begin{array}{ccc}
 & G(\varphi i') & \xrightarrow{f(i')} & G'(i') \\
 & \nearrow & & \downarrow \\
 G(k) & & & \\
 & \searrow & & \\
 & G(\varphi j') & \xrightarrow{f(j')} & G'(j')
 \end{array}$$

is commutative. Then $(\varphi, f(i'))^*(\alpha')$ is the morphism in $\text{pro}(G, \text{Sets})$ represented by the natural transformation $\theta_{\alpha'} * (f(i')G_{\varphi i'}^k) : F' f(i')G_{\varphi i'}^k \rightarrow H' f(j')G_{\varphi j'}^k$. It is easy to check that two choices of k represent the same morphism in $\text{pro}(G, \text{Sets})$. The functor $(\varphi, f(i'))^*$ has the property that if α' is in Σ' , then $(\varphi, f(i'))^*(\alpha')$ is an isomorphism. Therefore we have an induced functor

$$(\varphi, f(i'))^* : \text{pro}(G', \text{Sets}) \rightarrow \text{pro}(G, \text{Sets}).$$

We note that it $(\varphi, f(i'))$ and $(\psi, g(i'))$ represent the same morphism $f : G \rightarrow G'$, then the functor $(\varphi, f(i'))^*$ is isomorphic to $(\psi, g(i'))^*$. We will denote by $f^* : \text{pro}(G', \text{Sets}) \rightarrow \text{pro}(G, \text{Sets})$ one of these functors.

If $f : G \rightarrow G'$ and $g : G' \rightarrow G''$ are morphisms in pro Gpd represented by pairs $(\varphi, f(i'))$ and $(\psi, g(i''))$, then gf can be represented by $(\varphi\psi, g(i'')f(\psi i'))$. If $gf = \text{id}$ and $fg = \text{id}$, then $(\varphi\psi, g(i'')f(\psi i'))^*$ and $(\psi\varphi, f(i')g(\varphi i'))^*$ are isomorphic to identity functors. Therefore the functor $(\varphi, f(i'))^*$ is an equivalence of categories. We restate this fact in the following:

LEMMA 1.1. *If $f : G \rightarrow G'$ is an isomorphism in pro Gpd , then $f^* : \text{pro}(G', \text{Sets}) \rightarrow \text{pro}(G, \text{Sets})$ is an equivalence of categories.*

The following result will be useful:

LEMMA 1.2. *Let $f : G \rightarrow G'$ be a level morphism in pro Gpd such that for each $i \in I$, $f(i) : G(i) \rightarrow G'(i)$ is an equivalence. Then $f^* : \text{pro}(G', \text{Sets}) \rightarrow \text{pro}(G, \text{Sets})$ is an equivalence of categories.*

PROOF. In this case, the functor f^* is defined on objects by $f^*(G'(i), F')$ $= (G(i), F'f(i))$ and for morphisms one has

$$\begin{aligned}
 & \text{Hom}_{\text{pro}(G', \text{Sets})}((G'(i), F'), (G'(j), H')) \\
 &= \text{colim}_{I \downarrow \{i, j\}} \text{Hom}_{\text{Sets}^{G'(k)}}(F'G_i^k, H'G_j^k)
 \end{aligned}$$

$$\begin{aligned}
&\cong \operatorname{colim}_{I \downarrow \{i,j\}} \operatorname{Hom}_{\mathbf{Sets}^{G(k)}}(F'G'_i{}^k f(k), H'G'_j{}^k f(k)) \\
&\cong \operatorname{colim}_{I \downarrow \{i,j\}} \operatorname{Hom}_{\mathbf{Sets}^{G(k)}}(F'f(i)G_i^k, H'f(j)G_j^k) \\
&\cong \operatorname{Hom}_{\operatorname{pro}(G, \mathbf{Sets})}((G(i), F'f(i)), (G(j), H'f(j))).
\end{aligned}$$

Therefore f^* is a full faithful functor. On the other hand, if $(G(i), F)$ is an object in $\operatorname{pro}(G, \mathbf{Sets})$, we can take a quasi-inverse $g : G'(i) \rightarrow G(i)$ of the equivalence $f(i) : G(i) \rightarrow G'(i)$. Then $f^*(G'(i), Fg) = (G(i), Fgf(i))$. However, $(G(i), Fgf(i))$ is isomorphic to $(G(i), F)$. Thus we have shown that f^* is an equivalence of categories.

At the beginning of this section we have considered the categories \mathbf{Gpd} and $\pi_0\mathbf{Gpd}$ and the functor $\gamma : \mathbf{Gpd} \rightarrow \pi_0\mathbf{Gpd}$. This functor γ induces a functor $\gamma = \operatorname{pro} \gamma : \operatorname{pro} \mathbf{Gpd} \rightarrow \operatorname{pro} \pi_0\mathbf{Gpd}$. We have shown that two isomorphic objects G, G' in $\operatorname{pro} \mathbf{Gpd}$ induce equivalent categories $\operatorname{pro}(G, \mathbf{Sets})$, $\operatorname{pro}(G', \mathbf{Sets})$. Next we analyse this kind of questions for objects in the category $\operatorname{pro} \pi_0\mathbf{Gpd}$.

If $G : \mathbb{N} \rightarrow \pi_0\mathbf{Gpd}$ is an object in $\operatorname{tow} \pi_0\mathbf{Gpd}$, we can choose for each bonding morphism $G(i+1) \rightarrow G(i)$ in $\pi_0\mathbf{Gpd}$ a representative map \bar{G}_i^{i+1} ; in this way we obtain an object $\bar{G} : \mathbb{N} \rightarrow \mathbf{Gpd}$ in $\operatorname{tow} \mathbf{Gpd}$ such that $\bar{G}(i) = G(i)$ and $\gamma\bar{G} = G$. If we choose different bonding maps \tilde{G}_i^{i+1} , we have a new pro-groupoid $\tilde{G} : \mathbb{N} \rightarrow \mathbf{Gpd}$, but we can prove the following result:

LEMMA 1.3. *The category $\operatorname{pro}(\bar{G}, \mathbf{Sets})$ is equivalent to the category $\operatorname{pro}(\tilde{G}, \mathbf{Sets})$.*

Proof. For each $i \geq 0$, since \bar{G}_i^{i+1} is homotopic to \tilde{G}_i^{i+1} , we can choose a homotopy $L^{i+1} : G(i+1) \times [0, 1] \rightarrow G(i)$ such that $L^{i+1}\partial_0 = \bar{G}_i^{i+1}$ and $L^{i+1}\partial_1 = \tilde{G}_i^{i+1}$.

Consider the commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & G(i+1) & \xrightarrow{\bar{G}_i^{i+1}} & G(i) & \longrightarrow & \cdots \\
& & \downarrow \partial_0 & & \downarrow \partial_0 & & \\
\cdots & \longrightarrow & G(i+1) \times [0, 1] & \xrightarrow{(L^{i+1}, \operatorname{pr}_2)} & G(i) \times [0, 1] & \longrightarrow & \cdots \\
& & \uparrow \partial_1 & & \uparrow \partial_1 & & \\
\cdots & \longrightarrow & G(i+1) & \xrightarrow{\tilde{G}_i^{i+1}} & G(i) & \longrightarrow & \cdots
\end{array}$$

and denote by $G \times [0, 1]$ the ‘‘cylinder’’ pro-groupoid. We conclude that $\partial_0 : \bar{G} \rightarrow G \times [0, 1]$ and $\partial_1 : \tilde{G} \rightarrow G \times [0, 1]$ satisfy the conditions of Lemma 1.2, and therefore $\operatorname{pro}(\bar{G}, \mathbf{Sets})$ is equivalent to $\operatorname{pro}(\tilde{G}, \mathbf{Sets})$.

As a consequence of Lemma 1.3, if G is an object in $\text{tow } \pi_0 \mathbf{Gpd}$, the category $\text{pro}(\overline{G}, \mathbf{Sets})$ will be denoted by $\text{pro}(G, \mathbf{Sets})$.

LEMMA 1.4. *Let $f = \{f(i) : G(i) \rightarrow G'(i)\}$ be a level morphism in $\text{tow } \pi_0 \mathbf{Gpd}$ and assume that there are maps $g(i) : G'(i+1) \rightarrow G(i)$ such that for each $i \geq 0$ the diagram*

$$\begin{array}{ccc} G(i+1) & \xrightarrow{f(i+1)} & G'(i+1) \\ G_i^{i+1} \downarrow & \swarrow g(i) & \downarrow (G')_i^{i+1} \\ G(i) & \xrightarrow{f(i)} & G'(i) \end{array}$$

is commutative in $\pi_0 \mathbf{Gpd}$. Then $\text{pro}(G, \mathbf{Sets})$ is equivalent to $\text{pro}(G', \mathbf{Sets})$.

PROOF. Since $(G')_i^{i+1} = f(i)g(i)$ in $\pi_0 \mathbf{Gpd}$, from Lemma 1.3 it follows that the towers

$$\begin{aligned} \dots &\rightarrow G'(i+1) \xrightarrow{(G')_i^{i+1}} G'(i) \rightarrow \dots, \\ \dots &\rightarrow G'(i+1) \xrightarrow{f(i)g(i)} G'(i) \rightarrow \dots \end{aligned}$$

determine equivalent categories. Since the towers

$$\begin{aligned} \dots &\rightarrow G'(i+1) \xrightarrow{f(i)g(i)} G'(i) \rightarrow \dots, \\ \dots &\rightarrow G(i+1) \xrightarrow{f(i+1)} G'(i+1) \xrightarrow{g(i)} G(i) \xrightarrow{f(i)} G'(i) \rightarrow \dots, \\ \dots &\rightarrow G(i+1) \xrightarrow{g(i)f(i+1)} G(i) \rightarrow \dots \end{aligned}$$

are isomorphic in $\text{tow } \mathbf{Gpd}$ and $g(i)f(i+1) = (G')_i^{i+1}$ in $\pi_0 \mathbf{Gpd}$, Lemmas 1.1 and 1.3 show that $\text{pro}(G, \mathbf{Sets})$ is equivalent to $\text{pro}(G', \mathbf{Sets})$.

As a consequence of these lemmas, we have:

PROPOSITION 1.1. *Let G, G' be objects in $\text{tow } \pi_0 \mathbf{Gpd}$. If G is isomorphic to G' in $\text{tow } \pi_0 \mathbf{Gpd}$, then $\text{pro}(G, \mathbf{Sets})$ is equivalent to $\text{pro}(G', \mathbf{Sets})$.*

PROOF. Let $f : G \rightarrow G'$ be an isomorphism. The map f can be represented by a pair $(\varphi, f(i))$ such that $\varphi(i) \geq i$, $\varphi(i) > \varphi(j)$ if $i > j$, and for each $i \geq 0$, the diagram

$$\begin{array}{ccc} G(\varphi(i+1)) & \xrightarrow{f(i+1)} & G'(i+1) \\ \downarrow & & \downarrow \\ G(\varphi(i)) & \xrightarrow{f(i)} & G'(i) \end{array}$$

is commutative in $\pi_0 \mathbf{Gpd}$. Define an object G_1 in $\text{tow } \mathbf{Gpd}$ by $G_1(i) = G(\varphi(i))$

and $(G_1)_i^{i+1} = G_{\varphi(i)}^{\varphi(i+1)}$. We also have $f_1 : G_1 \rightarrow G'$ defined by $f_1(i) = f(i) : G_1(i) \rightarrow G'(i)$.

We know that G is isomorphic to G_1 in tow Gpd and $f_1 : G_1 \rightarrow G'$ is a level isomorphism in $\text{tow } \pi_0 \text{Gpd}$. By Lemma 1.1, $\text{pro}(G, \text{Sets})$ is equivalent to $\text{pro}(G_1, \text{Sets})$.

Since $f_1 : G_1 \rightarrow G'$ is a level isomorphism in $\text{tow } \pi_0 \text{Gpd}$, there is a map $g : G' \rightarrow G_1$ in $\text{tow } \pi_0 \text{Gpd}$ represented by $(\psi, g(i))$ such that $\psi(i) \geq i$, $\psi(i) < \psi(j)$ if $i < j$, and the diagram

$$\begin{array}{ccc} G_1(\psi^{i+1}0) & \xrightarrow{f_1(\psi^{i+1}0)} & G'(\psi^{i+1}0) \\ \downarrow & \swarrow g(\psi^i0) & \downarrow \\ G_1(\psi^i0) & \xrightarrow{f_1(\psi^i0)} & G'(\psi^i0) \end{array}$$

is commutative in $\pi_0 \text{Gpd}$, where ψ^k denotes the iterated map $\psi \circ \dots \circ \psi$. Define

$$\begin{aligned} G_2(i) &= G_1(\psi^i0), & G'_1(i) &= G'(\psi^i0), \\ f_2(i) &= f_1(\psi^i0), & g(i) &= g(\psi^i0). \end{aligned}$$

Now we see that G_1 is isomorphic to G_2 and G' is isomorphic to G'_1 in tow Gpd . By Lemma 1.1, $\text{pro}(G_1, \text{Sets})$ is equivalent to $\text{pro}(G_2, \text{Sets})$, and $\text{pro}(G', \text{Sets})$ is equivalent to $\text{pro}(G'_1, \text{Sets})$. Since $f_2 : G_2 \rightarrow G'_1$ and $g : G'_1 \rightarrow G_2$ satisfy the conditions of Lemma 1.4, it follows that $\text{pro}(G_2, \text{Sets})$ is equivalent to $\text{pro}(G'_1, \text{Sets})$. Thus we conclude that $\text{pro}(G, \text{Sets})$ is equivalent to $\text{pro}(G', \text{Sets})$.

EXAMPLE. We exhibit two non-isomorphic pro-groups F, F' which are isomorphic in $\text{tow } \pi_0 \text{Gpd}$. Therefore the categories $\text{pro}(F, \text{Sets})$ and $\text{pro}(F', \text{Sets})$ are equivalent. For $n \geq 0$ let $F(n)$ be the free group generated by $x_0, x_{n+1}, x_{n+2}, \dots$ and the bonding morphism $F(n+1) \rightarrow F(n)$ is defined to be the inclusion. The pro-group F' is defined by $F'(n) = F(n)$ and the bonding is $(F'_n)^{n+1}(a) = x_{n+1} a x_{n+1}^{-1}$. It is easy to check that $\lim F$ is the infinite cyclic group and $\lim F'$ is trivial. This implies that F is not isomorphic to F' . However, the bonding $F(n+1) \rightarrow F(n)$ is homotopic to the bonding $F'(n+1) \rightarrow F'(n)$. Therefore F is isomorphic to F' in the category $\text{tow } \pi_0 \text{Gpd}$ and by Proposition 1.1 above we see that $\text{pro}(F, \text{Sets})$ is equivalent to $\text{pro}(F', \text{Sets})$.

2. Classification of covering projections. In this section, we define a notion of covering projection that for locally connected spaces agrees with the notion given in Spanier's book [S]. The main result of this section is the determination of a pro-groupoid $\pi \text{crs}(X)$ such that the category of covering

projections of X is equivalent to the category $\text{pro}(\pi \text{ crs}(X), \text{Sets})$ defined in Section 1.

Given sets (or spaces) A, F, G a map $\theta : A \times F \rightarrow A \times G$ such that $\text{pr}_A \theta = \text{pr}_A$ is of the form $\theta(a, x) = (a, \theta_a(x))$ for $a \in A$ and $x \in F$, where $\theta_a : F \rightarrow G$ is a map which depends on $a \in A$. A map $\theta : A \times F \rightarrow A \times G$ such that $\text{pr}_A \theta = \text{pr}_A$ is said to be A -constant if $\theta_a = \theta_{a'}$ for all $a, a' \in A$.

Let $p : E \rightarrow X$ be a continuous map and let \mathcal{U} be an open covering of X . An atlas \mathcal{A} for $p : E \rightarrow X$ on \mathcal{U} consists of a family of homeomorphisms $\varphi_U : U \times F(U) \rightarrow p^{-1}U$, where $U \in \mathcal{U}$ and $F(U)$ is a discrete space, such that if $U, V \in \mathcal{U}$ and $\emptyset \neq W = U \cap V$, then the induced homeomorphism

$$W \times F(U) \xrightarrow{\varphi_U} p^{-1}W \xrightarrow{\varphi_V^{-1}} X \times F(V)$$

is W -constant, where φ_U, φ_V also denote the corresponding restrictions.

If $\mathcal{A} = \{\varphi_U : U \times F(U) \rightarrow p^{-1}U\}$ is an atlas on \mathcal{U} and $\mathcal{B} = \{\psi_V : V \times G(V) \rightarrow p^{-1}V\}$ is an atlas on \mathcal{V} , then \mathcal{A} is said to be *equivalent* to \mathcal{B} if there is an open covering \mathcal{W} which refines \mathcal{U} and \mathcal{V} and such that if $W \subset U \cap V$, then the induced homeomorphism

$$W \times F(U) \xrightarrow{\varphi_U} p^{-1}W \xrightarrow{\psi_V^{-1}} W \times G(V)$$

is W -constant.

DEFINITION 2.1. A *covering projection* $(p : E \rightarrow X, [\mathcal{A}])$ consists of a continuous map $p : E \rightarrow X$ and an equivalence class $[\mathcal{A}]$ of atlases.

REMARKS. (1) If X is a metrisable space and \mathcal{A} is an atlas on \mathcal{U} such that for any $U \in \mathcal{U}$, $F(U)$ is a finite set with d elements, then the map $p : E \rightarrow X$ is a d -fold overlay in the sense of Fox.

(2) If X is a locally connected space and \mathcal{A} is an atlas for $p : E \rightarrow X$ on \mathcal{U} and \mathcal{B} is an atlas for $p : E \rightarrow X$ on \mathcal{V} , then we can choose an open covering \mathcal{W} which refines \mathcal{U} and \mathcal{V} and such that each $W \in \mathcal{W}$ is connected. If $W \subset U \cap V$, then the homeomorphism

$$W \times F(U) \xrightarrow{\varphi_U} p^{-1}W \xrightarrow{\psi_V^{-1}} W \times G(V)$$

sends the connected components of $W \times F(U)$ into connected components of $W \times G(V)$. Therefore $\psi_V^{-1} \varphi_U$ is W -constant, and \mathcal{A} is equivalent to \mathcal{B} . Thus if X is a locally connected space, then a covering projection consists of a continuous map such that there is an open covering \mathcal{U} of X and for each $U \in \mathcal{U}$, $p^{-1}U = \coprod_{\alpha \in F(U)} U_\alpha$, where $F(U)$ is an index set, each U_α is an open subset of E and the restriction $p|_{U_\alpha} : U_\alpha \rightarrow U$ is a homeomorphism.

We shall use the following notion of covering transformation:

DEFINITION 2.2. Let $\Phi = (p : E \rightarrow X, [\mathcal{A}])$ and $\Phi' = (p' : E \rightarrow X, [\mathcal{A}'])$ be two covering projections. A *covering transformation* $f : \Phi \rightarrow \Phi'$ is a

continuous map $f : E \rightarrow E'$ such that $p'f = p$ and if $\mathcal{A} = \{\varphi_U \mid U \in \mathcal{U}\}$ and $\mathcal{A}' = \{\varphi_{U'} \mid U' \in \mathcal{U}'\}$ are two atlases for Φ and Φ' , respectively, then there is an open covering \mathcal{W} which refines \mathcal{U} and \mathcal{U}' and such that if $W \subset U \cap U'$, then the induced map

$$W \times F(U) \xrightarrow{\varphi_U} p^{-1}W \xrightarrow{f} p'^{-1}W \xrightarrow{\varphi_{U'}^{-1}} W \times F'(U')$$

is W -constant. We denote by $\text{Cov proj } X$ the category of covering projections and covering transformations of X .

REMARK. If X is a locally connected space and $p : E \rightarrow X$ and $p' : E' \rightarrow X$ are covering projections, then any continuous map $f : E \rightarrow E'$ such that $p'f = p$ is a covering transformation. Therefore, in this case the category $\text{Cov proj } X$ is equivalent to $(\text{Etale } X)_{\text{cp}}$, where $(\text{Etale } X)_{\text{cp}}$ denotes the full subcategory of $\text{Etale } X$ determined by covering projections.

If \mathcal{U} is an open covering X , we denote by $(\text{Cov proj } X)_{\mathcal{U}}$ the subcategory of $\text{Cov proj } X$ whose objects are those covering projections Φ which admit an atlas on \mathcal{U} . Given covering projections Φ and Φ' with atlases $\mathcal{A} = \{\varphi_U \mid U \in \mathcal{U}\}$ and $\mathcal{A}' = \{\varphi'_{U'} \mid U' \in \mathcal{U}'\}$, a morphism $f : \Phi \rightarrow \Phi'$ in $(\text{Cov proj } X)_{\mathcal{U}}$ is a covering transformation $f : \Phi \rightarrow \Phi'$ in $\text{Cov proj } X$ such that for any $U \in \mathcal{U}$ the map

$$U \times F(U) \xrightarrow{\varphi_U} p^{-1}U \xrightarrow{f} p'^{-1}U \xrightarrow{(\varphi'_{U'})^{-1}} U \times F'(U)$$

is U -constant. We note that if \mathcal{U} refines \mathcal{V} , then we have a faithful functor $(\text{Cov proj } X)_{\mathcal{V}} \rightarrow (\text{Cov proj } X)_{\mathcal{U}}$. One has the following result:

PROPOSITION 2.1. *Let Φ and Φ' be two covering projections and let $\mathcal{A} = \{\varphi_U \mid U \in \mathcal{U}\}$ and $\mathcal{A}' = \{\varphi_{U'} \mid U' \in \mathcal{U}'\}$ be two atlases of Φ and Φ' , respectively. Then*

$$\text{Hom}_{\text{Cov proj } X}(\Phi, \Phi') \cong \text{colim}_{\mathcal{W} \geq \mathcal{U}, \mathcal{W} \geq \mathcal{U}'} \text{Hom}_{(\text{Cov proj } X)_{\mathcal{W}}}(\Phi, \Phi').$$

PROOF. This follows directly from the definition of covering transformation.

In order to use locally constant sheaves to study the category $\text{Cov proj } X$, we recall and introduce some notions:

A family \mathcal{U} of open subsets of the space X is called a *sieve* on X if for every $U \in \mathcal{U}$ and every open subset $V \subset U$, we have $V \in \mathcal{U}$. If moreover $X = \bigcup_{U \in \mathcal{U}} U$, then \mathcal{U} is called a *covering sieve* on X . We denote by \mathcal{O} the covering sieve of all open subsets of X .

DEFINITION 2.3. Let \mathcal{U} be a family of non-empty open subsets of X such that if $U \in \mathcal{U}$ and $\emptyset \neq V \in \mathcal{O}$, $V \subset U$, then $V \in \mathcal{U}$. We then say that \mathcal{U} is a *reduced sieve* on X and if $X = \bigcup_{U \in \mathcal{U}} U$, then \mathcal{U} is a *covering reduced sieve* on X .

We note that if \mathcal{U} is a covering sieve on X , then $^*\mathcal{U} = \mathcal{U} \setminus \{\emptyset\}$ is a covering reduced sieve on X . Every open covering \mathcal{V} of X generates a covering sieve $s\mathcal{V} = \{U \in \mathcal{O} \mid \text{there is } V \in \mathcal{V} \text{ such that } U \subset V\}$ and the corresponding covering reduced sieve $^*s\mathcal{V} = \{U \in \mathcal{O} \mid U \neq \emptyset \text{ and there is } V \in \mathcal{V} \text{ such that } U \subset V\}$.

A (reduced) sieve \mathcal{U} can be considered as a small category, denoted again by \mathcal{U} , where the set of morphisms from U to V is given by $\text{Hom}_{\mathcal{U}}(U, V) = 1$ if $U \subset V$, and $\text{Hom}_{\mathcal{U}}(U, V) = \emptyset$ otherwise. A functor $P : \mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$ is said to be a *presheaf* on \mathcal{U} .

DEFINITION 2.4. Given a covering reduced sieve \mathcal{U} on X , a presheaf $P : \mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$ is said to be *locally constant* if P carries any arrow $U \subset V$ in \mathcal{U} into an isomorphism $P(V) \rightarrow P(U)$. We denote by $\mathbf{Sets}^{\mathcal{U}^{\text{op}}}$ the category of presheaves on \mathcal{U} and by $(\mathbf{Sets}^{\mathcal{U}^{\text{op}}})_{\text{lc}}$ the category of locally constant presheaves on \mathcal{U} .

Given a covering sieve \mathcal{U} , the canonical inclusion $\mathcal{U}^{\text{op}} \subset \mathcal{O}^{\text{op}}$ induces a restriction functor $\text{re} : \mathbf{Sets}^{\mathcal{O}^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{U}^{\text{op}}}$. We also have an extension functor $\text{ex} : \mathbf{Sets}^{\mathcal{U}^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{O}^{\text{op}}}$ which for a given presheaf $P : \mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$ is defined by

$$\text{ex}P(V) = \lim_{U \in \mathcal{U}^{\text{op}}} P(V \cap U).$$

It is routine to check

PROPOSITION 2.2. *The functor $\text{re} : \mathbf{Sets}^{\mathcal{O}^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{U}^{\text{op}}}$ is left adjoint to $\text{ex} : \mathbf{Sets}^{\mathcal{U}^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{O}^{\text{op}}}$.*

If \mathcal{U} is a covering reduced sieve and $P : \mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$ is a presheaf, we can consider the covering sieve $'\mathcal{U} = \mathcal{U} \cup \{\emptyset\}$ and the presheaf $'P : '\mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$ defined by the unique extension of P such that $P(\emptyset) = 1$. Therefore we also have an extension functor $\text{ex}' : \mathbf{Sets}^{\mathcal{U}^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{O}^{\text{op}}}$ defined by $\text{ex}'(P) = \text{ex}('P)$.

Let $\Lambda : \mathbf{Sets}^{\mathcal{O}^{\text{op}}} \rightarrow \mathbf{Etale} X$ be the functor which carries a presheaf $P : \mathcal{O}^{\text{op}} \rightarrow \mathbf{Sets}$ into the bundle ΛP of germs of P (see §1 and [M-M]). For a covering reduced sieve \mathcal{U} on X one has the composite

$$\mathbf{Sets}^{\mathcal{U}^{\text{op}}} \xrightarrow{\text{ex}'} \mathbf{Sets}^{\mathcal{O}^{\text{op}}} \xrightarrow{\Lambda} \mathbf{Etale} X.$$

If $P : \mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$ is a presheaf and $x \in X$ we have the set of germs of P at x ,

$$P_x = \text{colim}_{x \in U \in \mathcal{U}} P(U),$$

and the canonical map $\text{germ}_x^U : P(U) \rightarrow P_x$. We note that $\Lambda \text{ex}'P = (q(P) : E(P) \rightarrow X)$, where

$$E(P) = \bigsqcup_{x \in X} P_x, \quad (q(P))^{-1}x = P_x$$

and the topology of $E(P)$ is given by the base

$$\mathcal{B} = \{\dot{s}U \mid s \in P(U), U \in \mathcal{U}\}, \quad \dot{s}U = \{\text{germ}_x^U(s) \mid x \in U\}.$$

For each $U \in \mathcal{U}$, we consider the map

$$\varphi_U : U \times P(U) \rightarrow (q(P))^{-1}U$$

defined by $\varphi_U(x, s) = \text{germ}_x^U s$ for $x \in U$ and $s \in P(U)$. We note that for a fixed $s \in P(U)$, the restriction $\varphi_U(-, s) : U \times \{s\} \rightarrow \dot{s}U$ is a homeomorphism.

If P is a locally constant presheaf, then $\text{germ}_x^U : P(U) \rightarrow P_x$ is an isomorphism of discrete spaces. Therefore, in this case, φ_U is a homeomorphism.

Now we check that for a locally constant presheaf $P : \mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$, $\mathcal{A}(P) = \{\varphi_U \mid U \in \mathcal{U}\}$ is an atlas on \mathcal{U} for $q(P) : E(P) \rightarrow X$. If $U, V \in \mathcal{U}$ and $W = U \cap V$, then the map

$$W \times P(U) \xrightarrow{\varphi_U} (q(P))^{-1}W \xrightarrow{\varphi_V^{-1}} W \times P(V)$$

is W -constant, because if $x, y \in W$, $s \in P(U)$ and $t, t' \in P(V)$ are such that

$$\text{germ}_x^U s = \text{germ}_x^V t, \quad \text{germ}_y^U s = \text{germ}_y^V t'$$

then $t|_W = s|_W = t'|_W$, hence $t = t'$ ($t|_W$ denotes the image of $t \in P(V)$ under the restriction map $P(V) \rightarrow P(W)$).

Therefore for a locally constant presheaf $P : \mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$, one has the covering projection $\bar{A}P = (q(P) : E(P) \rightarrow X, [\mathcal{A}(P)])$. In order to construct a functor $\bar{A} : (\mathbf{Sets}^{\mathcal{U}^{\text{op}}})_{\text{lc}} \rightarrow (\text{Cov proj } X)_{\mathcal{U}}$ we recall that if $f : P \rightarrow P'$ is a natural transformation of presheaves defined on \mathcal{U} , then we have an induced map $\bar{A}f : E(P) \rightarrow E(P')$ defined by $\bar{A}f(\text{germ}_x^U s) = \text{germ}_x^U(f(U)s)$, where $f(U) : P(U) \rightarrow P'(U)$ are the ‘‘components’’ of f .

To show $\bar{A}f$ is a morphism in $(\text{Cov proj } X)_{\mathcal{U}}$ we have to check that

$$U \times P(U) \xrightarrow{\varphi_U} (q(P))^{-1}U \xrightarrow{\bar{A}f} (q(P'))^{-1}U \xrightarrow{(\varphi'_U)^{-1}} U \times P'(U)$$

is U -constant. If $(x, s) \in U \times P(U)$, then $(\varphi'_U)^{-1}(\bar{A}f)\varphi_U(x, s) = (x, f(U)(s))$ and $f(U)(s)$ does not depend on x . This implies that the above map is U -constant.

Thus we have constructed a functor $\bar{A} : (\mathbf{Sets}^{\mathcal{U}^{\text{op}}})_{\text{lc}} \rightarrow (\text{Cov proj } X)_{\mathcal{U}}$. Now we can prove that the category of covering projections and transformations which trivialise on \mathcal{U} is equivalent to the category of locally constant presheaves on \mathcal{U} .

THEOREM 2.1. *Given a covering reduced sieve \mathcal{U} on a space X , the functor $\bar{A} : (\mathbf{Sets}^{\mathcal{U}^{\text{op}}})_{\text{lc}} \rightarrow (\text{Cov proj } X)_{\mathcal{U}}$ is an equivalence of categories.*

Proof. First, we show that $\bar{\Lambda}$ is a faithful functor. Suppose that $f, g : P \rightarrow P'$ are natural transformations and P, P' are objects in $(\mathbf{Sets}^{\mathcal{U}^{\text{op}}})_{\text{lc}}$. If $\bar{\Lambda}f = \bar{\Lambda}g$, then for each $U \in \mathcal{U}$ and $s \in P(U)$, we have $\text{germ}_x^U(f(U)s) = \bar{\Lambda}f(\text{germ}_x^U s) = \bar{\Lambda}g(\text{germ}_x^U s) = \text{germ}_x^U(g(U)s)$. Since P and P' are locally constant, the maps of the form germ_x^U are isomorphisms, hence $f(U)s = g(U)s$ for each $s \in P(U)$. Therefore $f(U) = g(U)$ for all $U \in \mathcal{U}$. Thus $\bar{\Lambda}$ is a faithful functor.

Now assume that $h : E(P) \rightarrow E(P')$ is a covering transformation in $(\text{Cov proj } X)_{\mathcal{U}}$. Then the composite

$$U \times P(U) \xrightarrow{\varphi_U} (qP)^{-1}U \xrightarrow{h} (qP')^{-1}U \xrightarrow{(\varphi'_U)^{-1}} U \times P'(U)$$

is U -constant. Let $f(U) : P(U) \rightarrow P'(U)$ be the unique map such that $\text{id}_U \times f(U) = (\varphi'_U)^{-1}h\varphi_U$. It is easy to check that if $U, V \in \mathcal{U}$ and $U \subset V$ then $(P')_U^V f(V) = f(U)P_U^V$. Then $f(U) : P(U) \rightarrow P'(U)$, $U \in \mathcal{U}$, is a natural transformation from P to P' . The map $\bar{\Lambda}f$ has the property that for each $U \in \mathcal{U}$, the corresponding restrictions are such that $(\varphi'_U)^{-1}(\bar{\Lambda}f)\varphi_U = (\varphi'_U)^{-1}h\varphi_U$. This implies that for any U , $\bar{\Lambda}f|(qP)^{-1}U = h|(qP')^{-1}U$, and so that $\bar{\Lambda}f = h$. Thus we have shown that $\bar{\Lambda}$ is a full functor.

In order to check that $\bar{\Lambda}$ is an equivalence of categories, it suffices to prove that if $\Phi = (p : E \rightarrow X, [\mathcal{A}])$ is a covering projection in $(\text{Cov proj } X)_{\mathcal{U}}$, then there is a locally constant presheaf $F : \mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$ such that $\bar{\Lambda}F \cong \Phi$.

Suppose that $\mathcal{A} = \{\psi_U : U \times F(U) \rightarrow p^{-1}U\}$ is an atlas on \mathcal{U} . If $U' \subset U$, then

$$U' \times F(U') \xrightarrow{\psi_{U'}} p^{-1}U' \xrightarrow{\psi_U^{-1}} U' \times F(U)$$

is U' -constant. Denote by $F_{U'}^U : F(U) \rightarrow F(U')$ the unique bijective map such that $\text{id}_{U'} \times F_{U'}^U = \psi_U^{-1}\psi_{U'}$. It is easy to check that $F_{U'}^U = \text{id}_{F(U)}$ and if $U'' \subset U' \subset U$, then $F_{U''}^{U'} F_{U'}^U = F_{U''}^U$. Therefore F is a locally constant functor from \mathcal{U}^{op} to \mathbf{Sets} . Now we can consider the map $h : \bar{\Lambda}F \rightarrow \Phi$, where $h : E(\bar{\Lambda}F) \rightarrow E(\Phi)$ is defined by $h(\text{germ}_x^U s) = \psi_U(x, s)$ for $x \in U$ and $s \in F(U)$. We note that if $U' \subset U$ and $\text{germ}_x^U s = \text{germ}_x^{U'} s'$, then $s' = F_{U'}^U s$ and $\psi_U(x, s) = \psi_{U'}(x, F_{U'}^U s)$. Therefore h is well defined. We also see that the maps

$$U \times F(U) \xrightarrow{\varphi_U} (qF)^{-1}U \xrightarrow{h} p^{-1}U \xrightarrow{\psi_U^{-1}} U \times F(U)$$

satisfy $h\varphi_U(x, s) = h(\text{germ}_x^U s) = \psi_U(x, s)$. Then $\psi_U^{-1}h\varphi_U = \text{id}_U \times \text{id}_{F(U)}$ is U -constant and $h|(qF)^{-1}U$ is an isomorphism. Thus $h : \bar{\Lambda}F \rightarrow \Phi$ is an isomorphism in $(\text{Cov proj } X)_{\mathcal{U}}$. Therefore $\bar{\Lambda}$ is an equivalence of categories.

Given a covering reduced sieve \mathcal{U} , if we take the class Σ of all morphisms in \mathcal{U} , we have the corresponding category of fractions $\pi\mathcal{U} = \mathcal{U}[\Sigma^{-1}]$ which is a groupoid. We note the existence of natural isomorphisms $(\pi\mathcal{U})^{\text{op}} \cong \pi(\mathcal{U}^{\text{op}})$.

Therefore we use the notation $\pi\mathcal{U}^{\text{op}}$. For locally constant presheaves on \mathcal{U} , one has:

LEMMA 2.1. *Given a covering reduced sieve \mathcal{U} on X , the category $(\mathbf{Sets}^{\mathcal{U}^{\text{op}}})_{\text{lc}}$ of locally constant presheaves on \mathcal{U} is equivalent to the functor category $\mathbf{Sets}^{\pi\mathcal{U}^{\text{op}}}$.*

PROOF. Denote by $\gamma : \mathcal{U}^{\text{op}} \rightarrow \pi\mathcal{U}^{\text{op}}$ the projection functor. If $P : \mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$ is locally constant, then P carries each arrow of \mathcal{U}^{op} into an isomorphism. Therefore, P factors through $\pi\mathcal{U}^{\text{op}}$ as $P = \bar{P}\gamma$. Conversely, if $F : \pi\mathcal{U}^{\text{op}} \rightarrow \mathbf{Sets}$ is a functor, then because $\pi\mathcal{U}^{\text{op}}$ is a groupoid, F carries every arrow of $\pi\mathcal{U}^{\text{op}}$ into an isomorphism. Thus $F\gamma$ is a locally constant functor.

We recall that if \mathcal{U} and \mathcal{V} are open coverings of a space X , then we say that \mathcal{U} *refines* \mathcal{V} , $\mathcal{U} \geq \mathcal{V}$, if for every $U \in \mathcal{U}$, there is $V \in \mathcal{V}$ such that $U \subset V$. We note that for a given U , in general it is possible to find various $V \in \mathcal{V}$ such that $U \subset V$. It would be interesting to have a canonical way of finding a V for each U . We solve this problem if we work only with covering reduced sieves. We note that if \mathcal{U} and \mathcal{V} are two covering reduced sieves then \mathcal{U} refines \mathcal{V} if and only if $\mathcal{U} \subset \mathcal{V}$. If $U \in \mathcal{U}$, then there is $V \in \mathcal{V}$ such that $U \subset V$, but this implies that $U \in \mathcal{V}$. If $\mathcal{U} \subset \mathcal{V}$, then there is an induced functor $\pi_{\mathcal{V}}^{\mathcal{U}} : \pi\mathcal{U}^{\text{op}} \rightarrow \pi\mathcal{V}^{\text{op}}$ that again induces a functor $\mathbf{Sets}^{\pi\mathcal{V}^{\text{op}}} \rightarrow \mathbf{Sets}^{\pi\mathcal{U}^{\text{op}}}$.

Using the equivalence of categories $\mathbf{Sets}^{\pi\mathcal{U}^{\text{op}}} \xrightarrow{\gamma^*} (\mathbf{Sets}^{\mathcal{U}^{\text{op}}})_{\text{lc}}$ we have a new equivalence Λ' obtained as the composite $\Lambda' = \bar{\Lambda}\gamma^*$:

$$\mathbf{Sets}^{\pi\mathcal{U}^{\text{op}}} \xrightarrow{\gamma^*} (\mathbf{Sets}^{\mathcal{U}^{\text{op}}})_{\text{lc}} \xrightarrow{\bar{\Lambda}} (\text{Cov proj } X)_{\mathcal{U}}.$$

If \mathcal{U} refines \mathcal{V} one has the following:

PROPOSITION 2.3. *Let \mathcal{U} and \mathcal{V} be two covering reduced sieves on X . If $\mathcal{U} \subset \mathcal{V}$, then the functor diagram*

$$\begin{array}{ccc} \mathbf{Sets}^{\pi\mathcal{V}^{\text{op}}} & \xrightarrow{\Lambda'} & (\text{Cov proj } X)_{\mathcal{V}} \\ \downarrow & & \downarrow \\ \mathbf{Sets}^{\pi\mathcal{U}^{\text{op}}} & \xrightarrow{\Lambda'} & (\text{Cov proj } X)_{\mathcal{U}} \end{array}$$

is commutative up to natural isomorphism.

PROOF. If F is an object in $\mathbf{Sets}^{\pi\mathcal{V}^{\text{op}}}$, we have the presheaf $P = \gamma F$, which satisfies

$$\text{colim}_{x \in \mathcal{V} \in \mathcal{V}} P(V) \cong \text{colim}_{x \in \mathcal{U} \in \mathcal{U}} P(U).$$

This fact easily gives the existence of an isomorphism of functors in the diagram above.

Given a space X , we denote by $\text{COV}(X)$ the set of open coverings \mathcal{U} of X directed by refinement. We denote by $\text{CRS}(X)$ the set of covering reduce sieves of X directed by refinement or equivalently by “inclusion”; that is, if $\mathcal{U}, \mathcal{V} \in \text{CRS}(X)$ then $\mathcal{U} \geq \mathcal{V}$ if and only if $\mathcal{U} \subset \mathcal{V}$. Recall that Gpd denotes the category of groupoids. Using the directed set $\text{CRS}(X)$ as an indexing category we can consider the pro-groupoid

$$\pi \text{ crs}(X) : \text{CRS}(X) \rightarrow \text{Gpd}$$

defined by $\pi \text{ crs}(X)(\mathcal{U}) = \pi \mathcal{U}^{\text{op}}$. Associated with the pro-groupoid $\pi \text{ crs}(X)$ we have the category $\text{pro}(\pi \text{ crs}(X), \text{Sets})$ defined in §1. The main result of this section is the following:

THEOREM 2.2. *The category $\text{Cov proj } X$ of covering projections and transformations of a topological space X is equivalent to the category $\text{pro}(\pi \text{ crs}(X), \text{Sets})$.*

P r o o f. For the groupoid $\pi \text{ crs}(X)$, consider the category $(\pi \text{ crs}(X), \text{Sets})$ defined in §1. Now we are going to define a functor

$$A' : (\pi \text{ crs}(X), \text{Sets}) \rightarrow \text{Cov proj } X.$$

Suppose that $(\pi \mathcal{U}^{\text{op}}, F)$ and $(\pi \mathcal{V}^{\text{op}}, G)$ are objects in $(\pi \text{ crs}(X), \text{Sets})$ and $\alpha = (\mathcal{U} \subset \mathcal{V}, \theta_\alpha : F \rightarrow G\pi_{\mathcal{V}}^{\mathcal{U}})$ is a morphism in $(\pi \text{ crs}(X), \text{Sets})$. The functor A' carries $\alpha : (\pi \mathcal{U}^{\text{op}}, F) \rightarrow (\pi \mathcal{V}^{\text{op}}, G)$ to $A'\alpha : A'(\pi \mathcal{U}^{\text{op}}, F) \rightarrow A'(\pi \mathcal{V}^{\text{op}}, G)$, where $A'(\pi \mathcal{U}^{\text{op}}, F) = \bar{A}(F\gamma)$, $A'(\pi \mathcal{V}^{\text{op}}, G) = \bar{A}(G\gamma)$ and if $s \in F\gamma(U)$ and $x \in U \in \mathcal{U}$, then $A'\alpha(\text{germ}_x^U s) = \text{germ}_x^U(\theta_\alpha * \gamma(U)(s))$.

It is easy to check that if α is in Σ , then $A'\alpha$ is an isomorphism, hence there is an induced functor

$$A' : \text{pro}(\pi \text{ crs}(X), \text{Sets}) \rightarrow \text{Cov proj } X.$$

We note that if $\mathcal{U}, \mathcal{V} \in \text{CRS}(X)$, then $\mathcal{U} \cap \mathcal{V} = \{W \mid W \in \mathcal{U} \text{ and } W \in \mathcal{V}\} \in \text{CRS}(X)$. We also see that the inclusion $\text{CRS}(X) \rightarrow \text{COV}(X)$ of directed sets is cofinal.

If $(\pi \mathcal{U}^{\text{op}}, F)$ and $(\pi \mathcal{V}^{\text{op}}, G)$ are objects in $\text{pro}(\pi \text{ crs}(X), \text{Sets})$, then

$$\begin{aligned} & \text{Hom}_{\text{pro}(\pi \text{ crs}(X), \text{Sets})}((\pi \mathcal{U}^{\text{op}}, F), (\pi \mathcal{V}^{\text{op}}, G)) \\ & \cong \text{colim}_{\mathcal{W} \in \text{CRS}(X), \mathcal{W} \subset \mathcal{U} \cap \mathcal{V}} \text{Hom}_{\text{Sets}^{\pi \mathcal{W}^{\text{op}}}}(F\pi_{\mathcal{U}}^{\mathcal{W}}, G\pi_{\mathcal{V}}^{\mathcal{W}}) \\ & \cong \text{colim}_{\mathcal{W} \in \text{CRS}(X), \mathcal{W} \subset \mathcal{U} \cap \mathcal{V}} \text{Hom}_{(\text{Cov proj } X)_{\mathcal{W}}} (A'(F\pi_{\mathcal{U}}^{\mathcal{W}}), A'(G\pi_{\mathcal{V}}^{\mathcal{W}})) \\ & \cong \text{colim}_{\mathcal{W} \in \text{CRS}(X), \mathcal{W} \subset \mathcal{U} \cap \mathcal{V}} \text{Hom}_{(\text{Cov proj } X)_{\mathcal{W}}} (A'F, A'G) \\ & \cong \text{Hom}_{\text{Cov proj } X} (A'F, A'G). \end{aligned}$$

Thus we have shown that $A' : \text{pro}(\pi \text{ crs}(X), \text{Sets}) \rightarrow \text{Cov proj } X$ is a full faithful functor.

On the other hand, if $\Phi = (p : E \rightarrow X, [\mathcal{A}])$ is an object in $\text{Cov proj } X$ and \mathcal{A} is an atlas on a covering reduced sieve \mathcal{U} , then by Theorem 2.1 and Lemma 2.1, there is $F : \pi\mathcal{U}^{\text{op}} \rightarrow \text{Sets}$ such that $\Lambda'(\pi\mathcal{U}^{\text{op}}, F) \cong \Phi$. Therefore $\Lambda' : \text{pro}(\pi \text{crs}(X), \text{Sets}) \rightarrow \text{Cov proj } X$ is an equivalence of categories.

3. The subdivision of the Čech nerve. The Čech nerve $CX(\mathcal{U})$ of an open covering \mathcal{U} of X is defined to be the simplicial set whose q -simplexes are given by

$$CX(\mathcal{U})_q = \{(U_0, \dots, U_q) \mid U_0, \dots, U_q \in \mathcal{U}, U_0 \cap \dots \cap U_q \neq \emptyset\}.$$

The face and degeneracy operators are defined in the usual way. We note that if $\mathcal{U} \geq \mathcal{V}$, then we can choose a map $\varphi : \mathcal{U} \rightarrow \mathcal{V}$ such that $U \subset \varphi U$. This induces a simplicial map $(U_0, \dots, U_q) \rightarrow (\varphi U_0, \dots, \varphi U_q)$, which is denoted by $C\varphi : CX(\mathcal{U}) \rightarrow CX(\mathcal{V})$, and we also have the corresponding realization $|C\varphi| : |CX(\mathcal{U})| \rightarrow |CX(\mathcal{V})|$. If we choose a different map $\psi : \mathcal{U} \rightarrow \mathcal{V}$ such that $U \subset \psi U$, we have

$$\emptyset \neq U_0 \cap \dots \cap U_q \subset \varphi U_0 \cap \dots \cap \varphi U_q \cap \psi U_0 \cap \dots \cap \psi U_q,$$

hence there is a simplex $(\varphi U_0, \dots, \varphi U_q, \psi U_0, \dots, \psi U_q)$ having as faces $(\varphi U_0, \dots, \varphi U_q)$ and $(\psi U_0, \dots, \psi U_q)$. Then $|C\varphi|$ is contiguous to $|C\psi|$. Therefore $|C\varphi|$ is homotopic to $|C\psi|$, and we have $C\varphi = C\psi$ in the category $\text{Ho}(\text{SS})$ obtained by inverting the weak equivalences of the category of simplicial sets. As a consequence of these facts, we have a functor

$$CX : \text{COV}(X) \rightarrow \text{Ho}(\text{SS}).$$

It is interesting to observe that if \mathcal{U} and \mathcal{V} are covering reduced sieves and $\mathcal{U} \geq \mathcal{V}$, then there is a canonical map $\mathcal{U} \subset \mathcal{V}$. In this case we have an induced map $CX(\mathcal{U}) \rightarrow CX(\mathcal{V})$ and then we obtain a functor

$$CX : \text{CRS}(X) \rightarrow \text{SS}.$$

Let Cat denote the category of small categories and functors. There is a functor $l : \Delta \rightarrow \text{Cat}$ which carries an ordered set $[p] = \{0 \leq 1 \leq \dots \leq p\}$ to the small category $l[p] = \{0 \leftarrow 1 \leftarrow \dots \leftarrow p\}$. On the other hand, we consider the Yoneda embedding $y : \Delta \rightarrow \text{SS}$, $[p] \rightarrow \Delta[p] = \text{Hom}_\Delta(-, [p])$. Since Cat is a cocomplete category, we can apply [M-M, Th. I.5.2] to obtain a pair of functors $L : \text{SS} \rightarrow \text{Cat}$ and $\text{Ner} : \text{Cat} \rightarrow \text{SS}$ such that L is left adjoint to Ner and the diagram

$$\begin{array}{ccc} \Delta & & \\ \downarrow l & \searrow y & \\ \text{Cat} & \xleftarrow{L} & \text{SS} \end{array}$$

is commutative up to isomorphism.

Given a small category C , $\text{Ner } C$ is called the *nerve* of C ; we note that $\text{Ner } C_0 \cong C_0 \cong \text{Objects of } C$ and for $q > 0$,

$$(\text{Ner } C)_q = \{(f_0, \dots, f_{q-1}) \mid f_i \text{ is an arrow of } C \text{ and} \\ \text{domain}(f_i) = \text{codomain}(f_{i+1})\}.$$

If C is a category and X is an object of C , we denote by $C \downarrow X$ the category which has as objects those morphisms u of C such that $\text{codomain}(u) = X$. A morphism from $u : A \rightarrow X$ to $v : B \rightarrow X$ is a morphism $f : A \rightarrow B$ in C such that $vf = u$. We note that a morphism $g : X \rightarrow X'$ induces a functor $C \downarrow g : C \downarrow X \rightarrow C \downarrow X'$ defined by $C \downarrow g(u) = gu$.

In particular, for the category Δ , we have the functor $\Delta \downarrow - : \Delta \rightarrow \text{Cat}$, $[q] \rightarrow \Delta \downarrow [q]$, and we can consider the composite

$$\text{sd} = \text{Ner}(\Delta \downarrow -) : \Delta \rightarrow \text{Cat} \rightarrow \text{SS}.$$

Now we can apply [M-M, Th. I.5.2] to the functors $\text{sd} : \Delta \rightarrow \text{Cat}$ and $y : \Delta \rightarrow \text{SS}$ to obtain a pair of adjoint functors $\text{Sd} : \text{SS} \rightarrow \text{SS}$, $\text{Ex} : \text{SS} \rightarrow \text{SS}$ such that the diagram

$$\begin{array}{ccc} \Delta & & \\ \text{sd} \downarrow & \searrow y & \\ \text{SS} & \xleftarrow{\text{Sd}} & \text{SS} \end{array}$$

is commutative up to isomorphism. The left adjoint $\text{Sd} : \text{SS} \rightarrow \text{SS}$ is called the *subdivision functor*.

For each space X , we consider the functor $NX : \text{CRS}(X) \rightarrow \text{SS}$ defined by $NX(\mathcal{U}) = \text{Ner}(\mathcal{U}^{\text{op}})$, where \mathcal{U}^{op} is the opposite category of \mathcal{U} considered as a small category. A typical q -simplex of $NX(\mathcal{U})_q$ is of the form $U_0 \subset U_1 \subset \dots \subset U_q$ with $U_0, \dots, U_q \in \mathcal{U}$.

Next we prove that the subdivision $\text{Sd } CX$ of the Čech nerve is weakly equivalent to NX .

THEOREM 3.1. *There is a natural transformation ψ from the functor $\text{Sd } CX : \text{CRS}(X) \rightarrow \text{SS}$ to the functor $NX : \text{CRS}(X) \rightarrow \text{SS}$ such that for each \mathcal{U} in $\text{CRS}(X)$, $\psi(\mathcal{U}) : \text{Sd } CX(\mathcal{U}) \rightarrow NX(\mathcal{U})$ is a weak equivalence.*

Proof. Using the notation of [M-M] (see also §0), we recall that for an object \mathcal{U} of $\text{CRS}(X)$,

$$\text{Sd } CX(\mathcal{U}) = \text{colim}_{\Delta} \left(\int_{\Delta} CX(\mathcal{U}) \xrightarrow{\text{pr}_1} \Delta \xrightarrow{\text{sd}} \text{SS} \right).$$

Given an object $([n], (V_0, \dots, V_n))$ in $\int_{\Delta} CX(\mathcal{U})$, we have $\text{sd } \text{pr}_1([n], (V_0, \dots, V_n)) = \text{sd}[n]$. An element of $(\text{sd } \text{pr}_1([n], (V_0, \dots, V_n)))_q$ is determined by

a diagram in Δ of the form

$$\begin{array}{c} [p_0] \xleftarrow{\alpha_0} [p_1] \xleftarrow{\alpha_1} \dots [p_{q-1}] \xleftarrow{\alpha_{q-1}} [p_q] \\ \alpha \downarrow \\ [n] \end{array}$$

and it will be denoted by $\alpha(\alpha_0, \dots, \alpha_{q-1})$.

For each $([n], (V_0, \dots, V_n))$ we have the map

$$\psi([n], (V_0, \dots, V_n))_q : (\text{sd pr}_1([n], (V_0, \dots, V_n)))_q \rightarrow NX(\mathcal{U})_q$$

defined by

$$\begin{aligned} \psi([n], (V_0, \dots, V_n))_q & \alpha(\alpha_0, \dots, \alpha_{q-1}) \\ & = (V_{\alpha(0)} \cap \dots \cap V_{\alpha(p_0)} \subset V_{\alpha\alpha_0(0)} \cap \dots \cap V_{\alpha\alpha_0(p_1)} \subset \dots \\ & \quad \subset V_{\alpha\alpha_0 \dots \alpha_{q-1}(0)} \cap \dots \cap V_{\alpha\alpha_0 \dots \alpha_{q-1}(p_q)}). \end{aligned}$$

These maps induce a map

$$\psi(\mathcal{U}) : \text{Sd } CX(\mathcal{U}) \rightarrow NX(\mathcal{U}).$$

Now we define a transformation $\varphi(\mathcal{U}) : NX(\mathcal{U}) \rightarrow \text{Sd } CX(\mathcal{U})$. Given an element $U_0 \subset \dots \subset U_q$ of $NX(\mathcal{U})_q$, $([q], (U_0, \dots, U_q))$ is an object of $\int_{\Delta} CX(\mathcal{U})$. Consider the element $\beta(\beta_0, \dots, \beta_{q-1})$ of $(\text{sd pr}_1([q], (U_0, \dots, U_q)))_q$ determined by the diagram

$$\begin{array}{c} [q] \xleftarrow{\beta_0} [q-1] \xleftarrow{\beta_1} [q-2] \xleftarrow{\dots} [1] \xleftarrow{\beta_{q-1}} [0] \\ \beta \downarrow \\ [q] \end{array}$$

where $\beta = \text{id}_{[q]}$ and $\beta_i : [q-i-1] \rightarrow [q-i]$ is defined by $\beta_i(j) = j+1$ for $0 \leq j \leq q-i-1$. The element $\beta(\beta_0, \dots, \beta_{q-1})$ represents an element $[\beta(\beta_0, \dots, \beta_{q-1})]$ of $\text{Sd } CX(\mathcal{U})_q$. We define $\varphi(\mathcal{U})_q : NX(\mathcal{U})_q \rightarrow \text{Sd } CX(\mathcal{U})_q$ by

$$\varphi(\mathcal{U})_q (U_0 \subset \dots \subset U_q) = [\beta(\beta_0, \dots, \beta_{q-1})].$$

Because

$$\psi([q], (U_0, \dots, U_q))_q(\beta(\beta_0, \dots, \beta_{q-1})) = (U_0 \subset U_1 \subset \dots \subset U_q)$$

we have $\psi(\mathcal{U})\varphi(\mathcal{U}) = \text{id}_{NX(\mathcal{U})}$.

To see that $\psi(\mathcal{U})$ is a weak equivalence, it is sufficient to show that $|\varphi(\mathcal{U})| |\psi(\mathcal{U})|$ is homotopic to the identity. We note the following facts:

If we consider the realization functor $|-| : \mathbf{SS} \rightarrow \mathbf{Top}$, we find that for every object Y in \mathbf{SS} , $|\text{sd } Y|$ is homeomorphic to $|Y|$, and if y is an n -simplex

of Y and $\alpha(\alpha_0, \dots, \alpha_{q-1})$ is a q -simplex of $(\text{sd pr}_1([n], y))_q$ which represents a q -simplex $[\alpha(\alpha_0, \dots, \alpha_{q-1})]$ of $\text{sd } Y$, then $[[\alpha(\alpha_0, \dots, \alpha_{q-1})]] \subset |y|$, where $[[\alpha(\alpha_0, \dots, \alpha_{q-1})]]$ and $|y|$ denote the realizations of the corresponding simplexes.

For $Y = CX(\mathcal{U})$, if $[\alpha(\alpha_0, \dots, \alpha_{q-1})]$ is a q -simplex of $\text{Sd } CX(\mathcal{U})$ represented by the q -simplex $\alpha(\alpha_0, \dots, \alpha_{q-1})$ of $\text{sd pr}_1([n], (V_0, \dots, V_n))$, then we have

$$\begin{aligned} \psi(\mathcal{U})[\alpha(\alpha_0, \dots, \alpha_{q-1})] &= (U_0 \subset \dots \subset U_q), \\ \varphi(\mathcal{U})(U_0 \subset \dots \subset U_q) &= [\beta(\beta_0, \dots, \beta_{q-1})]. \end{aligned}$$

Consider the $(n+q+2)$ -simplex $(V_0, \dots, V_n, U_0, \dots, U_q)$ of $CX(\mathcal{U})$. The realizations $[[\alpha(\alpha_0, \dots, \alpha_{q-1})]]$ and $[[\beta(\beta_0, \dots, \beta_{q-1})]]$ are contained in $|(V_0, \dots, V_n, U_0, \dots, U_q)|$. This implies that $|\varphi(\mathcal{U})| |\psi(\mathcal{U})|$ is contiguous to $\text{id}_{|\text{Sd } CX(\mathcal{U})|}$ with respect to the simplicial decomposition given by $|CX(\mathcal{U})|$ (recall that $|CX(\mathcal{U})|$ is homeomorphic to $|\text{Sd } CX(\mathcal{U})|$). Therefore $|\varphi(\mathcal{U})| |\psi(\mathcal{U})|$ is homotopic to $\text{id}_{|\text{Sd } CX(\mathcal{U})|}$.

Finally, we note that if $\mathcal{U}, \mathcal{V} \in \text{CRS}(X)$ and $\mathcal{U} \subset \mathcal{V}$, then the diagram

$$\begin{array}{ccc} \text{Sd } CX(\mathcal{U}) & \xrightarrow{\psi(\mathcal{U})} & NX(\mathcal{U}) \\ \downarrow & & \downarrow \\ \text{Sd } CX(\mathcal{V}) & \xrightarrow{\psi(\mathcal{V})} & NX(\mathcal{V}) \end{array}$$

is commutative. Therefore $\psi : \text{Sd } CX \rightarrow NX$ is a natural transformation.

LEMMA 3.1. *Let \mathcal{C} be a small category. Then the fundamental groupoid $\pi \text{Ner } \mathcal{C}$ is isomorphic to the category of fractions $\mathcal{C}[\Sigma^{-1}]$, where Σ is the set of all morphisms of \mathcal{C} .*

Proof. The arrow of $\pi \text{Ner } \mathcal{C}$ represented by an arrow $A_0 \leftarrow A_1$ in \mathcal{C} is carried by the equivalence functor to the arrow of $\mathcal{C}[\Sigma^{-1}]$ induced by $A_0 \leftarrow A_1$. For details we refer the reader to [Go].

REMARK. For a given covering reduced sieve \mathcal{U} , the groupoid $\pi N\mathcal{U}^{\text{op}}$ is isomorphic to the category of fractions $\pi\mathcal{U}^{\text{op}}$.

COROLLARY 3.1. *Given a space X , the category $\text{Cov proj } X$ of covering projections of X is equivalent to the category $\text{pro}(\pi CX, \text{Sets})$.*

Proof. For each $\mathcal{U} \in \text{CRS}(X)$,

$$\psi(\mathcal{U}) : \text{Sd } CX(\mathcal{U}) \rightarrow NX(\mathcal{U})$$

is a weak equivalence. Since $|CX(\mathcal{U})|$ is homeomorphic to $|\text{Sd } CX(\mathcal{U})|$, we

also have the weak equivalences

$$CX(\mathcal{U}) \rightarrow \text{Sing} |CX(\mathcal{U})| \rightarrow \text{Sing} |\text{Sd} CX(\mathcal{U})| \leftarrow \text{Sd} CX(\mathcal{U}).$$

Therefore, one has the zig-zag weak equivalences of pro-groupoids

$$\pi CX(\mathcal{U}) \rightarrow \pi \text{Sing} |CX(\mathcal{U})| \rightarrow \pi \text{Sing} |\text{Sd} CX(\mathcal{U})| \leftarrow \pi \text{Sd} CX(\mathcal{U}).$$

Now, by Lemma 2.1, we conclude that $\text{pro}(\pi NX, \text{Sets})$ is equivalent to $\text{pro}(\pi CX, \text{Sets})$.

4. Covering projections for locally path-connected and semilocally 1-connected spaces. In this section we prove that under “good” local conditions of connectedness the pro-groupoid $\pi \text{crs}(X)$ is equivalent to the standard fundamental groupoid πX . In this way we obtain the well known equivalence of the category of covering projections of a “good” space and the functor category $\text{Sets}^{\pi X}$ (see [God]).

Given a space X , we denote by πX the fundamental groupoid of X . The groupoid πX has as objects the points of X and for $x_0, x_1 \in X$ an arrow from x_0 to x_1 is represented by a path from x_0 to x_1 (up to relative homotopy). If α is an arrow from x_0 to x_1 and β is an arrow from x_1 to x_2 the composite will be denoted by $\beta\alpha$. Let S be a subspace of X ; then the inclusion $S \subset X$ induces a canonical groupoid homomorphism $\pi S \rightarrow \pi X$. We say that $\pi S \rightarrow \pi X$ is *trivial* if any arrow α of πS such that $\text{domain}(\alpha) = \text{codomain}(\alpha) = s \in S$ is carried to the identity arrow by the functor $\pi S \rightarrow \pi X$.

We will consider open coverings \mathcal{U} of X such that

- (1) if $U \in \mathcal{U}$, then $U \neq \emptyset$, U is path-connected and $\pi U \rightarrow \pi X$ is trivial,
- (2) if V is a non-empty open subset of X such that V is path-connected, $\pi V \rightarrow \pi X$ is trivial and there is $U \in \mathcal{U}$ such that $V \subset U$, then $V \in \mathcal{U}$.

An open covering \mathcal{U} of X which satisfies conditions (1) and (2) is said to be a *trivial covering* of X . We denote by $\text{TCOV}(X)$ the family of all trivial coverings of X directed by refinement. We note that if $\mathcal{U}, \mathcal{V} \in \text{TCOV}(X)$, then $\mathcal{U} \geq \mathcal{V}$ if and only if $\mathcal{U} \subset \mathcal{V}$.

LEMMA 4.1. *Let X be a locally path-connected and semilocally 1-connected space. Then the map ${}^*s : \text{TCOV}(X) \rightarrow \text{CRS}(X)$ defined by ${}^*s\mathcal{U} = \{V \mid V \neq \emptyset, V \text{ is open, } \exists U \in \mathcal{U} \text{ such that } V \subset U\}$ is cofinal.*

PROOF. Let \mathcal{V} be a covering reduced sieve on X . For each $V \in \mathcal{V}$ and $x \in V$ there is an open subset $U(V, x)$ such that $x \in U(V, x) \subset V$, $U(V, x)$ is path-connected and $\pi U(V, x) \rightarrow \pi X$ is trivial. Let $\mathcal{U} = \{U \in \mathcal{O}(X) \mid U \neq \emptyset, \exists U(V, x) \text{ such that } U \subset U(V, x), U \text{ is path-connected and } \pi U \rightarrow \pi X \text{ is trivial}\}$. Since X is locally connected and semilocally 1-connected, it follows that $\mathcal{U} \in \text{TCOV}(X)$, and from the definition of \mathcal{U} we have ${}^*s\mathcal{U} \subset \mathcal{V}$.

COROLLARY 4.1. *Let X be a locally path-connected and semilocally 1-connected space. Then the category $\text{Cov proj } X$ is equivalent to the category*

$$\text{pro}(\{\pi(*s\mathcal{U}^{\text{op}}) \mid \mathcal{U} \in \text{TCOV}(X)\}, \text{Sets}).$$

PROOF. By Lemma 4.1, $\pi \text{ crs}(X) = \{\pi\mathcal{U}^{\text{op}} \mid \mathcal{U} \in \text{CRS}(X)\}$ is isomorphic to $\{\pi(*s\mathcal{U}^{\text{op}}) \mid \mathcal{U} \in \text{TCOV}(X)\}$ in pro Gpd . By Lemma 1.1, $\text{pro}(\pi \text{ crs}(X), \text{Sets})$ is equivalent to $\text{pro}(\{\pi(*s\mathcal{U}^{\text{op}}) \mid \mathcal{U} \in \text{TCOV}(X)\}, \text{Sets})$. Now from Theorem 2.2, it follows that $\text{Cov proj } X$ is equivalent to $\text{pro}(\{\pi(*s\mathcal{U}^{\text{op}}) \mid \mathcal{U} \in \text{TCOV}(X)\})$.

If a space X satisfies the usual connectedness conditions of this section, we can consider the following ‘‘maximal’’ open covering in $\text{TCOV}(X)$:

$$\mathcal{U}_0 = \{U \in \mathcal{O}(X) \mid U \neq \emptyset, U \text{ is path-connected and } \pi U \rightarrow \pi X \text{ is trivial}\}.$$

Now we can take a map $\eta : *s\mathcal{U}_0 \rightarrow X$ such that $\eta(U) = x_U \in U \in *s\mathcal{U}_0$. If $U, V \in \mathcal{U}_0$ and $U \subset V$, we can take in V a path α from x_V to x_U . We note that a different path α' determines the same arrow in πX . It is routine to check that this construction gives an equivalence $\eta : \pi(*s\mathcal{U}_0^{\text{op}}) \rightarrow \pi X$. If $\mathcal{U} \in \text{TCOV}(X)$, the inclusion $\mathcal{U} \subset \mathcal{U}_0$ induces the composite $\pi(*s\mathcal{U}^{\text{op}}) \rightarrow \pi(*s\mathcal{U}_0^{\text{op}}) \xrightarrow{\eta} \pi X$, which is also an equivalence of groupoids. Since the pro-groupoid $\{\pi(*s\mathcal{U}^{\text{op}}) \mid \mathcal{U} \in \text{TCOV}(X)\}$ and the constant pro-groupoid πX satisfy the conditions of Lemma 1.2, we deduce that $\text{pro}(\{\pi(*s\mathcal{U}^{\text{op}}) \mid \mathcal{U} \in \text{TCOV}(X)\}, \text{Sets})$ is equivalent to $\text{pro}(\pi X, \text{Sets})$. Therefore one has:

THEOREM 4.1. *Let X be a locally path-connected and semilocally 1-connected space. Then the category of covering projections of X is equivalent to $\text{Sets}^{\pi X}$, the category of functors from πX to Sets .*

5. Covering projections of compact metrisable spaces. The objective of this section is to reduce the pro-groupoid G of the category $\text{pro}(G, \text{Sets})$ to a pro-group, a tower of groups or even a pro-discrete topological group. For this purpose, we use compact metrisable spaces in order to obtain tower of groupoids, connectedness conditions to have a tower of groups, and finally pointed movability conditions to have a surjective tower of groups or a topological group.

In order to reduce a pro-groupoid to a tower of groupoids, we suppose that X is a compact metrisable space. Using the Lebesgue Lemma, we can construct a sequence

$$\dots \geq \mathcal{V}_{n+1} \geq \mathcal{V}_n \geq \dots \geq \mathcal{V}_0$$

of open coverings which is cofinal in $\text{COV}(X)$. This implies that

$$\dots \subseteq *s\mathcal{V}_{n+1} \subseteq *s\mathcal{V}_n \subseteq \dots \subseteq *s\mathcal{V}_0$$

is cofinal in $\text{CRS}(X)$. Therefore:

- (1) πCX is isomorphic to $\{\pi CX(*s\mathcal{V}_n)\}$ in pro Gpd ,
- (2) $\{\pi CX(*s\mathcal{V}_n)\}$ is isomorphic to $\{\pi CX(\mathcal{V}_n)\}$ in $\text{tow } \pi_0\text{Gpd}$.

From Lemma 1.1, Proposition 1.1 and Corollary 3.1, we obtain the following:

THEOREM 5.1. *Let X be a compact metrisable space and suppose that $\dots \geq \mathcal{V}_{n+1} \geq \mathcal{V}_n \geq \dots \geq \mathcal{V}_0$ is a cofinal sequence in $\text{COV}(X)$. Then the category $\text{Cov proj } X$ is equivalent to $\text{pro}(\{\pi CX(\mathcal{V}_n)\}, \text{Sets})$.*

If X is a connected compact metrisable space and \mathcal{U} is an open covering, then $|CX(\mathcal{U})|$ is 0-connected. We can check this fact as follows: For $x, y \in X$, a \mathcal{U} -path from x to y is a finite family U_0, \dots, U_m such that $x \in U_0, y \in U_m$ and $U_0 \cap U_1 \neq \emptyset, \dots, U_{m-1} \cap U_m \neq \emptyset$. If $U, U' \in \mathcal{U}$, a \mathcal{U} -path from U to U' is a finite family U_0, \dots, U_m such that $U = U_0, U' = U_m$ and $U_0 \cap U_1 \neq \emptyset, \dots, U_{m-1} \cap U_m \neq \emptyset$. For $x_0 \in X$ it is easy to check that $C_0 = \{x \mid \text{there is a } \mathcal{U}\text{-path from } x_0 \text{ to } x\}$ is open and closed in X , and since X is connected it follows that $C_0 = X$. Now for a given $U_0 \in \mathcal{U}$ if U is also in \mathcal{U} we can choose $x_0 \in U_0$ and $x \in U$. Because $C_0 = X$ there is a \mathcal{U} -path V_0, \dots, V_m from x_0 to x . Therefore U_0, V_0, \dots, V_m, U is a \mathcal{U} -path from U_0 to U . This implies that the groupoid $\pi CX(\mathcal{U})$ is connected. For a connected groupoid π and an object U in π let π_1 be the group of endomorphisms of U . Then if π_1 is considered as a groupoid with one object, the inclusion $\pi_1 \rightarrow \pi$ is an equivalence of groupoids or equivalently an isomorphism in $\pi_0\text{Gpd}$.

Suppose that X is a connected compact metrisable space and we have a given point $x \in X$. Then there exists a sequence $\dots \geq \mathcal{V}_{n+1} \geq \mathcal{V}_n \geq \dots$ of open coverings; we can also assume that there are open subsets $V_n \in \mathcal{V}_n$ such that $x \in V_n \subset V_{n+1}$. Denote by $\pi_1 CX(\mathcal{V}_i, V_i)$ the fundamental group of the groupoid $\pi CX(\mathcal{V}_i)$ based at the object V_i . In this way, the sequence of pointed open coverings induced by the pointed space (X, x) determines a tower $\{\pi_1 CX(\mathcal{V}_i, V_i)\}$ of groups that will be denoted by $\pi_1 C(X, x)$. Then $\pi_1 C(X, x)$ is isomorphic to $\{\pi CX(\mathcal{V}_i)\}$ in $\text{tow } \pi_0\text{Gpd}$. From Proposition 1.1 and Theorem 5.1, we have:

THEOREM 5.2. *Let (X, x) be a pointed connected compact metrisable space. Then $\text{Cov proj } X$ is equivalent to $\text{pro}(\pi_1 C(X, x), \text{Sets})$.*

The category $G\text{-Sets}$ for G a group (see §0) can be generalised for a pro-group G as follows: Objects are morphisms $\eta : G \rightarrow \text{Aut}(F)$ in pro Gps , where F is a set and $\text{Aut}(F)$ is the group of automorphisms, which can be considered as a pro-group. A morphism $f : \eta \rightarrow \eta'$ is given by a map $\widehat{f} : F \rightarrow F'$ and a morphism $\eta_f : G \rightarrow \text{Aut}(\widehat{f})$ such that the diagram

$$\begin{array}{ccc}
 & \text{Aut}(F) & \\
 & \nearrow \eta & \uparrow \\
 G & \xrightarrow{\eta_f} & \text{Aut}(\widehat{f}) \\
 & \searrow \eta' & \downarrow \\
 & & \text{Aut}(F')
 \end{array}$$

is commutative.

Recall that in §1 for a pro-group G we have constructed the category $\text{pro}(G, \text{Sets})$ as a category of right fractions $(G, \text{Sets})\Sigma^{-1}$. An object $(G(i), F)$ in (G, Sets) is determined by a homomorphism $F : G(i) \rightarrow \text{Aut}(F(*))$, where $F(*)$ is the set obtained by the functor $F : G(i) \rightarrow \text{Sets}$ when it is applied to the unique object $*$ of the category $G(i)$ (we denote by F both the functor and the homomorphism). It is clear that F represents a unique morphism $\eta_F : G \rightarrow \text{Aut}(F(*))$, which is an object in $G\text{-Sets}$. On the other hand, a morphism $\alpha = (i \rightarrow j, \theta_\alpha : F \rightarrow HG_j^i)$ from $(G(i), F)$ to $(G(j), H)$ induces a map $(\theta_\alpha)_* : F(*) \rightarrow H(*)$ and a homomorphism $(F, HG_j^i) : G(i) \rightarrow \text{Aut}((\theta_\alpha)_*)$ which represents a morphism $\eta_\alpha : \eta_F \rightarrow \eta_H$. This functor factorizes to $\text{pro}(G, \text{Sets})$ and one can check the following result:

PROPOSITION 5.1. *For a given pro-group G the category $\text{pro}(G, \text{Sets})$ is equivalent to the category $G\text{-Sets}$.*

As a consequence of this description and Theorem 5.2, for a finite set $\{1, \dots, d\}$ one has the following version of the fundamental theorem of overlay theory proved by Fox [F1, F2].

COROLLARY 5.1. *Let X be a connected compact metrisable space X and let x be a point of X . Then the d -fold covering projections X are in bi-unique correspondence with the representations up to conjugation of the fundamental pro-group $\pi_1 C(X, x)$ in the symmetric group Σ_d of degree d .*

PROOF. Two d -fold covering projections p, p' are isomorphic in the category $\text{Cov proj } X$ if and only if the corresponding objects $\eta, \eta' : \pi_1 C(X, x) \rightarrow \Sigma_d$ are isomorphic in the category $\pi_1 C(X, x)\text{-Sets}$. If $f : \eta \rightarrow \eta'$ is an isomorphism given by $\widehat{f} \in \Sigma_d$ and $\eta_f : \pi_1 C(X, x) \rightarrow \text{Aut}(\widehat{f})$, then for some pointed open covering (\mathcal{V}_i, V_i) and each element $g \in \pi_1 CX(\mathcal{V}_i, V_i)$, we have $\eta_f(g) = (\eta(g), \eta'(g))$; that is, the following diagram commutes, where $F = \{1, \dots, d\}$:

$$\begin{array}{ccc}
 F & \xrightarrow{\eta(g)} & F \\
 \widehat{f} \downarrow & & \downarrow \widehat{f} \\
 F & \xrightarrow{\eta'(g)} & F
 \end{array}$$

This implies that the representations η and η' are conjugate. Conversely, if η and η' are conjugate, then they are isomorphic in the category $G\text{-Sets}$.

REMARK. Notice that in this corollary we have considered connected compact metrisable spaces instead of connected separable metrisable spaces of Fox's overlay theorem (see [F1]). The notion of fundamental pro-group in the corollary corresponds to the one of the fundamental tropes of the space considered by Fox. It is not hard to give a version of our main Theorem 2.2 for a connected space with a base point. In this case the pro-groupoid reduces to a pro-group and a version of Fox's theorem can be obtained for covering projections over a pointed connected space with a fibre F . That is, we can avoid the separable metrisable condition and we can consider infinite fibres.

For an object G in tow Gps , $\lim G$ can be provided with the inverse limit topology and we have the category of continuous $\lim G$ -sets defined in §0. The canonical map $\theta : \lim G \rightarrow G$ induces a functor $\theta^* : G\text{-Sets} \rightarrow \lim G\text{-Sets}$ as follows:

If $f : \eta \rightarrow \eta'$ is a morphism in $G\text{-Sets}$ represented by a map $\hat{f} : F \rightarrow F'$ and a commutative diagram

$$\begin{array}{ccc} & & \text{Aut}(F) \\ & \nearrow \eta & \uparrow \\ G & \xrightarrow{\eta_f} & \text{Aut}(\hat{f}) \\ & \searrow \eta' & \downarrow \\ & & \text{Aut}(F') \end{array}$$

we consider $\theta^*(f) = f\theta : \eta\theta \rightarrow \eta'\theta$ represented by the diagram

$$\begin{array}{ccc} & & \text{Aut}(F) \\ & \nearrow \eta\theta & \uparrow \\ \lim G & \xrightarrow{\eta_f\theta} & \text{Aut}(\hat{f}) \\ & \searrow \eta'\theta & \downarrow \\ & & \text{Aut}(F') \end{array}$$

If the bonding maps, $G(i+1) \rightarrow G(i)$, are surjective, the map $\theta : \lim G \rightarrow G$ is an epimorphism in tow Gps . Then if $\theta^*(f) = \theta^*(g)$, we have $\eta_f\theta = \eta_g\theta$. Therefore $\eta_f = \eta_g$ and this implies that $f = g$.

On the other hand, if $\eta : \lim G \rightarrow \text{Aut}(\hat{f})$ is a continuous homomorphism, since $\text{Aut}(\hat{f})$ has the discrete topology, $\eta^{-1}\{1\}$ is an open neighbourhood

of 1, and so there is $\theta(i) : \lim G \rightarrow G(i)$ such that $\text{Ker } \theta(i) \subset \eta^{-1}\{1\}$. Hence η factors as

$$\begin{array}{ccc} \lim G & \xrightarrow{\eta} & \text{Aut}(\widehat{f}) \\ & \searrow \theta & \nearrow \bar{\eta}_f \\ & & G(i) \end{array}$$

And $\bar{\eta}_f$ defines a morphism $(\bar{\eta}_f, \widehat{f}) : \eta \rightarrow \eta'$ such that $\theta^*(\bar{\eta}_f, \widehat{f}) = (\eta, \widehat{f})$. Therefore one has:

PROPOSITION 5.2. *Let G be an object in tow Gps such that the bonding morphisms of G are surjective maps. Then the category $G\text{-Sets}$ is equivalent to the category $\lim G\text{-Sets}$, where $\lim G$ is provided with the inverse limit topology.*

REMARK. The proof given above only works for surjective towers of groups. For a more general surjective pro-group G , $\lim G$ can be trivial (see [M]). In this case, Moerdijk notes that a similar result is obtained by taking the inverse limit in the category of localic groups.

In order to obtain pro-groups it is convenient to work with pointed spaces and if we want to have surjective pro-groups, it will be useful to recall some notion of pointed movability. A *pointed open covering* is a pair (\mathcal{U}, U_0) where $U_0 \in \mathcal{U}$; (\mathcal{V}, V_0) refines (\mathcal{U}, U_0) if $\mathcal{V} \geq \mathcal{U}$ and $V_0 \subset U_0$.

DEFINITION 5.2. A pointed space (X, x) is said to be *1-movable* if for every pointed open covering (\mathcal{U}, U_0) there is a finer pointed open covering $(\mathcal{V}, V_0) \geq (\mathcal{U}, U_0)$ such that for any pointed open covering $(\mathcal{W}, W_0) \geq (\mathcal{U}, U_0)$, and for any pointed map $h : (P, *) \rightarrow (|CX(\mathcal{V})|, *)$, where P is a CW -complex with $\dim P \leq 1$, there exists a pointed map $r : (P, *) \rightarrow (|CX(\mathcal{W})|, *)$ such that $|CX|_{\mathcal{U}}^{\mathcal{W}} r$ is homotopic to $|CX|_{\mathcal{U}}^{\mathcal{V}} h$ relative to the base point.

Now we refer the reader to Theorem 2 of [M-S, Ch. II, §8.1] to deduce that if (X, x) is 1-movable, then the pro-group $\pi_1 C(X, x)$ satisfies the Mittag-Leffler condition, and by Theorem 7 of [M-S, Ch. II, §6.2], $\pi_1 C(X, x)$ is isomorphic to a pro-group G whose bonding maps are surjective. Moreover, if $\pi_1 C(X, x)$ is isomorphic to a tower, then we can suppose that G is a tower.

THEOREM 5.3. *Let X be a connected compact metrisable space and let x be a point of X . Suppose that (X, x) is 1-movable and denote by $\tilde{\pi}_1(X, x)$ the Čech fundamental group of (X, x) , $\tilde{\pi}_1(X, x) = \lim \pi_1 C(X, x)$, provided with the inverse limit topology. Then the category of covering projections of X is equivalent to the category of continuous $\tilde{\pi}_1(X, x)$ -sets.*

Proof. This follows from Theorem 5.2 and Propositions 5.1 and 5.2.

REMARKS. (1) Let X be a pointed connected, compact and metrisable space and let $\pi_1 CX$ denote its fundamental pro-group. For a set F , the Brown \mathcal{P} functor (see [H]) carries a morphism $\eta : \pi_1 CX \rightarrow \text{Aut } F$ to a morphism of the form $\mathcal{P}\eta : \mathcal{P}\pi_1 CX \rightarrow \mathcal{P}\text{Aut } F$ and then we have the composite $\mathcal{P}\pi_1 CX \rightarrow \mathcal{P}\text{Aut } F \rightarrow \text{Aut } \mathcal{P}F$. Since $\mathcal{P}\pi_1 CX$ is isomorphic to the Quigley inward group ${}^I\pi_1^Q(X)$, the category of covering projections of X with fibre F is also equivalent to a category of “distinguished” representations of the Quigley inward group ${}^I\pi_1^Q(X)$ with “fibre” $\mathcal{P}F$. In [H], we have constructed a space $\overline{\mathcal{P}}^R CX$ whose fundamental group is isomorphic to the Quigley inward group ${}^I\pi_1^Q(X)$. Thus the category of covering projections of X with fibre F is equivalent to a category of “distinguished” covering projections of $\overline{\mathcal{P}}^R CX$ with fibre $\mathcal{P}F$.

(2) If the pointed space X of remark (1) is also 2-movable (see [M-S]) then the fundamental group of the homotopy limit $\lim^R CX$ is isomorphic to the Čech fundamental group of X . Therefore the category of covering projections of X is equivalent to a category of “distinguished” covering projections of $\lim^R CX$.

(3) Some additional results about classification of principal G -bundles of a space X can be obtained in terms of morphisms of the form $\pi CX \rightarrow G$, or $\lim \pi CX \rightarrow G$, even in the case where G is a pro-discrete group. This is also connected with the first cohomology set of a pro-groupoid or a topological group with coefficients in a discrete or pro-discrete group.

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