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TWO EQUIVALENT PROBLEMS: GYROSTATS IN FREE MOTION AND PARAMETRIC QUADRATIC HAMILTONIANS

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Introduction

A gyrostat $\mathcal{G}$ is a mechanical system made of a rigid body $\mathcal{P}$ called the *platform* and other bodies $\mathcal{R}$ called the *rotors*, connected to the platform and in such a way that the motion of the rotors does not modify the distribution of mass of the gyrostat $\mathcal{G}$. Due to this double spinning, the gyrostat on the one hand and the rotors on the other, the gyrostat is also known with the name of *dual-spin* body.

Such a model was employed by Volterra [1] to study the rotation of the earth. Ever since, the problem has attracted the attention of physicists, and a number of theoretical results have been obtained (see the textbook of Leimanis [2] and references therein). More recently, aerospace engineers used the gyrostat model for controlling the attitude dynamics of spacecrafts and for stabilizing their rotations. See, for instance, [3]—[6] and also references contained in [4]. In the last years, the gyrostat problem has regained interest and some papers [7]—[9] appeared in the literature, mainly dealing with qualitative aspects of the motion, such as bifurcations, splitting of separatrices, etc.. For the unperturbed problem (gyrostat in free motion), most of the papers above mentioned use canonical formalism, either in Euler or in Serret-Andoyer variables. However, in the free motion, the angular momentum vector is an integral, and thus, it is possible to represent the phase flow on a sphere (as it was pointed out by Hubert [10]), in a way very similar to the Euler-Poinsot case of the rigid body [11].

In a different context, Elipe and Lanchares [12]—[15] studied the bifurcations of several parametric quadratic Hamiltonians on the unit sphere. The classification of the two parameters case was established in [16] and this work was generalized by Frauendiener [17] for all possible parameters.

In this communication, we show that the gyrostat in free motion may be formulated as a multi parametric quadratic Hamiltonian on the unit sphere. To each case (depending on the moments of inertia and the rotors that act simultaneously) corresponds one case of the parametric quadratic Hamiltonian. Hence a real physical problem is identified with a theoretical one.

Rotation of a Gyrostat: the basics

Let us assume that the gyrostat has a fixed point O, that we will identify with the center of mass of the gyrostat and centered on it there are two orthonormal reference frames:

- $\mathcal{S}$, the space frame $O\boldsymbol{s}_1\boldsymbol{s}_2\boldsymbol{s}_3$, fixed in the space.
- $\mathcal{B}$, the body frame $O\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$, fixed in the platform.

Since by definition the motion of the rotors does not alter the distribution of mass of the gyrostat, there is a constant inertia tensor associated to $\mathcal{G}$ and we may assume that the body frame is precisely the frame of principal axes of inertia of the gyrostat.

The attitude of $\mathcal{B}$ in $\mathcal{S}$ results in three rotations by means of the Euler angles (ϕ, ϑ, ψ).

Following the steps described in [11, pp. 603–606], we overlay a symplectic structure in such a way that the conjugate moments (Φ, Θ, Ψ) of the Euler angles are the projections of the total angular vector $\boldsymbol{G}$ onto the non orthonormal basis $\boldsymbol{s}_3, \boldsymbol{l}, \boldsymbol{b}_3$, that is, that

$$\Phi = \boldsymbol{G}\cdot\boldsymbol{s}_3, \qquad \Theta = \boldsymbol{G}\cdot\boldsymbol{l}, \qquad \Psi = \boldsymbol{G}\cdot\boldsymbol{b}_3,$$

where the node vector $\boldsymbol{l}$ is defined as $\boldsymbol{l} = \boldsymbol{s}_1\cos\phi + \boldsymbol{s}_2\sin\phi$. Hence, by inversion, there results

$$g_1 = \left(\frac{\Phi - \Psi\cos\vartheta}{\sin\vartheta}\right)\sin\psi + \Theta\cos\psi, \quad g_2 = \left(\frac{\Phi - \Psi\cos\vartheta}{\sin\vartheta}\right)\cos\psi - \Theta\sin\psi, \quad g_3 = \Psi,$$
$$G_1 = \Theta\cos\phi + \left(\frac{\Psi - \Phi\cos\vartheta}{\sin\vartheta}\right)\sin\phi, \quad G_2 = \Theta\sin\phi - \left(\frac{\Psi - \Phi\cos\vartheta}{\sin\vartheta}\right)\cos\phi, \quad G_3 = \Phi,$$

for the components in the body frame (g_i) and in the space frame (G_i).

It is just a matter of computing partial derivatives to check that the Poisson brackets satisfy the identities

$$\begin{aligned} (g_1\,;\,g_2) &= -g_3, \quad (g_2\,;\,g_3) = -g_1, \quad (g_3\,;\,g_1) = -g_2, \\ (G_1\,;\,G_2) &= G_3, \quad (G_2\,;\,G_3) = G_1, \quad (G_3\,;\,G_1) = G_2. \end{aligned} \tag{1}$$

Now, we proceed to compute the kinetic energy of the gyrostat. Let $\boldsymbol{x} = x_1\boldsymbol{b}_1 + x_2\boldsymbol{b}_2 + x_3\boldsymbol{b}_3$ be the position vector of a particle P of the gyrostat with mass dm; its absolute velocity is $d\boldsymbol{x}/dt = \boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{x}$, where $\boldsymbol{v} = \dot{x}_1\boldsymbol{b}_1 + \dot{x}_2\boldsymbol{b}_2 + \dot{x}_3\boldsymbol{b}_3$. If the particle belongs to the platform (recall that it is a rigid body), then $\boldsymbol{v} = 0$. Taking into account that $\mathcal{G} = \mathcal{P}\cup\mathcal{R}$, the kinetic energy of the gyrostat is obtained by computing the volume quadrature

$$\begin{aligned} T &= \frac{1}{2}\int_{\mathcal{G}}(\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{x})^2\,dm \\ &= \frac{1}{2}\int_{\mathcal{G}}(\boldsymbol{\omega}\times\boldsymbol{x})^2\,dm + \boldsymbol{\omega}\cdot\int_{\mathcal{R}}(\boldsymbol{x}\times\boldsymbol{v})\,dm + \frac{1}{2}\int_{\mathcal{R}}\boldsymbol{v}^2\,dm \\ &= \frac{1}{2}\boldsymbol{\omega}\cdot\mathbb{I}\,\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\boldsymbol{f} + T_{\mathcal{R}}, \end{aligned} \tag{2}$$

where $\mathbb{I}$ is the diagonal tensor of inertia of the gyrostat $\mathcal{G}$, while $\boldsymbol{f} = f_1\boldsymbol{b}_1 + f_2\boldsymbol{b}_2 + f_3\boldsymbol{b}_3$ is the angular moment of the rotors and $T_{\mathcal{R}}$ is the kinetic energy of the rotors in their relative motion.

The Hamiltonian is the Legendre transformation with respect to the velocities of the Lagrangian function. Let us call the abbreviation $\boldsymbol{q} = (\phi, \theta, \psi)$ to denote the set of Eulerian angles; the kinetic energy of the gyrostat (2) is made of the addition of a pure quadratic term $(\frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\,\boldsymbol{\omega})$ in the velocities $\dot{\boldsymbol{q}}$, plus a linear part $(\boldsymbol{\omega} \cdot \boldsymbol{f})$ in the velocities $\dot{\boldsymbol{q}}$ (for $\boldsymbol{f}$ does not depend on the Euler angles), plus a function of the time $(T_{\mathcal{R}}(t))$. By virtue of Euler's theorem for homogeneous functions, the Hamiltonian is simply

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\,\boldsymbol{\omega}, \tag{3}$$

Rather than expressing $\mathcal{H}$ in terms of the canonical coordinates and moments, we will express it in terms of the total angular moment.

The independent variable t does not appear explicitly on the Hamiltonian (3), hence the kinetic energy of the gyrostat $\mathcal{G}$ considered as a rigid body (3) is preserved along the motion, whereas the total energy T is not.

In an analogous way, we find that the angular momentum vector of the gyrostat in the body frame is

$$\begin{aligned}
\boldsymbol{G} &= \int_{\mathcal{G}} (\boldsymbol{x} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}))\, dm \\
&= \int_{\mathcal{P}} (\boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x}))\, dm + \int_{\mathcal{R}} (\boldsymbol{x} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}))\, dm \\
&= \int_{\mathcal{G}} (\boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x}))\, dm + \int_{\mathcal{R}} (\boldsymbol{x} \times \boldsymbol{v})\, dm \\
&= \mathbb{I}\,\boldsymbol{\omega} + \boldsymbol{f}.
\end{aligned}$$

Let us denote by a_j the inverse or the principal moment of inertia I_j, that is $a_j = 1/I_j$. Assume moreover that $0 < a_1 \leq a_2 \leq a_3$. With these conventions, there is a diagonal tensor $\boldsymbol{A}$, such that the angular velocity is $\boldsymbol{\omega} = \boldsymbol{A}(\boldsymbol{G} - \boldsymbol{f})$. The Hamiltonian (3) in these notations is

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{G} - \boldsymbol{f}) \cdot \boldsymbol{A}(\boldsymbol{G} - \boldsymbol{f}). \tag{4}$$

From here on, we shall assume that the rotor moment is constant ($f_i =$ constant, $i = 1, 2, 3$). Hence, expanding the expression (4), and after dropping the constant terms $(\sum a_i f_i^2)$, we find the following expression for the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(a_1 g_1^2 + a_2 g_2^2 + a_3 g_3^2) - (a_1 g_1 f_1 + a_2 g_2 f_2 + a_3 g_3 f_3). \tag{5}$$

The Poisson structure (1) gives rise to the equations of the motion

$$\begin{aligned}
\dot{g}_1 &= (g_1\,;\, \mathcal{H}) = (a_3 - a_2) g_2 g_3 + a_2 f_2 g_3 - a_3 f_3 g_2, \\
\dot{g}_2 &= (g_2\,;\, \mathcal{H}) = (a_1 - a_3) g_1 g_3 + a_3 f_3 g_1 - a_1 f_1 g_3, \\
\dot{g}_3 &= (g_3\,;\, \mathcal{H}) = (a_2 - a_1) g_1 g_2 + a_1 f_1 g_2 - a_2 f_2 g_1.
\end{aligned} \tag{6}$$

From this system, one can easily check that the norm of the angular momentum vector $\boldsymbol{G}$ is an integral

$$\|\boldsymbol{G}\|^2 = g_1^2 + g_2^2 + g_3^2 = G^2 = \text{constant}; \tag{7}$$

the history of the rotation of $\boldsymbol{G}$ in the body frame is represented as a curve on the S^2 sphere of constant radius G.

In sum, the equations (6) admit two integrals, the kinetic energy (5) and the norm of the total angular momentum (7), therefore it is integrable. The phase space of (6) may be regarded as a foliation of invariant manifolds

$$S^2(G) = \{(g_1, g_2, g_3) \mid g_1^2 + g_2^2 + g_3^2 = G^2\}.$$

By using the angular momentum $\boldsymbol{G}$ instead of the angular velocity $\boldsymbol{\omega}$, the geometric model depicting the rotations of $\boldsymbol{G}$ is a sphere with constant radius. Most importantly, unlike Poinsot's ellipsoids, the underlying model is independent of the ellipsoid of inertia. More details about the derivation of the above formulas may be found in [9].

The triaxial gyrostat

The Hamiltonian of the gyrostat (5), when expressed in terms of the components g_i of the angular momentum, belongs to a general class of Hamiltonian systems, the one of the type

$$\mathcal{H} = \mathcal{T}_2 + \mathcal{T}_1, \quad \text{with} \quad \mathcal{T}_2 = \frac{1}{2} \sum_{1 \le i,j \le 3} A_{ij}\, \xi_i \xi_j \quad \text{and} \quad \mathcal{T}_1 = \sum_{1 \le i \le 3} B_i\, \xi_i.$$

The unknowns ξ have the Poisson structure (1)

$$(\xi_i ; \xi_j) = \sum_{1 \le k \le 3} \epsilon_{i,j,k}\, \xi_k,$$

where $\epsilon_{i,j,k}$ stands for the Levi-Civita symbol.

The class depends on 9 parameters, but it is possible to reduce it to 6 standard classes [17] : this is done by rotations in the phase space (g_1, g_2, g_3). Let us see that there is an equivalence among this class of Hamiltonians and the set of gyrostats.

If gyrostat is triaxial, the three moments of inertia are different and, thus, the three eigenvalues of the quadratic form $\boldsymbol{A}$ are different; we will suppose without loss of generality $a_1 < a_2 < a_3$. By means of the equivalence transformations (see [17]), that is to say, time scaling or adding a constant to the Hamiltonian, it is possible to shift one of the eigenvalues to 0 and to scale one of the two remaining to 1, without changing the simplectic structure of the variables g_i.

The number of essential parameters depends on the vector $\boldsymbol{A} \cdot \boldsymbol{f}$ and, more precisely, on the non zero components of $\boldsymbol{f}$, since the matrix $\boldsymbol{A}$ is diagonal. Let us analyze the different possibilities of the gyrostat depending on the number of spinning rotors (i.e. depending on $\boldsymbol{f}$).

One spinning rotor

If only one of the rotors is spinning about one of the principal axes of inertia, it was established in [9] that the Hamiltonian (5) reduces to one of the generic biparametric Hamiltonians

$$\mathcal{H} = \tfrac{1}{2}u^2 + \tfrac{1}{2}Pv^2 + Qu. \tag{8}$$

The phase flow and bifurcation lines in the parametric plane PQ have been determined by the authors in [14].

Although this Hamiltonian (8) does not depend on the axis where the spinning rotor is located, the parameter P is restricted to belong to different intervals according with the axis of rotation. Furthermore, to cover the whole parametric plane (P, Q), the three cases (each one with the spin about either $\boldsymbol{b}_1$ or $\boldsymbol{b}_2$ or $\boldsymbol{b}_3$) must be accounted. Indeed, let us consider a gyrostat with one spinning rotor about the biggest axis of inertia $\boldsymbol{b}_1$,

$$\mathcal{H} = \tfrac{1}{2}(a_1 g_1^2 + a_2 g_2^2 + a_3 g_3^2) - a_1 f_1 g_1.$$

The additive transformation

$$\mathcal{K} = \mathcal{H} - \frac{a_2}{2}(g_1^2 + g_2^2 + g_3^2) \tag{9}$$

yields to

$$\mathcal{K} = \tfrac{1}{2}(a_1 - a_2)g_1^2 + \tfrac{1}{2}(a_3 - a_2)g_3^2 - a_1 f_1 g_1,$$

and by means of the scaling transformation $\mathcal{H} = \mathcal{K}/(a_1 - a_2)$ we get

$$\mathcal{H} = \tfrac{1}{2}u^2 + \tfrac{1}{2}Pv^2 + Qu.$$

where now $P = (a_3 - a_2)/(a_1 - a_2)$, $Q = -a_1 f_1/(a_1 - a_2)$ and $(g_1, g_2, g_3) \longrightarrow (u, w, v)$. Since $a_1 < a_2 < a_3$, the parameter $P \in (-\infty, 0)$ and the variables (u, v, w) satisfy the simplectic relations

$$\{u; v\} = w, \quad \{v; w\} = u, \quad \{w; u\} = v. \tag{10}$$

The other two cases are reduced to (8) in analogous way, but the parameter P belongs to the intervals listed in Table 1.

axis	P	$P \in$	Q	variables
biggest ($\boldsymbol{b}_1$)	$\dfrac{a_3 - a_2}{a_1 - a_2}$	$(-\infty, 0)$	$\dfrac{-a_1 f_1}{a_1 - a_2}$	$(g_1, g_2, g_3) \longrightarrow (u, w, v)$
smallest ($\boldsymbol{b}_3$)	$\dfrac{a_2 - a_1}{a_3 - a_1}$	$(0, 1)$	$\dfrac{-a_3 f_3}{a_3 - a_1}$	$(g_1, g_2, g_3) \longrightarrow (w, v, u)$
intermediate ($\boldsymbol{b}_2$)	$\dfrac{a_1 - a_3}{a_2 - a_3}$	$(1, \infty)$	$\dfrac{-a_2 f_2}{a_2 - a_3}$	$(g_1, g_2, g_3) \longrightarrow (v, u, w)$

Table 1: Reduction of the case of one spinning rotor to the generic Hamiltonian $\mathcal{H} = \frac{1}{2}u^2 + \frac{1}{2}Pv^2 + Qu$.

It is worth to notice that an additive transformation, different from (9), yields to Hamiltonian (8), but the simplectic structure (10) changes its sign. Besides, the interval where the parameter P is located is interchanged in the cases of rotations about the smallest and the biggest axes of inertia, but this due to the fact that a $\pi/2$ rotation about the intermediate axis of inertia interchanges these axes.

Two spinning rotors

In the case of two axial spinning rotors the Hamiltonian (5) reduces to the generic one

$$\mathcal{H} = \tfrac{1}{2}u^2 + \tfrac{1}{2}Pv^2 + Qu + Rv. \tag{11}$$

The phase flow and bifurcation surfaces — in the parametric space PQR — have been studied in detail by Lanchares *et al.* [15].

As in the preceding case, the three possible selections of two spinning rotors ($\boldsymbol{b}_1\boldsymbol{b}_2$, $\boldsymbol{b}_1\boldsymbol{b}_3$ or $\boldsymbol{b}_2\boldsymbol{b}_3$) reduce to the generic Hamiltonian (11), but, depending on the case selected, the parameter P belongs to different intervals. Moreover, to cover the whole parameter space ($(P, Q, R) = \mathbb{R}^3$), the three cases are needed. As a matter of fact, the region to which the parameter P belongs, is originated by the additive and scaling transformations necessary to eliminate the non essential parameters.

Let us consider, for instance, two spinning rotors about the biggest and the intermediate axes of inertia ($f_3 = 0$). The Hamiltonian is now

$$\mathcal{H} = \tfrac{1}{2}(a_1 g_1^2 + a_2 g_2^2 + a_3 g_3^2) - a_1 f_1 g_1 - a_2 f_2 g_2.$$

The additive transformation $\mathcal{H} - (g_1^2 + g_2^2 + g_3^2)\, a_3/2$, and the scaling transformation $\mathcal{H}/(a_2 - a_3)$ yield to the reduced form (11), where the parameters are $P = (a_1 - a_3)/(a_2 - a_3)$, $Q = -a_2 f_2/(a_2 - a_3)$, $R = -a_1 f_1/(a_2 - a_3)$ and $(g_1, g_2, g_3) \longrightarrow (v, u, w)$. In this case the parameter P belongs to the open interval $(1, \infty)$ and the simplectic structure of the variables (u, v, w) verifies (10).

In Table 2 we summarize the reductions of the three cases of two spinning rotors. Notice that the reduction is not unique and other equivalence transformations yield to the generic Hamiltonian (11). Indeed, a different scale transformation ($\mathcal{H}/(a_1 - a_3)$), in the case of $f_3 = 0$, leads to a Hamiltonian with $P \in (0, 1)$ and the variables (u, v, w) satisfying the simplectic structure (10), but the sign. This can be overcome either considering the time going in the reverse sense, or performing a rotation of $\pi/2$ about the g_2 axis.

Case	P	$P \in$	Q	R	variables
$f_3 = 0$	$\dfrac{a_2 - a_3}{a_1 - a_3}$	$(0, 1)$	$\dfrac{a_1 f_1}{a_1 - a_3}$	$\dfrac{-a_2 f_2}{a_1 - a_3}$	$(g_1, g_2, g_3) \longrightarrow (w, v, u)$
$f_2 = 0$	$\dfrac{a_3 - a_2}{a_1 - a_2}$	$(-\infty, 0)$	$\dfrac{-a_1 f_1}{a_1 - a_2}$	$\dfrac{-a_3 f_3}{a_1 - a_2}$	$(g_1, g_2, g_3) \longrightarrow (u, w, v)$
$f_1 = 0$	$\dfrac{a_2 - a_1}{a_3 - a_1}$	$(0, 1)$	$\dfrac{-a_3 f_3}{a_3 - a_1}$	$\dfrac{-a_2 f_2}{a_3 - a_1}$	$(g_1, g_2, g_3) \longrightarrow (w, v, u)$

Table 2: Reduction of the case of two spinning rotors to the generic Hamiltonian $\mathcal{H} = \frac{1}{2}u^2 + \frac{1}{2}Pv^2 + Qu + Rv$.

THREE SPINNING ROTORS

In this case, the problem is equivalent to the parametric quadratic Hamiltonians, where the number of essential parameters is maximum, that is

$$\mathcal{H} = \tfrac{1}{2}u^2 + \tfrac{1}{2}Pv^2 + Qu + Rv + Sw. \tag{12}$$

This Hamiltonian, to our knowledge, is yet to be analyzed.

By means of two equivalence transformations, an additive plus a scaling one, it is easy to obtain the Hamiltonian (12). As in the other cases above considered, the choice of the transformations limits the parameter P to range one of the three open intervals $(-\infty, 0)$, $(0, 1)$ or $(1, \infty)$.

The axially symmetric gyrostat

In this section, we suppose now that the gyrostat is axially symmetric, that is to say, two of the principal moments of inertia are equal. Let us assume, for instance, that $a_1 = a_2 < a_3$. In this case the number of essential parameters is, at most, two. Indeed, no essential parameters are found in the quadratic part; an additive transformation shifts to 0 the two common eigenvalues corresponding to the two equal moments of inertia; a scaling transformation puts to 1 the remaining eigenvalue.

Let us apply to Hamiltonian (5) the additive transformation

$$\mathcal{H} - \frac{a_1}{2}(g_1^2 + g_2^2 + g_3^2)$$

and the scaling transformation $\mathcal{H}/(a_3 - a_1)$; this yields

$$\mathcal{H} = \tfrac{1}{2}g_3^2 - \frac{a_1 f_1}{a_3 - a_1}g_1 - \frac{a_2 f_2}{a_3 - a_1}g_2 - \frac{a_3 f_3}{a_3 - a_1}g_3.$$

Taking into account that any vector perpendicular to the axis of symmetry is itself principal axis of inertia, a rotation about the axis of symmetry $\boldsymbol{b}_3$ and angle $\alpha = \arctan(-a_2 f_2/a_1 f_1)$ about the symmetry axis reduces Hamiltonian to

$$\mathcal{H} = \tfrac{1}{2}u^2 + Pu + Qv, \tag{13}$$

that is, with only two essential parameters P and Q defined by where

$$P = -\frac{a_3 f_3}{a_3 - a_1} \quad \text{and} \quad Q = \frac{a_2 f_2}{a_3 - a_1}(\sin\alpha - \cos\alpha).$$

The phase flow and bifurcation lines of this Hamiltonian (13) were obtained by the authors in [13]. The axially symmetrical gyrostat with only one spinning rotor, its bifurcations and the integration of the trajectories in terms of elliptic functions have been studied in detail in [18].

The different cases for an axially symmetrical gyrostat are summarized in in Table 3

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rotors	axes	$\mathcal{H}$
1	$\boldsymbol{b}_1$	$\frac{1}{2}u^2 + Pv$
1	$\boldsymbol{b}_2$	$\frac{1}{2}u^2 + Pv$
1	$\boldsymbol{b}_3$	$\frac{1}{2}u^2 + Pu$
2	$\boldsymbol{b}_1, \boldsymbol{b}_2$	$\frac{1}{2}u^2 + Pv$
2	$\boldsymbol{b}_1, \boldsymbol{b}_3$	$\frac{1}{2}u^2 + Pu + Qv$
2	$\boldsymbol{b}_2, \boldsymbol{b}_3$	$\frac{1}{2}u^2 + Pu + Qv$
3	$\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$	$\frac{1}{2}u^2 + Pu + Qv$

Table 3: Reduction of the axially symmetric gyrostat with $a_1 = a_2 < a_3$

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