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Computing the homology of groups: The geometric way[☆]

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A B S T R A C T

In this paper, we present several algorithms related with the computation of the homology of groups, from a geometric perspective (that is to say, carrying out the calculations by means of simplicial sets and using techniques of Algebraic Topology). More concretely, we have developed some algorithms which, making use of the *effective homology* method, construct the homology groups of Eilenberg–MacLane spaces $K(G, 1)$ for different groups G, allowing one in particular to determine the homology groups of G.

Our algorithms have been programmed as new modules for the Kenzo system, enhancing it with the following new functionalities:

- construction of the effective homology of $K(G, 1)$ from a given finite type free resolution of the group G;
- construction of the effective homology of $K(A, 1)$ for every finitely generated Abelian group A (as a consequence, the effective homology of $K(A, n)$ is also available in Kenzo, for all $n \in \mathbb{N}$);
- computation of homology groups of some 2-types;
- construction of the effective homology for central extensions.

In addition, an *inverse* problem is also approached in this work: given a group G such that $K(G, 1)$ has effective homology, can a finite type free resolution of the group G be obtained? We provide some algorithms to solve this problem, based on a notion of *norm* of a group, allowing us to control the convergence of the process when building such a resolution.

1. Introduction

When using homological algebra techniques to study group theory, two different (but related) alternatives are possible (see Brown (1982) for details on the following discussion). One is algebraic

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and is based on the notion of *resolution* (replacing the group under study by an acyclic object in a suitable category of modules). The other alternative is geometric in nature. It consists in finding a contractible topological space with a free action of a group G. Then the space of orbits of the action can be endowed with a convenient quotient topology, in such a way that we obtain an *aspherical* space (that is to say, a space whose only non-null homotopy group is the first, fundamental one). The homology of this space is, by definition, the homology of G, and it does not depend on the choosing of the contractible space or of the action. Each aspherical space (unique up to homotopy type) is a particular *Eilenberg–MacLane space* for G, and is generically denoted by $K(G, 1)$.

If we move to *computational* mathematics, the preferred via chosen was the algebraic one, as exemplified by the package HAP (Ellis, 2009) of the computer algebra system GAP (The GAP Group, 2008), which contains an impressive number of algorithms dealing with resolutions. The geometric way has been up to now neglected from the algorithmic point of view. The reason is that the contractible spaces to be constructed are very frequently of infinite type (even in cases where the group G is not too complicated), apparently closing the possibility of a computational treatment.

This view changed drastically when Sergeraert introduced at the end of the 1980's his theory of *effective homology* (Sergeraert, 1994). His methods allow the programmer to deal with spaces of infinite dimension, encoded in a lazy functional programming style, producing a complete revision of Algebraic Topology from a constructive point of view (see Rubio and Sergeraert (2006) for recent developments of this theory). Perhaps more important from a practical point of view was Sergeraert's construction of the *Kenzo* system, a Common Lisp program implementing the effective homology methods (Dousson et al., 1999). Since then, the programmer can work on a computer with simplicial sets, loop spaces, fibrations, classifying spaces and many other Algebraic Topology constructions, computing, at the end, homology groups of complicated spaces (under the combinatorial form of *simplicial sets*).

Taking into account this new situation, this paper represents a first step to take up again the geometrical way of approaching *group homology*, by means of techniques from effective homology and using, and extending, Kenzo as a computing platform.

Our proposal is not opposed to the algebraic view. Our aim is rather to take the best of both worlds. Therefore, and as a first module, we programmed, in collaboration with Graham Ellis (see Romero et al., 2009), an OpenMath link between HAP and Kenzo, allowing Kenzo to import from HAP resolutions of groups. Once a resolution of a group G is internally stored in Kenzo, an algorithm allows us to construct the Eilenberg–MacLane space $K(G, 1)$, with *effective homology*. This provides not only access to some homology groups of G, but also makes it possible to apply on the space $K(G, 1)$ all the powerful tools available in Kenzo, and construct in this way further spaces.

This via is explored in this paper. We show two applications in Algebraic Topology, and another one in Homological Algebra. As a first application, we develop a Kenzo package to compute, as *objects with effective homology*, the generalized Eilenberg–MacLane spaces $K(G, n)$ for any finitely generated Abelian group G and for all $n \in \mathbb{N}$. These objects are very important in Algebraic Topology, to study and compute homotopy groups, through Whitehead and Postnikov towers (see May (1967) and Rubio and Sergeraert (2006)).

As a second topological application, we compute mechanically (for the first time, up to our knowledge) some homology groups of 2-types, the second step (the first one consists of Eilenberg–MacLane spaces) towards the difficult problem of characterizing homotopy types.

Our last application provides programs to deal with the effective homology of central extensions of groups. The theoretical algorithms were known some time ago (see Rubio, 1997), but only now the technological tools explained before allow us to tackle the problem of programming them. Let us observe that this *algebraic* application has also positive consequences on topological problems, since it enlarges the field of application of our 2-types package: we can also compute with 2-types whose fundamental group is a central extension.

Finally, we have also approached an *inverse* problem: how to obtain a resolution of a group G from the knowledge of an effective homology of $K(G, 1)$. The results in this area are still partial, and more research will be needed to get fully satisfactory algorithms, and to proceed to implement them as Kenzo modules.

The organization of the paper is as follows. The next section is devoted to preliminaries. Section 3 contains our main algorithm, which constructs the effective homology of $K(G, 1)$ from a finite type resolution of a group G, and then Section 4 collects some interesting fields of application of this result. Section 5 explains how our algorithms have been translated to Common Lisp and comments on experimental results. In Section 6 an inverse problem is considered: given a group G with effective homology, it is (sometimes) possible to determine a resolution for G. The paper ends with conclusions, open problems and the bibliography.

2. Definitions and preliminaries

2.1. Some fundamental notions about homology of groups

The following definitions and important results about homology of groups can be found in MacLane (1963) and Brown (1982).

Definition 1. Given a ring R, a *chain complex* of R-modules is a pair of sequences $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$ where, for each degree $n \in \mathbb{Z}$, C_n is an R-module and $d_n : C_n \to C_{n-1}$ (*the differential map*) is an R-module morphism such that $d_{n-1} \circ d_n = 0$ for all n.

Definition 2. Let $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$ be a chain complex of R-modules, with R a general ring. For each degree $n \in \mathbb{Z}$, the *nth homology module* of C_* is defined to be the quotient module $H_n(C_*) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$. A chain complex C_* is *acyclic* if $H_n(C_*) = 0$ for all n.

Definition 3. Let G be a group and $\mathbb{Z}G$ the free $\mathbb{Z}$-module generated by the elements of G. The multiplication in G extends uniquely to a $\mathbb{Z}$-bilinear product $\mathbb{Z}G \times \mathbb{Z}G \to \mathbb{Z}G$ which makes $\mathbb{Z}G$ a ring. This is called the *integral group ring* of G.

Definition 4. A *resolution* F_* for a group G is an acyclic chain complex of $\mathbb{Z}G$-modules

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} F_{-1} = \mathbb{Z} \longrightarrow 0$$

where $F_{-1} = \mathbb{Z}$ is considered a $\mathbb{Z}G$-module with the trivial action and $F_i = 0$ for $i < -1$. The map $\varepsilon : F_0 \to F_{-1} = \mathbb{Z}$ is called *augmentation*. If F_i is free for each $i \geq 0$, then F_* is said to be a *free resolution*.

Very frequently, resolutions come equipped with a *contracting homotopy* h, which is a set of Abelian group morphisms $h_n : F_n \to F_{n+1}$ for each $n \geq -1$ (in general not compatible with the G-action), such that

$$\begin{aligned} \varepsilon h_{-1} &= \mathrm{Id}_{\mathbb{Z}} \\ h_{-1}\varepsilon + d_1 h_0 &= \mathrm{Id}_{F_0} \\ h_{i-1} d_i + d_{i+1} h_i &= \mathrm{Id}_{F_i}, \quad i > 0. \end{aligned}$$

The existence of the contracting homotopy for F_* assures in particular the exactness of the resolution.

Given a free resolution F_*, one can consider the chain complex of $\mathbb{Z}$-modules (that is to say, Abelian groups) $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ defined by

$$C_n = (\mathbb{Z} \otimes_{\mathbb{Z}G} F_*)_n, \quad n \geq 0$$

(where $\mathbb{Z} \equiv C_*(\mathbb{Z}, 0)$ is the chain complex with only one non-null $\mathbb{Z}G$-module in dimension 0, $C_0 = \mathbb{Z}$) with differential maps $d_{C_n} : C_n \to C_{n-1}$ induced by $d_n : F_n \to F_{n-1}$.

Let us emphasize the difference between the chain complexes F_* and $C_* = \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$. The elements of F_n ($n \geq 0$) can be seen as *words* $\sum \lambda_i(g_i, z_i)$ where $\lambda_i \in \mathbb{Z}$, $g_i \in G$ and z_i is a generator of F_n (which is a *free* $\mathbb{Z}G$-module). On the other hand, the associated chain complex $C_* = \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$ of Abelian groups has elements in degree n of the form $\sum \lambda_i z_i$ with $\lambda_i \in \mathbb{Z}$ and z_i a generator of the free $\mathbb{Z}$-module C_n.

Although the chain complex of $\mathbb{Z}G$-modules F_* is acyclic, $C_* = \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$ is in general not exact and its homology groups are thus not null. An important result in homology of groups claims that these homology groups are independent of the chosen resolution for G.

Theorem 5 (Brown, 1982). *Let G be a group and F_*, F'_* two free resolutions of G. Then*

$$H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*) \cong H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F'_*) \quad \text{for all } n \in \mathbb{N}.$$

The hypothesis that F_* and F'_* are free can in fact be relaxed; it suffices for the modules F_* and F'_* to be *projective*. This theorem leads to the following definition.

Definition 6. Given a group G, the *homology groups* $H_n(G)$ are defined as

$$H_n(G) = H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*), \quad n \in \mathbb{N}$$

where F_* is any free (or projective) resolution for G.

Let G be a group, how can we determine a free resolution F_*? One approach is to consider the *Bar resolution* $B_* = \mathrm{Bar}_*(G)$ (explained, for instance, in MacLane (1963)) whose associated chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B_*$ can be viewed as the chain complex of the Eilenberg–MacLane space $K(G, 1)$ (see Brown (1982), for details). The homology groups of $K(G, 1)$ are those of the group G and this space has a big structural richness. But it has a serious drawback: its size. If $n > 1$, then $K(G, 1)_n = G^n$. In particular, if $G = \mathbb{Z}$, the space $K(G, 1)$ is of infinite type in each dimension. This fact is an important obstacle to using $K(G, 1)$ as a means for computing the homology groups of G. It would be therefore convenient to construct *smaller* resolutions.

For some particular cases, small (or minimal) resolutions can be directly constructed. For instance, let G be the cyclic group of order m with generator t, $G = C_m$. The resolution F_* for G

$$\cdots \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

where N denotes the *norm element* $1 + t + \cdots + t^{m-1}$ of $\mathbb{Z}G$, produces the chain complex of Abelian groups

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

and therefore

$$H_i(G) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/m\mathbb{Z} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even and } i > 0. \end{cases}$$

But in general it is not so easy to obtain a resolution for a group G, and in fact this problem provides an interesting research field where many papers and works have appeared trying to determine resolutions for different kinds of groups. As we will see later, the GAP package HAP has been designed as a tool for constructing resolutions for a wide variety of groups. On the other hand, our work shows that the *effective homology* method, introduced in the following section, could also be helpful in order to compute the homology of some groups.

2.2. Effective homology

We now present the general ideas of the effective homology method, devoted to the computation of homology groups of *spaces*. See Rubio and Sergeraert (2002) and Rubio and Sergeraert (2006) for more details.

Definition 7. A *reduction* ρ between two chain complexes $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{N}}$ (which is denoted $\rho : C_* \Rightarrow\!\!\!\Rightarrow D_*$) is a triple (f, g, h) where: (a) the components f and g are chain complex morphisms $f : C_* \to D_*$ and $g : D_* \to C_*$; (b) the component h is a homotopy operator $h : C_* \to C_{*+1}$ (a graded group morphism of degree $+1$); (c) the following relations are satisfied: $fg = \mathrm{Id}_D$; $d_C h + h d_C = \mathrm{Id}_C - gf$; $fh = 0$; $hg = 0$; $hh = 0$.

These properties express that C_* is the direct sum of D_* and an acyclic complex. This decomposition is simply $C_* = \mathrm{Ker} f \oplus \mathrm{Im}\, g$, with $\mathrm{Im}\, g \cong D_*$ and $H_*(\mathrm{Ker} f) = 0$. In particular, this implies that the graded homology groups $H_*(C_*)$ and $H_*(D_*)$ are canonically isomorphic.

Remark 8. A reduction is in fact a particular case of chain equivalence in the classical sense (see MacLane (1963), page 40), where the homotopy operator on the small chain complex D_* is the null map.

Definition 9. A *(strong chain) equivalence* ε between two chain complexes C_* and D_*, denoted by $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$, is a triple (B_*, ρ_1, ρ_2) where B_* is a chain complex, and ρ_1 and ρ_2 are reductions $\rho_1 : B_* \Rrightarrow C_*$ and $\rho_2 : B_* \Rrightarrow D_*$.

Remark 10. We need the notion of *effective* chain complex: it is essentially a free chain complex C_* where each group C_n is finitely generated, and a provided algorithm returns a (distinguished) $\mathbb{Z}$-basis in each degree n; in particular, its homology groups are elementarily computable (for details, see Rubio and Sergeraert (2002)).

Definition 11. An *object with effective homology* X is a quadruple $(X, C_*(X), HC_*, \varepsilon)$ where $C_*(X)$ is a chain complex canonically associated with X (which allows us to study the homological nature of X), HC_* is an effective chain complex, and ε is an equivalence $\varepsilon : C_*(X) \Lleftarrow\!\!\Rrightarrow HC_*$.

It is important to understand that in general the HC_* component of an object with effective homology is *not* made of the homology groups of X; this component HC_* is a free $\mathbb{Z}$-chain complex of finite type, in general with a non-null differential, whose homology groups $H_*(HC_*)$ can be determined by means of an elementary algorithm. From the equivalence ε one can deduce the isomorphism $H_*(X) := H_*(C_*(X)) \cong H_*(HC_*)$, which allows one to compute the homology groups of the initial space X. In this way, the notion of object with effective homology provides a method to compute homology groups of complicated spaces by means of homology groups of effective complexes.

The effective homology technique is based on the following idea: given some topological spaces $X_1, \ldots, X_n$, a topological constructor Φ produces a new topological space X. If effective homology versions of the spaces $X_1, \ldots, X_n$ are known, then one should be able to build an effective homology version of the space X, and this version would allow us to compute the homology groups of X. A typical example of this kind of situation is the loop space constructor. Given a 1-reduced simplicial set X with effective homology, it is possible to determine the effective homology of the loop space $\Omega(X)$, which in particular allows one to compute the homology groups $H_*(\Omega(X))$. Moreover, if X is m-reduced, this process may be iterated m times, producing an effective homology version of $\Omega^k(X)$, for $k \leq m$. Effective homology versions are also known for classifying spaces or total spaces of fibrations, see Rubio and Sergeraert (2006) for more information.

All these constructions have been implemented in the Kenzo system (Dousson et al., 1999), a Common Lisp program which makes use of the effective homology method to determine homology groups of complicated spaces; it has obtained some results (for example homology groups of iterated loop spaces of a loop space modified by a cell attachment, components of complex Postnikov towers, etc.) which had never been determined before. Furthermore, Kenzo implements Eilenberg–MacLane spaces $K(G, n)$ for every n but only for $G = \mathbb{Z}$ and $G = \mathbb{Z}/2\mathbb{Z}$ (these spaces appear in different constructions of Algebraic Topology), although in principle it is not designed to determine the homology of groups and it does not know how to work with resolutions.

These ideas suggest that the effective homology technique and the Kenzo program should have a role in the computation of the homology of a group G. To this end, we can consider the Eilenberg–MacLane space $K(G, 1)$, whose homology groups coincide with those of G. The size of this space makes it difficult to calculate the groups in a direct way, but it is possible to operate with this simplicial set making use of the *effective homology* technique: if we construct the effective homology of $K(G, 1)$ then we would be able to *efficiently* compute the homology groups of $K(G, 1)$, which are those of G. Furthermore, it should be possible to extend many group theoretic constructions to effective homology constructions of Eilenberg–MacLane spaces. We thus introduce the following definition.

Definition 12. A group G is a *group with effective homology* if $K(G, 1)$ is a simplicial set with effective homology.

The problem is, given a group G, how can we determine the effective homology of $K(G, 1)$? If the group G is finite, the simplicial set $K(G, 1)$ is effective too, so that it can be considered with effective homology in a trivial way. However, the enormous size of this space makes it difficult to obtain real calculations, and therefore even in this case we will try to obtain an equivalence $C_*(K(G, 1)) \Lleftarrow\!\!\Rrightarrow E_*$ where E_* is effective and (much) smaller than the initial complex. Section 3 of this paper presents an algorithm that computes this desired equivalence provided that the group G is endowed with a finite type resolution.

3. Effective homology of a group from a resolution

This section is devoted to an algorithm computing the effective homology of a group G given a (small) free $\mathbb{Z}G$-resolution. This algorithm was the main theoretical result included in the work Romero et al. (2009) and has been implemented in Common Lisp enhancing the Kenzo system. We will see some examples of use of these new programs in Section 5. A brief description of the construction of the algorithm is included in the following paragraphs. For more details, see Romero et al. (2009).

Let G be a group and F_* a free (augmented) finite type resolution for G with a contracting homotopy h. We want to construct the effective homology of the space $K(G, 1)$, that is to say, a (strong chain) equivalence $C_*(K(G, 1)) \Lleftarrow\!\!\Rrightarrow E_*$ where E_* is an effective chain complex.

We begin by considering the Bar resolution $B_* = \mathrm{Bar}_*(G)$ for G, with augmentation ε' and contracting homotopy h' (the definition of these maps can be found in Brown (1982)). As B_* and the given resolution F_* are free resolutions for G, it is well known (see Brown, 1982) that one can explicitly construct morphisms of chain complexes of $\mathbb{Z}G$-modules $f : B_* \to F_*$ and $g : F_* \to B_*$ which are homotopy equivalences. Moreover, one can construct graded morphisms of $\mathbb{Z}G$-modules

$$k : F_* \to F_{*+1}, \qquad k' : B_* \to B_{*+1}$$

such that $d_F k + k d_F = \mathrm{Id}_F - fg$ and $d_B k' + k' d_B = \mathrm{Id}_B - gf$.

We have therefore a homotopy equivalence (in the classical sense):

$$\begin{tikzcd} B_* \arrow[loop left, "k'"] \arrow[r, bend left, "f"] & F_* \arrow[loop right, "k"] \arrow[l, bend left, "g"] \end{tikzcd}$$

in which the four components f, g, k and k' are compatible with the action of the group G.

If we now apply the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} -$, which is additive, we obtain an equivalence of chain complexes (of $\mathbb{Z}$-modules):

$$\begin{tikzcd} \mathbb{Z} \otimes_{\mathbb{Z}G} B_* \arrow[loop above, "k'"] \arrow[r, bend left, "f"] & \mathbb{Z} \otimes_{\mathbb{Z}G} F_* \arrow[loop above, "k"] \arrow[l, bend left, "g"] \end{tikzcd}$$

where both chain complexes provide us the homology of the initial group G, that is,

$$H_*(\mathbb{Z} \otimes_{\mathbb{Z}G} B_*) \cong H_*(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*) \equiv H_*(G).$$

In order to obtain a strong chain equivalence (in other words, a pair of reductions, following the framework of effective homology), we make use of the mapping cylinder construction (see Weibel, 1994). This allows one to produce a (strong chain) equivalence

$$\mathbb{Z} \otimes_{\mathbb{Z}G} B_* \overset{\rho'}{\Lleftarrow} \mathrm{Cylinder}(f)_* \overset{\rho}{\Rrightarrow} \mathbb{Z} \otimes_{\mathbb{Z}G} F_*.$$

The definitions of the different components of both reductions are included in Romero et al. (2009).

Now we recall that the left chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B_*$ is equal to $C_*(K(G, 1))$. On the other hand, if we suppose that the initial resolution F_* is of finite type (and small), then the right chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} F_* \equiv E_*$ is effective (and small too), so that we have obtained the desired effective homology of $K(G, 1)$,

$$C_*(K(G, 1)) \Lleftarrow\!\!\Rrightarrow E_*.$$

We have constructed in this way an algorithm computing the effective homology of a group, formally described in Algorithm 1.

Algorithm 1 Computation of the effective homology of a group

Require: a group G and a free (augmented) finite type resolution F for G with a contracting homotopy h.
Ensure: a (strong chain) equivalence $\varepsilon : C_*(K(G,1)) \Longleftarrow\!\!\Longrightarrow E$ where E is an effective chain complex.

1: $B = \mathit{BarResolution}(G)$

[Compute the Bar resolution of the group G and store it in the variable B]

2: $f = 2\mathit{ResolutionsRightZGMorphism}(B, F)$

[$f : B \to F$ is the morphism of chain complexes of $\mathbb{Z}G$-modules between both resolutions]

3: $g = 2\mathit{ResolutionsLeftZGMorphism}(B, F)$

[$g : F \to B$ is the morphism of chain complexes of $\mathbb{Z}G$-modules between both resolutions]

4: $k = 2\mathit{ResolutionsRightHomotopy}(B, F)$

[$k : F_* \to F_{*+1}$ is the graded morphism of chain complexes of $\mathbb{Z}G$-modules such that $d_F k + k d_F = \mathrm{Id}_F - fg$]

5: $k' = 2\mathit{ResolutionsLeftHomotopy}(B, F)$

[$k' : B_* \to B_{*+1}$ is the graded morphism of chain complexes of $\mathbb{Z}G$-modules such that $d_B k' + k' d_B = \mathrm{Id}_B - gf$]

6: $\rho = \mathit{CylinderRightReduction}(f)$

[Computes a reduction $\rho : \mathrm{Cylinder}(f)_* \Rightarrow\!\!\Rightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$. Only the parameter f is necessary]

7: $\rho' = \mathit{CylinderLeftReduction}(f, g, k', k)$

[Computes a reduction $\rho' : \mathrm{Cylinder}(f)_* \Rightarrow\!\!\Rightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} B_*$]

8: $\varepsilon = \mathit{BuildHomotopyEquivalence}(\rho', \rho)$

[Constructs a (strong chain) equivalence from two reductions. The result is a reduction $C_*(K(G,1)) \Longleftarrow\!\!\Longrightarrow E_*$]

All the functions included in Algorithm 1 are new functions which have been implemented in Common Lisp enhancing the Kenzo system, with the exception of the function `BuildHomotopyEquivalence` which was already included in Kenzo. As we will see in Section 5 the names of our real Lisp functions are not exactly the same as those included in this algorithm. Our implementation follows in most cases Kenzo's habit of using only the first consonants involved in the description of a function; for the general algorithm we have preferred to include a complete name for being more intuitive. The same will be done for all algorithms in this paper.

The strong chain equivalence determined by Algorithm 1 makes it possible to determine the homology groups of G, and, what is more useful, once we have $K(G,1)$ with its effective homology we could apply different constructors and obtain the effective homology of the results. This could allow one, for instance, to determine the homology of some groups (obtained from other initial groups with effective homology) without constructing a resolution for them. Some fields of application of our algorithm are introduced in the following section.

4. Applications

4.1. 2-types

Let A be an Abelian group and G a group acting on A; a 2-*type* for G and A is a (topological) space with $\pi_1(X) = G$, $\pi_2(X) = A$, and $\pi_n(X) = 0$ for all $n \geq 3$; the computation of the homology groups of these spaces is a difficult problem in the field of group homology (Ellis, 1992). It is well known that a 2-type X for G and A corresponds to a cohomology class $[f]$ in $H^3(G, A)$, and there exists a fibration

$$K(A, 2) \hookrightarrow X \rightarrow K(G, 1).$$

The theoretical existence of this fibration can be made *constructive* as follows, when the action of G on A is trivial. A cohomology class $[f]$ is given by a 3-cocycle f, which is a map $f : K(G, 1)_3 \rightarrow A$ (satisfying some properties). This map induces a simplicial morphism $f : K(G, 1) \rightarrow K(A, 3)$, which can be composed with the universal fibration (see May, 1967) $K(A, 2) \hookrightarrow E \rightarrow K(A, 3)$ in order to construct the desired fibration. In this way, we obtain a twisting operator $\tau_f : K(G, 1)_* \rightarrow K(A, 2)_{*-1}$ which allows one to express the total space X as a twisted Cartesian product

$$X = K(A, 2) \times_f K(G, 1).$$

Supposing now that the group G is given with a finite type free resolution, our Algorithm 1 can be applied in order to produce the effective homology of $K(G, 1)$. Analogously, provided a finite type resolution for A, we can determine the effective homology of $K(A, 1)$. Since $K(A, 1)$ is a simplicial Abelian group one can apply the classifying space constructor B that gives us $B(K(A, 1)) = K(A, 2)$, which is also a simplicial Abelian group. Furthermore, the effective homology version of the classifying space constructor B (see Rubio and Sergeraert (2006), for details) provides us the effective homology of the space $K(A, 2)$ from the effective homology of $K(A, 1)$ (iterating the process, $K(A, n) = B(K(A, n-1))$ has effective homology for every $n \in \mathbb{N}$). In this way, both spaces $K(A, 2)$ and $K(G, 1)$ are *objects with effective homology*. Finally, the effective homology version for a fibration (described also in Rubio and Sergeraert (2006)), makes use of the effective homologies of $K(A, 2)$ and $K(G, 1)$ and of the twisting operator $\tau_f : K(G, 1)_* \rightarrow K(A, 2)_{*-1}$ and gives us the effective homology of the total space $X = K(A, 2) \times_f K(G, 1)$. In particular, this leads to the desired homology groups of the 2-type X. The process is formally described in our Algorithm 2.

Algorithm 2 Computation of the effective homology of a 2-type

Require: an Abelian group A with a free resolution F_A of finite type (with a contracting homotopy); a group G (acting trivially on A) with a free resolution F_G of finite type (with a contracting homotopy); a cohomology class $[f] \in H^3(G, A)$ given by a 3-cocycle $f : K(G, 1)_3 \rightarrow A$.
Ensure: $\varepsilon : C_*(X) \Longleftarrow\!\!\!\Rightarrow E_*$ where E_* is an effective chain complex and $X = K(A, 2) \times_f K(G, 1)$.

1: $X = 2Type(A, G, f)$

[New function which computes the 2-type associated to the groups A and G and the 3-cocycle f, that is, $X = K(A, 2) \times_f K(G, 1)$; the implementation follows the ideas explained at the beginning of this subsection]

2: $efhmKG1 = Algorithm1(G, F_G)$

[Apply Algorithm 1 to the group G with the finite type resolution F_G; the effective homology of $K(G, 1)$ is obtained]

3: $efhmKA2 = ClassifyingSpaceEfhm(K(A, 1), Algorithm1(A, F_A))$

[Apply Algorithm 1 to compute the effective homology of $K(A, 1)$; then we use the Kenzo function computing the effective homology of the classifying space of a simplicial Abelian group from its effective homology, which in this case produces the effective homology of $B(K(A, 1)) = K(A, 2)$]

4: $\varepsilon = FibrationEfhm(X, efhmKA2, efhmKG1)$

[Use the Kenzo function which computes the effective homology of a fibration from the effective homologies of the two factors, in this case $K(A, 2)$ and $K(G, 1)$]

This algorithm has been implemented in Common Lisp as part of our new module for the Kenzo system dealing with homology of groups. See Section 5.3 for some examples of calculations.

If the group G acts non-trivially on A, an action $K(G, 0) \times K(A, 2) \rightarrow K(A, 2)$ must also be considered in the fibration $K(A, 2) \hookrightarrow X \rightarrow K(G, 1)$. The explicit construction of the twisting operator which describes the fibration cannot be obtained as easily as in the previous case, and a more deep study of the fibration is necessary. It should be done as a further work.

4.2. Central extensions

Let $0 \to A \to E \to G \to 1$ be a central extension of groups (that is, A is an Abelian group and G acts on A in a trivial way). Then, it is well-known (see Brown, 1982) there exists a set-theoretic map $f : G \times G \to A$ which satisfies:

(1) $f(g, 1) = 0 = f(1, g)$
(2) $f(gh, k) = f(h, k) - f(g, h) + f(g, hk)$

In addition, the initial extension is equivalent to another extension

$$0 \to A \to A \times_f G \to G \to 1$$

where the elements of $A \times_f G$ are pairs (a, g) with $a \in A$ and $g \in G$, and the group law is defined by

$$(a_1, g_1)(a_2, g_2) \equiv (a_1 + a_2 + f(g_1, g_2), g_1 g_2).$$

The set-theoretic map f is called the 2-*cocycle* of the extension, since it corresponds to a map $f : K(G, 1)_2 \to A$ in $H^2(G, A)$.

Very frequently, the groups G and A are not complicated and their homology groups are known. On the contrary, the homology groups of $E \cong A \times_f G$ are not always easy to obtain. The effective homology technique and our Algorithm 3 will provide a method computing the desired homology groups of E from finite type resolutions of G and A. In this way, it will not be necessary to determine a finite type resolution for E.

As explained in a previous work of the second author of this paper (see Rubio, 1997), given a 2-cocycle f defining a central extension of a group G by an Abelian group A, one can (explicitly) construct a fibration

$$K(A, 1) \hookrightarrow X \to K(G, 1)$$

where the total space X can be seen as a twisted Cartesian product $K(A, 1) \times_\tau K(G, 1)$. Furthermore, it can be proved that this space is in fact isomorphic to the Eilenberg–MacLane space $K(A \times_f G, 1)$, whose homology groups are those of the group $A \times_f G \cong E$. The simplicial morphisms $\Phi : K(A \times_f G, 1) \to K(A, 1) \times_\tau K(G, 1)$ and $\Phi^{-1} : K(A, 1) \times_\tau K(G, 1) \to K(A \times_f G, 1)$ can be found in Rubio (1997).

On the other hand, in the case where both the fiber and base spaces of the fibration, $K(A, 1)$ and $K(G, 1)$, are objects with effective homology, the effective homology version of a fibration (see Rubio and Sergeraert, 2006) provides the effective homology of the total space $K(A, 1) \times_\tau K(G, 1)$, which in particular will make it possible to obtain the homology groups of E. Finally, if the groups G and A are given with finite type (small) resolutions, our Algorithm 1 provides the necessary effective homologies of $K(G, 1)$ and $K(A, 1)$. We obtain therefore the following algorithm.

This algorithm has also been implemented in Common Lisp and in particular it allows us to determine the homology groups of central extensions of finitely generated Abelian groups. In Section 5.4 we include some examples of calculations.

5. New modules for Kenzo and experimental results

As already mentioned in Section 2.2, Kenzo (Dousson et al., 1999) is a Common Lisp program devoted to Symbolic Computation in Algebraic Topology, developed by Francis Sergeraert and some co-workers. This system makes use of the effective homology method to determine homology groups of complicated spaces and has obtained some results which had never been determined before. In principle Kenzo was not intended to compute homology of groups but we have enhanced this system with a new module dealing with groups, resolutions, and Eilenberg–MacLane spaces (which were already implemented in Kenzo for the particular cases $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$) and we have written

Algorithm 3 Computation of the effective homology of a central extension

Require: groups G and A (A is Abelian) and for both of them free resolutions of finite type F_G and F_A with the corresponding contracting homotopies (or more generally, G and A with effective homology); a 2-cocycle f defining an extension of G by A.

Ensure: $\varepsilon : C_*(K(A \times_f G, 1)) \Longleftarrow\!\!\Longrightarrow E_*$ where E_* is an effective chain complex.

1: $X = \mathit{CentralExtensionFibration}(A, G, f)$

[Compute the fibration associated with a central extension; in other words, $X = K(A, 1) \times_\tau K(G, 1)$. The implementation follows the construction of Rubio (1997)]

2: $\mathit{efhmKG1} = \mathit{Algorithm1}(G, F_G)$

[Apply Algorithm 1 to the group G with the finite type resolution F_G; the effective homology of $K(G, 1)$ is obtained]

3: $\mathit{efhmKA1} = \mathit{Algorithm1}(A, F_A)$

[Apply Algorithm 1 to the group A with the finite type resolution F_A; the effective homology of $K(A, 1)$ is obtained]

4: $\varepsilon' = \mathit{FibrationEfhm}(X, \mathit{efhmKA1}, \mathit{efhmKG1})$

[Use the Kenzo function which computes the effective homology of a fibration from the effective homologies of the two factors, in this case $K(A, 1)$ and $K(G, 1)$]

5: $\Phi = \mathit{CentralExtensionLeftIsomorphism}(A, G, f)$

[Produce the map $\Phi : K(A \times_f G, 1) \to K(A, 1) \times_\tau K(G, 1)$ as explained in Rubio (1997)]

6: $\Phi' = \mathit{CentralExtensionRightIsomorphism}(A, G, f)$

[Produce the map $\Phi^{-1} : K(A, 1) \times_\tau K(G, 1) \to K(A \times_f G, 1)$ as explained in Rubio (1997)]

7: $\varepsilon = \mathit{Composition}(\varepsilon', \Phi, \Phi')$

[Compose the equivalence $\varepsilon' : C_*(X) \Longleftarrow\!\!\Longrightarrow E_*$ with the isomorphism $K(A \times_f G, 1) \cong K(A, 1) \times_\tau K(G, 1) = X$ given by the maps Φ and Φ'. We obtain an equivalence $C_*(K(A \times_f G, 1)) \Longleftarrow\!\!\Longrightarrow E_*$]

---

the corresponding programs implementing Algorithm 1, which produces the effective homology of the space $K(G, 1)$ given a finite type resolution for the group G. Since the construction of a finite type resolution for a group is not always an easy task, we have allowed Kenzo to connect with the GAP package HAP and obtain a resolution from it. Furthermore, as already announced, we provide programs which determine the homology groups of some 2-types and central extensions. All the programs presented in this section can be found in http://www.unirioja.es/cu/anromero/research2.html.

5.1. Interoperating with GAP

GAP (The GAP Group, 2008) is a system for computational discrete algebra with particular emphasis on Computational Group Theory. In our work we consider the HAP homological algebra library (Ellis, 2009) for use with GAP; it was written by Graham Ellis and is still under development. The initial focus of HAP is on computations related to the cohomology of groups. A range of finite and infinite groups are handled, with particular emphasis on integral coefficients. It also contains some functions for the integral (co)homology of: Lie rings, Leibniz rings, cat-1-groups and digital topological spaces. And in particular, HAP allows one to obtain (small) resolutions of many different groups, although it does not implement the Bar resolution nor Eilenberg–MacLane spaces $K(G, 1)$.

As presented in Romero et al. (2009), in a joint work with Graham Ellis, we have developed a new module making it possible to export resolutions from HAP and import them into Kenzo. As interchange language we have used OpenMath (The OpenMath Society, 2004), an XML standard for representing mathematical objects. There exist OpenMath translators from several Computer Algebra systems, and in particular GAP includes a package (Solomon and Costantini, 2009) which produces OpenMath code from some GAP elements (lists, groups...). We have extended this package in order to represent resolutions, including a new GAP function which provides the OpenMath code of these elements. In Romero et al. (2009) a detailed description of our OpenMath representation for resolutions can be found.

The communication between HAP and Kenzo is done as follows: given a group G, the system HAP produces a $\mathbb{Z}G$-resolution (including the homotopy operator). This resolution can be automatically translated to OpenMath code thanks to our new function added to the OpenMath package for GAP, and this code is written in a text file. Then Kenzo imports the file (and translates the OpenMath code into Kenzo elements thanks to the corresponding parser) so that one can use the resolution directly in Kenzo without the need of programming it in Common Lisp. Once the resolution is defined in Kenzo, we can use it to determine the effective homology of $K(G, 1)$ as explained in Section 3. In this way, if the construction of a resolution for a group G is complicated, we can avoid programming it by hand; it will be automatically implemented in Kenzo by obtaining it from HAP.

5.2. Computations with $K(G, n)$'s

Let G be a group and let us suppose that Kenzo knows a resolution for it (for some particular groups the system can construct it directly; for others, it could obtain it from HAP). Making use of our main Algorithm 1 one can determine the effective homology of the simplicial Abelian group $K(G, 1)$, and in particular, compute its homology groups.

Let us consider, for instance, $G = C_5$, the cyclic group of order 5. As already seen in Section 2.1, in this case it is not difficult to construct a small resolution F_* of G. The group can be built in Kenzo with the function `cyclicGroup`; the program computes automatically the well-known resolution for G (coded as a reduction $F_* \Rrightarrow \mathbb{Z}$) and stores it in the slot `resolution` of the group.

```
> (setf C5 (cyclicGroup 5))
[K1 Abelian-Group]
> (resolution C5)
[K10 Reduction K2 => K5]
```

This resolution is then used by our programs, following Algorithm 1, to determine the effective homology of $K(G, 1)$. The corresponding homotopy equivalence is spontaneously computed and stored in the slot `efhm`. In this way one can obtain the homology groups of this Eilenberg–MacLane space.

```
> (setf KC51 (K-G-1 C5))
[K11 Abelian-Simplicial-Group]
> (efhm KC51)
[K50 Homotopy-Equivalence K11 <= K40 => K31]
> (homology KC51 0 5)
Homology in dimension 0 :
Component Z
---done---
Homology in dimension 1 :
Component Z/5Z
---done---
Homology in dimension 2 :
---done---
Homology in dimension 3 :
Component Z/5Z
---done---
Homology in dimension 4 :
---done---
```

Moreover, since $G = C_5$ is Abelian, $K(G, 1)$ is a simplicial Abelian group, and we can apply the classifying space constructor B (already implemented in Kenzo) which gives us $B(K(G, 1)) = K(G, 2)$, a new simplicial Abelian group *with effective homology*.

```
> (setf KC52 (classifying-space KC51))
[K51 Abelian-Simplicial-Group]
> (efhm KC52)
[K190 Homotopy-Equivalence K51 <= K180 => K176]
> (homology KC52 3 6)
Homology in dimension 3 :
---done---
Homology in dimension 4 :
Component Z/5Z
---done---
Homology in dimension 5 :
---done---
```

Iterating the process, $K(G, n) = B(K(G, n-1))$ has effective homology for every $n \in \mathbb{N}$. Our new Kenzo function `K-Cm-n` allows us to directly construct $K(C_m, n)$; we observe that the slot `efhm` is automatically created.

```
> (setf KC42 (K-Cm-n 4 2))
[K204 Abelian-Simplicial-Group]
> (efhm KC42)
[K378 Homotopy-Equivalence K204 <= K368 => K364]
> (homology KC42 4)
Homology in dimension 4 :
Component Z/8Z
---done---
```

The construction of Eilenberg–MacLane spaces $K(G, n)$ for every cyclic group $G = C_m$ (with the corresponding effective homology) is an important enhancing of the Kenzo system, which previously was only able to deal with cases $G = \mathbb{Z}$ and $G = \mathbb{Z}/2\mathbb{Z} = C_2$. The homology groups obtained for some of these new spaces have been tested comparing them with the results shown in Alain Clément's thesis (Clément, 2002). It is important to stress that Clément's tables, computed by using a direct algorithm created by Henry Cartan (see Cartan, 1954–1955), contain much more groups than those that can be computed with our programs in its current state. Nevertheless, Clément's tables give only the homology groups of the spaces, while our approach provides the *effective homology*. Our information is much more complete, giving access to geometrical generators of the homology and, in fact, fully solving the *homological problem* of these groups (see Rubio and Sergeraert, 2006). And, perhaps more important, our programs allow us to continue working with the corresponding $K(G, n)$, to produce new interesting topological spaces and to determine their homology groups. The information computed by Clément is not enough to carry out this further work.

The same technique explained for cyclic groups can be used to compute the effective homology of spaces $K(G, n)$, where G is a finitely generated Abelian group. In this case, the homology of $K(G, n)$ is one of the main ingredients to compute homotopy groups of spaces (see Rubio and Sergeraert (2002) and Rubio and Sergeraert (2006) for details).

5.3. *An example of homology of a* 2*-type*

Let us consider now $G = C_3$ the cyclic group of order 3. Let $A = \mathbb{Z}/3\mathbb{Z}$ be the Abelian group of three elements with trivial G-action (the groups G and A are in fact isomorphic; different notations are used to distinguish multiplicative and additive operations). Then the third cohomology group of G with coefficients in A is

$$H^3(G, A) = \mathbb{Z}/3\mathbb{Z}.$$

The classes $[f]$ of this cohomology group correspond to 2-types with $\pi_1 = G, \pi_2 = A$, and one such 2-type X can be seen as a twisted Cartesian product $X = K(A, 2) \times_f K(G, 1)$. It can be constructed by Kenzo in the following way:

```
> (setf KC31 (K-Cm-n 3 1))
[K380 Abelian-Simplicial-Group]
> (setf chml-clss (chml-clss KC31 3))
[K427 Cohomology-Class on K407 of degree 3]
> (setf tau (zp-whitehead 3 KC31 chml-clss))
```

```
[K442 Fibration K380 -> K428]
> (setf X (fibration-total tau))
[K448 Kan-Simplicial-Set]
```

As seen in the previous section, $K(A, 2)$ and $K(G, 1)$ are objects with effective homology. From the two equivalences $C_*(K(A, 2)) \Longleftarrow\!\!\!\Rightarrow E_*$ and $C_*(K(G, 1)) \Longleftarrow\!\!\!\Rightarrow E'_*$, Kenzo knows how to construct the effective homology of the twisted Cartesian product $X = K(A, 2) \times_f K(G, 1)$, which allows one to determine its homology groups.

```
> (efhm X)
[K660 Homotopy-Equivalence K448 <= K650 => K646]
> (homology X 5)
Homology in dimension 5 :
component Z/3Z
---done---
```

In the same way, the homology groups of $X = K(A, 2) \times_f K(G, 1)$ can be determined for all groups A and G with given (small) resolutions and cohomology classes $[f]$ in $H^3(G, A)$. Up to now, only the homology of finitely presented groups has been considered, restricting the kind of 2-types that can be studied with our methods, since only spaces with Abelian fundamental group would be in its scope. The range of groups which can be considered is considerably enlarged with the central extension constructions, as explained in the following subsection.

5.4. Central extensions

Let us introduce an interesting example of central extension extracted from Leary (1991). Let E be the group defined by the following presentation:

$$E = \langle x, y, z | x^p = y^p = z^{p^{n-2}} = [x, z] = [y, z] = 1; [x, y] = z^{p^{n-3}} \rangle.$$

This group can be seen as a central extension of the groups

$$A = \langle z | z^{p^{n-2}} = 1 \rangle,$$

isomorphic to the cyclic group with p^{n-2} elements, and

$$G = \langle x, y | x^p = y^p = [x, y] = 1 \rangle,$$

which is the direct sum of two cyclic groups of cardinality p. A 2-cocycle of the extension is defined by

$$f(x^{p_1} y^{q_1}, x^{p_2} y^{q_2}) = z^{q_1 p_2 (p-1) p^{n-3}}.$$

As already explained, the group $A \cong C_{p^{n-2}}$ has effective homology. On the other hand, the effective homology of $G \cong C_p \oplus C_p$ can be easily obtained from the effective homology of the cyclic group C_p (a direct sum of two groups can in fact be considered as a particular case of central extension, where the 2-cocycle is trivial, so that its effective homology can be computed given the effective homologies of the two factors). In this way, Algorithm 3 can be applied to obtain the effective homology of E and then compute its homology groups.

Let us consider, for instance, $p = 3$ and $n = 4$. The following Kenzo instructions construct the group E.

```
> (progn
     (setf p 3 n 4)
     (setf A (cyclicGroup (expt  p (- n 2))))
     (setf G (gr-crts-prdc (cyclicGroup p) (cyclicGroup p)))
     (setf cocycle #'(lambda (crpr1 crpr2)
                    (with-grcrpr (x1 y1) crpr1
                      (with-grcrpr (x2 y2) crpr2
                        (mod (* y1 x2 (1- p) (expt p (- n 3))) (expt p (- n 2)))))))
     (setf E (gr-cntr-extn A G cocycle)))
[K663 Group]
```

The spaces $K(A, 1)$ and $K(G, 1)$ can be constructed with the function K-G-1; both of them are Abelian simplicial groups with effective homology.

```
> (setf KA1 (K-G-1 A))
[K664 Abelian-Simplicial-Group]
> (efhm KA1)
[K710 Homotopy-Equivalence K664 <= K700 => K691]
> (setf KG1 (K-G-1 G))
[K711 Abelian-Simplicial-Group]
> (efhm KG1)
[K775 Homotopy-Equivalence K711 <= K765 => K745]
```

Given the effective homologies of $K(A, 1)$ and $K(G, 1)$, our Algorithm 3 returns the effective homology of $K(E, 1)$, which is then stored in the corresponding slot efhm.

```
> (setf KE1 (K-G-1 E))
[K776 Simplicial-Group]
> (efhm KE1)
[K884 Homotopy-Equivalence K776 <= K870 => K866]
> (homology KE1 0 5)
Homology in dimension 0 :
Component Z
---done---
Homology in dimension 1 :
Component Z/3Z
Component Z/3Z
Component Z/3Z
---done---
Homology in dimension 2 :
Component Z/3Z
Component Z/3Z
---done---
Homology in dimension 3 :
Component Z/9Z
Component Z/3Z
Component Z/3Z
Component Z/3Z
---done---
Homology in dimension 4 :
Component Z/3Z
Component Z/3Z
Component Z/3Z
---done---
```

In this way, one can determine the homology groups of the central extension E. The computations obtained by our programs have been compared with Leary's theoretical results for different values of p and n; the same groups have been obtained by both methods. We can repeat here the discussion made at the end of Section 5.2 with respect to Clément's computations for $H_*(K(G, n))$: Leary's methods give more groups than our techniques, but with less information. In particular, our results allow us to compute the homology of 2-types whose fundamental groups are central extensions, while Leary's groups are not enough for this task.

6. The inverse problem: recovering a resolution from the effective homology of a group

In Section 3 we have presented an algorithm which, given a group G with a free finite type resolution F_*, constructs the effective homology of the simplicial Abelian group $K(G, 1)$. This effective homology allows one to determine the homology groups of G and, as seen in Sections 4 and 5, makes it possible to use the space $K(G, 1)$ as initial data for some constructions in Algebraic Topology, computing in this way homology groups of other interesting objects.

We consider now the inverse problem: let G be a group such that an equivalence

$$C_*(K(G, 1)) \Lleftarrow\!\!\Rrightarrow E_*$$

is given, E_* being a finite type chain complex of Abelian groups. Is it possible to obtain a finite type free resolution for the group G? It seems, in principle, that the answer should be negative in the general case; since no condition is imposed on the arrows, they surely do not respect the G-action and, thus, it would not be possible to build a $\mathbb{Z}G$-resolution. We have proved, however, that supposing that some additional conditions for the given chain equivalence are satisfied, one can construct the desired resolution with the corresponding contracting homotopy.

The algorithm we have developed makes use of the *Basic Perturbation Lemma* (BPL), one of the fundamental results in Constructive Algebraic Topology. The general idea of this theorem is that given a reduction $\rho = (f, g, h) : C_* \Rightarrow\!\!\!\!\Rightarrow D_*$, if we modify the initial differential d_C of the *big* complex C_* by adding some *perturbation*, then it is possible to perturb the differential d_D in the *small* chain complex D_* so that we obtain a new reduction between the perturbed complexes. But the result is not always true, a necessary condition must be satisfied: the composite function $h \circ d_C$ must be *locally nilpotent*. An endomorphism $\alpha : C_* \to C_*$ is locally nilpotent if for every $x \in C_*$ there exists $m \in \mathbb{N}$ such that $\alpha^m(x) = 0$. The condition of local nilpotency ensures the convergence of a formal series used in the BPL to build the perturbed differential on the small complex (and, in fact, to construct also all the arrows defining the new reduction between the perturbed complexes). The Basic Perturbation Lemma was discovered by Shih (1962), and then generalized by Brown (1967). In its modern form it was formulated by Gugenheim (1972) and its essential use in Kenzo has been documented in Rubio and Sergeraert (2006).

Let us suppose that G is a group and we have an equivalence $C_*(K(G,1)) \overset{\rho_1}{\Leftarrow\!\!\!\!\Leftarrow} D_* \overset{\rho_2}{\Rightarrow\!\!\!\!\Rightarrow} E_*$, where $\rho_1 = (f_1, g_1, h_1)$, $\rho_2 = (f_2, g_2, h_2)$, E_* is an effective chain complex, and the composition $h_2 g_1 \partial_n f_1$ is locally nilpotent (∂_n is the face of index n over the elements of $K(G,1)_n$, which can be extended to $C_n(K(G,1))$). We want to construct a free resolution F_* for G of finite type with a contracting homotopy h.

Let us start by considering the universal fibration $K(G,0) \to K(G,0) \times_\tau K(G,1) \to K(G,1)$ (see May, 1967). The total space $K(G,0) \times_\tau K(G,1)$ is acyclic and one can construct a reduction

$$C_*(K(G,0) \times_\tau K(G,1)) \Rightarrow\!\!\!\!\Rightarrow \mathbb{Z}$$

where $\mathbb{Z}$ represents the chain complex (of Abelian groups) $C_*(\mathbb{Z},0)$ with a unique non-null component $\mathbb{Z}$ in dimension 0.

On the other hand, one can consider the Eilenberg–Zilber theorem (Eilenberg and Zilber, 1953), which relates the chain complex of a Cartesian product with the tensor product of the chain complexes of the two components, and allows one to build a reduction

$$C_*(K(G,0) \times K(G,1)) \Rightarrow\!\!\!\!\Rightarrow C_*(K(G,0)) \otimes C_*(K(G,1)).$$

Applying the BPL (it can be proved that the nilpotence condition is satisfied) we obtain a *perturbed* reduction (this is in fact the *twisted* Eilenberg–Zilber theorem, see May (1967))

$$C_*(K(G,0) \times_\tau K(G,1)) \Rightarrow\!\!\!\!\Rightarrow C_*(K(G,0)) \otimes_t C_*(K(G,1))$$

where $C_*(K(G,0)) \otimes_t C_*(K(G,1))$ is a chain complex with the same underlying graded module as the tensor product $C_*(K(G,0)) \otimes C_*(K(G,1))$, but its differential is modified to take account of the twisting operator τ.

Now, from the given equivalence $C_*(K(G,1)) \Leftarrow\!\!\!\!\Leftarrow D_* \Rightarrow\!\!\!\!\Rightarrow E_*$, it is not difficult to construct a new equivalence

$$C_*(K(G,0)) \otimes C_*(K(G,1)) \Leftarrow\!\!\!\!\Leftarrow C_*(K(G,0)) \otimes D_* \Rightarrow\!\!\!\!\Rightarrow C_*(K(G,0)) \otimes E_*$$

and applying again the BPL, provided that $h_2 g_1 \partial_n f_1$ is locally nilpotent, we obtain

$$C_*(K(G,0)) \otimes_t C_*(K(G,1)) \Leftarrow\!\!\!\!\Leftarrow C_*(K(G,0)) \otimes_t D_* \Rightarrow\!\!\!\!\Rightarrow C_*(K(G,0)) \otimes_t E_*.$$

Finally, one can observe that $C_*(K(G,0)) \equiv \mathbb{Z}G$ and composing the reductions $C_*(K(G,0) \times_\tau K(G,1)) \Rightarrow\!\!\!\!\Rightarrow \mathbb{Z}$ and $C_*(K(G,0) \times_\tau K(G,1)) \Rightarrow\!\!\!\!\Rightarrow C_*(K(G,0)) \otimes_t C_*(K(G,1))$ with the last equivalence, we get a contracting homotopy on $\mathbb{Z}G \otimes_t E_*$ which is a resolution for G.

This construction can be formalized by means of our Algorithm 4.

Algorithm 4 *Inverse* algorithm

Require: a group G and a (strong) chain equivalence $\varepsilon : C_*(K(G,1)) \overset{\rho_1}{\Lleftarrow} D \overset{\rho_2}{\Rrightarrow} E$, where $\rho_1 = (f_1, g_1, h_1)$, $\rho_2 = (f_2, g_2, h_2)$, E is an effective chain complex, and the composition $h_2 g_1 \partial_n f_1$ is locally nilpotent.

Ensure: F is a free resolution for G of finite type with a contracting homotopy h.

1: $X = \mathit{UniversalFibration}(K(G,0))$

[Construct the acyclic space $K(G,0) \times_\tau K(G,1)$ (May, 1967). This is a new function]

2: $\tau = \mathit{UniversalFibrationPerturbation}(K(G,0))$

[Compute the perturbation τ in $K(G,0) \times_\tau K(G,1)$]

3: $\rho 1 = \mathit{UniversalFibrationReduction}(K(G,0))$

[Construct the reduction $K(G,0) \times_\tau K(G,1) \Rrightarrow \mathbb{Z}$. New function]

4: $\rho 2 = \mathit{TwistedEilenbergZilber}(K(G,0), K(G,1), \tau)$

[Kenzo function which computes the reduction $C_*(K(G,0) \times_\tau K(G,1)) \Rrightarrow C_*(K(G,0)) \otimes_t C_*(K(G,1))$. It makes use of the Basic Perturbation Lemma as explained before]

5: $C = \mathit{BottomChainComplex}(\rho 2)$

[Kenzo returns the bottom chain complex in a reduction; in our case it is the space $C_*(K(G,0)) \otimes_t C_*(K(G,1))$]

6: $\rho 3 = \mathit{CompositionAsReduction}(\rho 2, \rho 1)$

[Since the bottom chain complex in the reduction $\rho 1$ is $\mathbb{Z}$, one can compute a new reduction $\rho 3 : C_*(K(G,0)) \otimes_t C_*(K(G,1)) \Rrightarrow \mathbb{Z}$. This is a new function not included in Kenzo]

7: $\varepsilon 1 = \mathit{TwistedTensorProductEfhm}(C, \mathit{TrivialEfhm}(K(G,0)), \varepsilon)$

[Kenzo knows how to determine the effective homology of a (twisted) tensor product from the effective homologies of the two components. In our case, $K(G,0)$ has trivial effective homology and the effective homology of $K(G,1)$ is the given equivalence ε. In this way the new object $\varepsilon 1$ is an equivalence $\varepsilon 1 : C_*(K(G,0)) \otimes_t C_*(K(G,1)) \Lleftarrow\!\!\Rrightarrow C_*(K(G,0)) \otimes_t E$]

8: $\rho 4 = \mathit{CompositionAsReduction}(\rho 3, \varepsilon 1)$

[The composition of the reduction $\rho 3 : C_*(K(G,0) \otimes_t K(G,1)) \Rrightarrow \mathbb{Z}$ with the equivalence $\varepsilon 1 : C_*(K(G,0)) \otimes_t C_*(K(G,1)) \Lleftarrow\!\!\Rrightarrow C_*(K(G,0)) \otimes_t E$ leads to a reduction $\rho 4 : C_*(K(G,0)) \otimes_t E \Rrightarrow \mathbb{Z}$]

9: $F = \mathit{TopChainComplex}(\rho 4)$

[The top chain complex in the reduction $\rho 4$ is $C_*(K(G,0)) \otimes_t E \equiv \mathbb{Z}G \otimes_t E$ which can be seen as a chain complex of $\mathbb{Z}G$-modules]

10: $h = \mathit{hMorphism}(\rho 4)$

[The component h in the reduction $\rho 4$ is a contracting homotopy for F]

As a first possible application of this algorithm, one can consider the integer group $G = \mathbb{Z}$ and the well known effective homology of $K(\mathbb{Z},1)$, given by a reduction $C_*(K(\mathbb{Z},1)) \Rrightarrow C_*(S^1)$, where S^1 denotes a simplicial model for the sphere of dimension 1. In this case it is not difficult to prove the desired condition, $h_2 g_1 \partial_n f_1$ is locally nilpotent, and therefore one can construct a finite type resolution for $G = \mathbb{Z}$, as a reduction $\mathbb{Z}G \otimes_t C_*(S^1) \Rrightarrow \mathbb{Z}$.

A natural question which appears in this context is whether, given a group G and an equivalence $C_*(K(G,1)) \Lleftarrow\!\!\Rrightarrow E_*$ which has been obtained by means of our Algorithm 1 from a finite type resolution F_*, the necessary condition of $h_2 g_1 \partial_n f_1$ being locally nilpotent is satisfied or not. The answer is positive if the group G and the resolution F_* satisfy some particular properties. More concretely, we suppose that a *norm* is defined on G and it can be extended to F_* in the following *natural* way.

Definition 13. Let G be a group. A *norm* for G is a map $\|.\| : G \to \mathbb{N}$ such that

- $\|g\| > 0$ for each $g \in G$ and $\|g\| = 0$ if and only if $g = 1$;
- $\|g_1 g_2\| \leq \|g_1\| + \|g_2\|$ for all $g_1, g_2 \in G$.

We suppose that the resolution is reduced ($F_0 = \mathbb{Z}G$) and define $\|.\| : F_0 = \mathbb{Z}G \to \mathbb{N}$ as $\|\sum \lambda_i g_i\| = \max\{\|g_i\|\}$. We say that the norm is *compatible* with the resolution F_* if for each $n \geq 1$ we can also define $\|.\| : F_n \to \mathbb{N}$ such that

- $\|(g, z)\| = \|g\| + \|z\|$ for all $g \in G$ and z a generator of F_n;
- there exists $i_n \in \mathbb{Z}$ such that $\|h_n(x)\| \leq \|x\| - i_n$ and $\|d_{n+1}(x')\| \leq \|x'\| + i_n$ for all $x \in F_n, x' \in F_{n+1}$.

The last condition introduces a control measure on the contracting homotopy h, with respect to the structure of the group, allowing us (as shown in the following result) to ensure in this case the convergence of the Basic Perturbation Lemma. Examples of resolutions with this kind of norm are the Bar resolution, the canonical small resolution for $G = \mathbb{Z}$ and, for instance, the small resolutions for cyclic groups introduced in Section 2.1.

Theorem 14. *Let G be a group and F_* a free resolution for G with contracting homotopy h. Let us suppose that G is provided with a norm $\|.\| : G \to \mathbb{N}$ which is compatible with the resolution. Then the effective homology of $K(G, 1)$ obtained from F_* by our Algorithm 1 satisfies the necessary condition of $h_2 g_1 \partial_n f_1$ being locally nilpotent, and therefore it is possible to construct a (new) free finite type resolution for G.*

Proof. Let us recall that the effective homology of $K(G, 1)$ given by our Algorithm 1 is given by an equivalence:

$$C_*(K(G,1)) \overset{\rho'}{\Lleftarrow} \mathrm{Cylinder}(f)_* \overset{\rho}{\Rrightarrow} E_*$$

obtained from an equivalence in the classical sense:

$$\begin{tikzcd} C_*(K(G,1)) \arrow[loop above, "k'"] \arrow[r, shift left, "f"] & E_* \arrow[l, shift left, "g"] \arrow[loop right, "k"] \end{tikzcd}$$

Taking into account the definition of the different components of the reductions ρ and ρ' (included in Romero et al. (2009)), one can observe that in this case the composition $f_1 h_2 g_1$ is in fact the morphism $k' : C_*(K(G, 1)) \to C_{*+1}(K(G, 1))$, and therefore the condition of $h_2 g_1 \partial_n f_1$ being locally nilpotent is equivalent to $\partial_n k'$ being locally nilpotent.

We recall too that the morphism k' is obtained by tensorizing the map $k' : \mathrm{Bar}_*(G) \to \mathrm{Bar}_{*+1}(G)$, which, as explained in Romero et al. (2009), is defined over the generators u^n_α of $B_n \equiv \mathrm{Bar}_n(G)$ as

$$k'_0(u^0_\alpha) = h'(u^0_\alpha) - h'gf(u^0_\alpha)$$
$$k'_n(u^n_\alpha) = h'(\mathrm{Id} - gf - k'_{n-1}d(u^n_\alpha))$$

where h' is the contracting homotopy of the Bar resolution $\mathrm{Bar}_*(G) \equiv B_*$.

The norm $\|.\| : G \to \mathbb{N}$ can be extended to B_* as follows: $\|.\| : B_* \to \mathbb{N}$ given by $\|g.[g_1|\cdots|g_n]\| = \|g\| + \sum_j \|g_j\|$ and $\|\sum_i \lambda_i(g^i.[g^i_1|\cdots|g^i_n])\| = \max_i\{\|g^i\| + \sum_j \|g^i_j\|$. From the definitions of the differential map d in B_* and the contracting homotopy h' (Brown, 1982) one can easily observe that both maps preserve the norm $\|.\|$. Furthermore, one can prove in a recursive way that the composition gf preserves the norm $\|.\|$ too, so that using an inductive reasoning one has that k' preserves $\|.\|$ too. Finally it is not difficult to observe that ∂_n decreases $\|.\|$ at least in one unit, and then the composition $\partial_n k'$ is locally nilpotent, as desired. $\square$

The new resolution F'_* given by Algorithm 4 has in this case the same structural components as the initial resolution F_*; in other words, $F_n = F'_n$ for all $n \in \mathbb{N}$. However, the differential and contracting homotopy maps could be different.

A final example of application of Algorithm 4 and Theorem 14 is the following one.

Theorem 15. *Let G, G' be groups with free resolutions F_* and F'_* (with contracting homotopies h and h' respectively). Let us suppose that there exists norms on G and G' which are compatible with the corresponding resolutions. Then the effective homology of $G \oplus G'$ (obtained from those of G and G' as a particular case of central extension) satisfies that $h_2 g_1 \partial_n f_1$ is locally nilpotent, so that it is also possible to determine a resolution for the direct sum $G \oplus G'$.*

Again, we know the graded part of the output resolution, but it is still unknown if the differential and contracting homotopy constructed have some good geometrical behavior.

7. Conclusions and further work

In this paper we have defended this proposal: the *geometric way* for computing group homology can be sensible and fruitful. To this aim, we have worked inside Sergeraert's *effective homology*, and added packages devoted to group homology in Sergeraert's Kenzo system.

In their current state our methods have a performance penalty when compared with the more standard algebraic approach (based on *resolutions*). Nevertheless, this claim is only true for computations reachable by previous means. Furthermore, what is more important, to get available the homology of a group G through an Eilenberg–MacLane space $K(G, 1)$ with *effective homology* allows us to use that space for further topological constructions. The poorer performance is therefore balanced with the richer information we get.

The paper illustrates our approach with concrete computer experiments for general Eilenberg–MacLane spaces $K(G, n)$, for central extensions of groups and for 2-types. In the first two applications, the computer results have been compared with previously published works. In the case of the homology groups of 2-types computed with Kenzo, no comparison is possible, because no other source of results is known by us.

Furthermore we have explored the problem of computing a resolution of G from the effective homology of $K(G, 1)$, obtaining some partial algorithmic results which have not yet been implemented.

Some of the lines opened in this paper have not been completely closed, signaling clear lines of further work. Starting from the end, the scope of the methods to compute resolutions from effective homologies should be enlarged, and more examples should be worked out. In particular, a comparison between the initial resolutions and the ones constructed in the case of *normed* groups should be undertaken, trying to elucidate if our output resolution is better in some geometrical sense.

In the area of 2-types, the more important task would be to extend our approach to 2-types with non-trivial action of the fundamental group. The main obstacle here is to obtain a fibration expressed as a twisted Cartesian product, in order to be able to apply the previous Kenzo infrastructure.

Another challenge consists in trying to get better algorithms from the efficiency point of view, in such a way that our programs can compete with other approaches. In particular, we should improve the algorithm to construct the effective homology from a resolution, at least in certain cases, to obtain execution times closer to those of the source system, HAP. For finitely generated Abelian groups (which are the building blocks to start many of our constructions) a more direct approach, much more efficient, could be extracted from the original papers by Eilenberg and MacLane (1953, 1954a,b).

Finally, the application of our methods for wider classes of groups (for instance, extensions beyond the central extensions dealt with in this paper) is likely possible and surely an interesting research topic.

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References

Brown, K.S., 1982. Cohomology of Groups. Springer-Verlag.

Brown, R., 1967. The twisted Eilenberg-Zilber theorem. Celebrazioni Archimedi de Secolo XX, Simposio di Topologia, 34–37.

Cartan, H., 1954–1955. Algèbres d'Eilenberg–MacLane et homotopie. Séminaire H. Cartan, École Normal Supérieure, Paris, exposés 2–11.

Clément, A., 2002. Integral cohomology of finite postnikov towers. Ph.D. Thesis, Université de Lausanne.

Dousson, X., Rubio, J., Sergeraert, F., Siret, Y., 1999. The Kenzo program. Institut Fourier, Grenoble. http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo.

Eilenberg, S., MacLane, S., 1953. On the groups $H(\pi, n)$, I. Annals of Mathematics 58, 55–106.

Eilenberg, S., MacLane, S., 1954a. On the groups $H(\pi, n)$, II. Annals of Mathematics 70, 49–139.

Eilenberg, S., MacLane, S., 1954b. On the groups $H(\pi, n)$, III. Annals of Mathematics 60 (3), 513–557.

Eilenberg, S., Zilber, J.A., 1953. On products of complexes. American Journal of Mathematics 75, 200–204.

Ellis, G.J., 1992. Homology of 2-types. Journal of the London Mathematical Society 46 (2), 1–27.

Ellis, G.J., 2009. HAP package for GAP. http://www.gap-system.org/Packages/hap.html.

Gugenheim, V.K.A.M., 1972. On the chain complex of a fibration. Illinois 16 (3), 398–414.

Leary, I., 1991. The integral cohomology rings of some p-groups. Mathematical Proceedings of the Cambridge Philosophical Society 110, 25–32.

MacLane, S., 1963. Homology. In: Grundleren der Mathematischen Wissenschaften, vol. 114. Springer.

May, J.P., 1967. Simplicial Objects in Algebraic Topology. In: Van Nostrand Mathematical Studies, vol. 11.

Romero, A., Ellis, G., Rubio, J., 2009. Interoperating between Computer Algebra systems: computing homology of groups with Kenzo and GAP. In: Proceedings ISSAC 2009. ACM, New York, pp. 303–310.

Rubio, J., 1997. Integrating functional programming and symbolic computation. Mathematics and Computers in Simulation 44, 505–511.

Rubio, J., Sergeraert, F., 2002. Constructive algebraic topology. Bulletin des Sciences Mathématiques 126 (5), 389–412.

Rubio, J., Sergeraert, F., 2006. Constructive Homological Algebra and Applications. In: Lecture Notes Summer School on Mathematics, Algorithms, and Proofs. University of Genova. http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Genova-Lecture-Notes.pdf.

Sergeraert, F., 1994. The computability problem in Algebraic Topology. Advances in Mathematics 104 (1), 1–29.

Shih, W., 1962. Homologie des espaces fibrés. Publications mathématiques de l'Institut des Hautes Études Scientifiques 13.

Solomon, A., Costantini, M., 2009. OpenMath package for GAP. http://www.gap-system.org/Packages/openmath.html.

The GAP Group, 2008. GAP – Groups, Algorithms, and Programming, Version 4.4.12. http://www.gap-system.org.

The OpenMath Society, 2004. OpenMath. http://www.openmath.org/.

Weibel, C.A., 1994. An introduction to homological algebra. In: Cambridge Studies in Advanced Mathematics, vol. 38. Cambridge University Press.