SAFE DOMAIN FOR PROJECTILE TRAJECTORIES IN A MEDIUM WITH QUADRATIC DRAG FORCE

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Abstract. In this paper we provide an expression for the border of the safe domain associated with projectile trajectories in a medium with a quadratic drag force. The curve defining the safe domain is given in parametric coordinates involving some integrals. These integrals have to be evaluated numerically after solving an integral equation. We show that not all the trajectories are necessary to obtain the safe domain.

1. Introduction and main result

The main target of this paper is the analysis of the safe domain for projectiles subject to the action of a quadratic drag force. We say that D is the safe domain of a family of trajectories $\{T_\alpha\}_{\alpha\in\Lambda}$, if any object inside D cannot be reached by a projectile describing any trajectory T_α. Let us suppose that the projectiles are set off from a point O, that we consider the origin of a coordinate system, with an initial velocity whose modulus v_0 is fixed and forming an angle $\alpha \in (0,\pi)$ with the horizontal axes. Then the safe domain will be the complement in the upper half-plane of the region covered by all the trajectories, and this region will be limited by the envelope of the family of trajectories. An analysis of safe domains in very different settings can be seen in [4].

The first study on the trajectory of a projectile was performed by Galileo Galilei in his book *Discorsi e Dimostrazioni Matematiche intorno à Due Nuove Scienze attenenti alla Mecanica & i Movimienti Locali* (1638). He analysed horizontal launches under the influence of gravity force $F_g = -mg$. He determined the parabolic trajectory of the projectile in this case but the solution was acceptable only in a vacumm.

The research for projectiles launched with an angle $\alpha \in (0,\pi)$ was developed by E. Torricelli in his work *Opera Geométrica* (1644). He proved that all the trajectories were given by the family of parabolas

$$y = x\left(\tan\alpha - \frac{g}{2v_0^2\cos^2\alpha}x\right).$$

Moreover he showed that all the trajectories were inside the parabola

$$y = \frac{g}{2v_0^2}\left(\left(\frac{v_0^2}{g}\right)^2 - x^2\right),$$

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which is know as the safety parabola, see Figure 1. So the safe domain in this case is the exterior of the safety parabola.

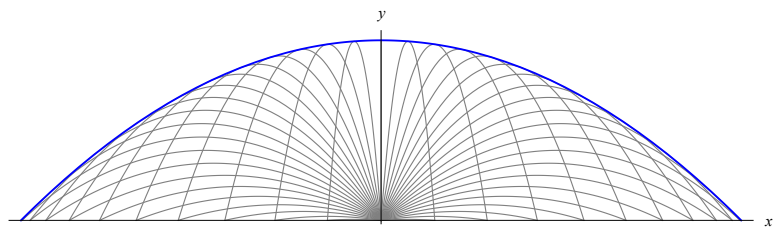

FIGURE 1. Some projectile trajectories in classical parabolic motion and the safety parabola.

It is known that a projectile moving in a medium is subject to a drag force $\mathbf{F}_d$, which opposes the movement. In the case of a sphere, which we can suppose a projectile to be, drag force acts in the opposite direction to projectile velocity $\mathbf{v}$ and it depends on the dimensionless Reynolds number Re. When $\mathrm{Re} < 1$, the modulus of the drag force is proportional to speed, namely $F_d = kv$. This case is named the linear drag model and $\mathbf{F}_d = -k\mathbf{v}$. Sometimes this force is known as Stokes flow.

For the linear drag model, the projectile trajectories and safe domain are well-known and we can find them, for instance, in [3]. In this case the trajectories are given by

$$y = \left(\tan\alpha + \frac{gm}{kv_0\cos\alpha}\right)x + \frac{gm^2}{k^2}\log\left(1 - \frac{kx}{mv_0\cos\alpha}\right),$$

and the envelope by

$$y = \frac{v_0^2}{g} + \frac{(g^2x^2 + v_0^4)(mg^2 - k^2v_0^2)}{k^2gv_0^4 + kg^2v_0\sqrt{(g^2m^2 - k^2v_0^2)x^2 + m^2v_0^4}} + \frac{gm^2}{k^2}\log\left(\frac{-g^2kx^2 + gv_0\sqrt{(g^2m^2 - k^2v_0^2)x^2 + m^2v_0^4)}}{kv_0^4 + gv_0\sqrt{(g^2m^2 - k^2v_0^2)x^2 + m^2v_0^4)}}\right).$$

We can see the family of trajectories and the envelope delimiting the safe domain for this case in Figure 2.

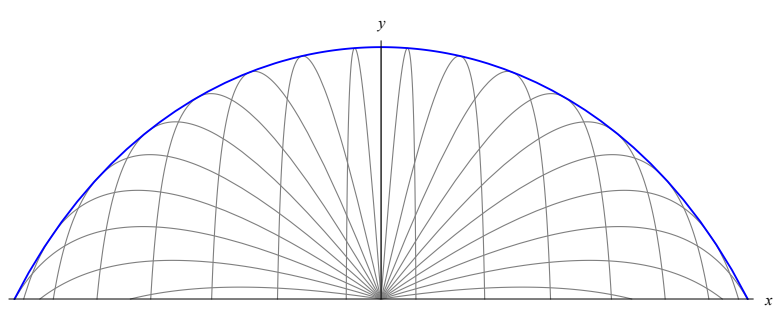

FIGURE 2. The projectile trajectories and the safe domain when we consider a linear drag model.

On the other hand, when $10^3 < \mathrm{Re} < 3 \times 10^5$ the modulus of the drag force is proportional to the square speed $F_d = kv^2$. This case is known as the quadratic drag model, and $\mathbf{F}_d = -kv\mathbf{v}$. Although Newton studied this model in *Principia Mathematica* (Book II, Proposition XXXVIII and Scholium to Proposition XL), Leonhard Euler was the first one who explained it in his paper *Research on the curve that is described by bodies shot throught the air or in other fluid* (1753). Various properties of the projectile trajectories in the quadratic drag model have been analysed; see [2] and [5]. For instance, in [2] the parametric equations of the trajectories are given with an integral expression; see (1) below. With those equations the author determines the maximum projectile range (the value of x at the impact point) and other interesting characteristics of the trajectories. The safe domain has been treated in [1] but from a heuristic point of view. Also in [3] we can find a numerical approach to the safe domain with quadratic drag force but the authors do not provide an analytical expression for it. Here we focus our attention on a rigorous deduction of the shape of the region.

Concerning the quadratic drag model, according to Hayen [2] and taking units for length and time such that $v_0 = g = 1$, the parametric equations for the projectile trajectories are

$$x = \int_{-\xi}^{u(t)} \frac{dp}{f(p,\xi)} \quad \text{and} \quad y = -\int_{-\xi}^{u(t)} \frac{p}{f(p,\xi)}\, dp, \tag{1}$$

where $u(t) = -\tan\phi(t)$ is a parameter which depends on the local trajectory angle ϕ in every instant t, and $\xi = \tan\alpha$. Function f is given by $f(x,y) = A(g(x) + g(y)) + 1 + y^2$, where $g(x) = x\sqrt{1+x^2} + \arg\sinh x$ and $A = k/m$. Moreover, $u(t)$ depends on t by means of

$$t = \int_{-\xi}^{u} \frac{dp}{\sqrt{f(p,\xi)}}.$$

It is clear that $u \in [-\xi, \infty)$ because $t = 0$ when $u = -\xi$. We remark that equations (1) cannot be written in terms of elementary analytic functions.

The main result of this paper is the following one.

Theorem 1. *According to previous notation, the safe domain of the projectile trajectories in a medium with quadratic drag force is delimited by the curve given in parametric coordinates*

$$\begin{cases} x = \displaystyle\int_{-\xi}^{u(\xi)} \frac{dp}{f(p,\xi)}, \\ y = -\displaystyle\int_{-\xi}^{u(\xi)} \frac{p}{f(p,\xi)}\, dp, \end{cases} \tag{2}$$

for $\xi \in (\xi_0, \infty)$, with

$$\xi_0 = \inf_{\xi \in (0,\infty)} \left\{ 1 - 2(1+\xi^2)\left(\xi + A\sqrt{1+\xi^2}\right) \int_{-\xi}^{\infty} \frac{1}{f^2(p,\xi)}\, dp < 0 \right\},$$

and where $u(\xi)$ is the unique solution of the equation

$$\xi + u + 2(1+\xi^2)\left(\xi + A\sqrt{1+\xi^2}\right) \int_{-\xi}^{u} \frac{p-u}{f^2(p,\xi)}\, dp = 0, \tag{3}$$

in the interval $(-\xi, \infty)$.

At this point, a comment about the meaning of our result is in order. The envelope of a family of curves is another curve in which every point belongs to a curve of the given family; moreover, in that point both curves have a common tangent line. With the value $u(\xi)$ defined by the condition (3) we can determine the point in each trajectory belonging to the envelope and, for each $\xi \in (\xi_0, \infty)$, that point is given by (2). Then, we have obtained a parametric equation for the envelope of the family of trajectories (1).

Note that the trajectories associated with a launch angle such that $\xi \leq \xi_0$ do not have tangent point with the envelope because the projectile falls down very quickly due to the action of the quadratic drag force and the gravity force.

The safe domain, with the projectile trajectories, is plotted in Figure 3. This image has been created by using numerical methods to approximate the envelope given by (2). First, it is necessary to know the minimum slope ξ_0 which limits the trajectories having a common point with the envelope, and the value $u(\xi)$ defined by the relation (3). To obtain these values we have used a quadrature rule to approximate the integrals and the secant method to find the required value in each case. The integrals in (2) have been evaluated by using a quadrature rule again.

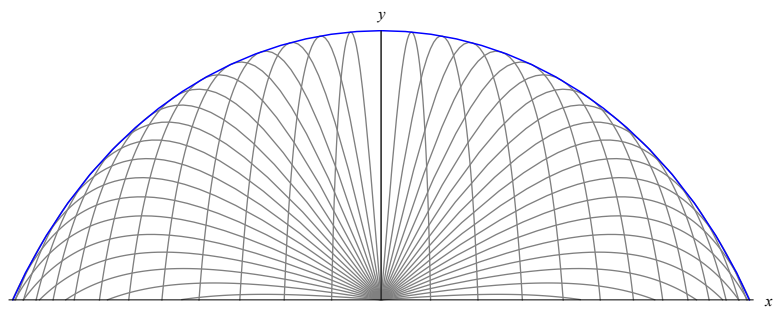

FIGURE 3. The projectile trajectories and the safe domain when we consider a quadratic drag model.

2. Proof of Theorem 1

The starting point of the proof of Theorem 1 will be the equation for the envelope of a family of curves. When the curves are given by $\mathcal{F}_\alpha(x, y) = 0$, with $\alpha \in \Lambda$, then the envelope is the solution to the system of equations

$$\mathcal{F}_\alpha(x, y) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial \alpha}\mathcal{F}_\alpha(x, y) = 0. \tag{4}$$

If the family of curves is given in parametric form, $x = g_\alpha(t)$ and $y = h_\alpha(t)$, with $\alpha \in \Lambda$ and $t \in I \subseteq \mathbb{R}$, the system (4) becomes

$$x = g_\alpha(t), \qquad y = h_\alpha(t) \qquad \text{and} \qquad \frac{\partial g_\alpha}{\partial t}\frac{\partial h_\alpha}{\partial \alpha} - \frac{\partial h_\alpha}{\partial t}\frac{\partial g_\alpha}{\partial \alpha} = 0. \tag{5}$$

To obtain (5) from (4) we have to use the implicit function theorem. Let us suppose that there exists $\mathcal{F}$ such that, at least locally, $\mathcal{F}(g_\alpha, h_\alpha) = 0$: then

$$\frac{\partial}{\partial \alpha}\mathcal{F}(g_\alpha, h_\alpha) = \frac{\partial \mathcal{F}}{\partial x}\frac{\partial g_\alpha}{\partial \alpha} + \frac{\partial \mathcal{F}}{\partial y}\frac{\partial h_\alpha}{\partial \alpha}.$$

Now, it is clear that

$$\frac{\partial \mathcal{F}}{\partial x}\frac{\partial g_\alpha}{\partial t}+\frac{\partial \mathcal{F}}{\partial y}\frac{\partial h_\alpha}{\partial t}=0.$$

Then, from the two previous identities,

$$\frac{\partial}{\partial \alpha}\mathcal{F}(g_\alpha,h_\alpha)=\frac{\frac{\partial \mathcal{F}}{\partial y}}{\frac{\partial g_\alpha}{\partial t}}\left(\frac{\partial g_\alpha}{\partial t}\frac{\partial h_\alpha}{\partial \alpha}-\frac{\partial h_\alpha}{\partial t}\frac{\partial g_\alpha}{\partial \alpha}\right),$$

and the condition

$$\frac{\partial}{\partial \alpha}\mathcal{F}(g_\alpha,h_\alpha)=0$$

implies

$$\frac{\partial g_\alpha}{\partial t}\frac{\partial h_\alpha}{\partial \alpha}-\frac{\partial h_\alpha}{\partial t}\frac{\partial g_\alpha}{\partial \alpha}=0.$$

The family of trajectories for projectiles under the action of a quadratic drag force is given in parametric form, so we will deduce the envelope by using (5).

Proof. Consider the trajectories given by (1) as a family of the parameter ξ. Then, by solving the system (5), the envelope of this family, which allows us to obtain the safe domain, is given by

$$\begin{cases} x=\displaystyle\int_{-\xi}^{u}\frac{dp}{f(p,\xi)},\\ y=-\displaystyle\int_{-\xi}^{u}\frac{p}{f(p,\xi)}\,dp,\end{cases} \tag{6}$$

with the condition

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial \xi}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial \xi}=0. \tag{7}$$

To obtain an appropriate expression for (7), we compute the partial derivatives of x and y with respect to u and ξ. By the fundamental theorem of calculus

$$\frac{\partial x}{\partial u}=\frac{1}{f(u,\xi)}\quad\text{and}\quad\frac{\partial y}{\partial u}=-\frac{u}{f(u,\xi)}.$$

Since, for nice functions,

$$\frac{d}{dz}\int_{z_0}^{z}h(t,z)\,dt=h(z,z)+\int_{z_0}^{z}\frac{\partial}{\partial z}h(t,z)\,dt,$$

we have

$$\frac{\partial x}{\partial \xi}=\frac{1}{1+\xi^2}-2\left(\xi+A\sqrt{1+\xi^2}\right)\int_{-\xi}^{u}\frac{dp}{f^2(p,\xi)},$$

and

$$\frac{\partial y}{\partial \xi}=\frac{\xi}{1+\xi^2}+2\left(\xi+A\sqrt{1+\xi^2}\right)\int_{-\xi}^{u}\frac{p}{f^2(p,\xi)}\,dp.$$

Then, (7) becomes

$$\xi+u+\eta(\xi)\int_{-\xi}^{u}\frac{p-u}{f^2(p,\xi)}\,dp=0, \tag{8}$$

where $\eta(\xi)=2(1+\xi^2)\left(\xi+A\sqrt{1+\xi^2}\right)$.

Now, let us see that for $\xi \in (\xi_0, \infty)$ the equation (8) has a unique solution in $(-\xi, \infty)$. We define the function

$$F(u) = \xi + u + \eta(\xi) \int_{-\xi}^{u} \frac{p-u}{f^2(p,\xi)}\, dp.$$

It is clear that $F(-\xi) = 0$, but $u = -\xi$ is not a valid solution for our purposes. By elementary calculus we can check that $F(u)$ has a root in $(-\xi, \infty)$. Indeed,

$$F''(u) = -\frac{\eta(\xi)}{f^2(u,\xi)} < 0, \qquad \xi \in (0, \infty),$$

because $f(u,\xi) > 0$ and $\eta(\xi) > 0$ for $u \in [-\xi, \infty)$ and $\xi \in (0, \infty)$. This implies that $F'(u)$ is a decreasing function for $u \in [-\xi, \infty)$. On the other hand,

$$F'(u) = 1 - \eta(\xi) \int_{-\xi}^{u} \frac{dp}{f^2(p,\xi)}.$$

Then $F'(-\xi) > 0$ for $\xi \in (0, \infty)$ and

$$\lim_{u\to\infty} F'(u) = 1 - \eta(\xi) \int_{-\xi}^{\infty} \frac{dp}{f^2(p,\xi)} < 0, \qquad \xi \in (\xi_0, \infty).$$

In this way, we can assure that for each $\xi \in (\xi_0, \infty)$ there exists a unique $u_m \in (-\xi, \infty)$ such that $F'(u_m) = 0$. Obviously u_m is the maximum of the function and $F(u)$ increases on $[-\xi, u_m)$ and decreases on (u_m, ∞). Moreover, we have $F(u_m) > 0$ (because $F(-\xi) = 0$), and

$$\lim_{u\to\infty} F(u) = \xi + \eta(\xi) \int_{-\xi}^{\infty} \frac{p}{f^2(p,\xi)}\, dp + \left(1 - \eta(\xi) \int_{-\xi}^{\infty} \frac{dp}{f^2(p,\xi)}\right) \lim_{u\to\infty} u = -\infty < 0,$$

for $\xi \in (\xi_0, \infty)$, because

$$\xi + \eta(\xi) \int_{-\xi}^{\infty} \frac{p}{f^2(p,\xi)}\, dp < \infty.$$

In this way we obtain that for each $\xi \in (\xi_0, \infty)$ there exists a unique $u(\xi) \in (-\xi, \infty)$ such that $F(u(\xi)) = 0$.

Taking $u = u(\xi)$ in (6) we get the tangent point between the envelope and the projectile trajectory associated with ξ, for each $\xi \in (\xi_0, \infty)$. The set of all tangent points form the envelope which is the border of the safe domain.

To complete the proof we have to check that ξ_0 is a finite value. Obviously, $\xi_0 \geq 0$. Taking the function

$$G(\xi, A) = 1 - 2(1+\xi^2)\left(\xi + A\sqrt{1+\xi^2}\right) \int_{-\xi}^{\infty} \frac{1}{f^2(p,\xi)}\, dp,$$

it is clear that

$$\begin{aligned} G(1, A) &< 1 - 4(1 + A\sqrt{2}) \int_{-1}^{1} \frac{dp}{(A(g(p) + g(1)) + 2)^2} \\ &< 1 - \sqrt{2}(1 + A\sqrt{2}) \int_{-1}^{1} \frac{g'(p)}{(A(g(p) + g(1)) + 2)^2}\, dp, \end{aligned}$$

where in the last step we have used that $0 < g'(p) = 2\sqrt{1+p^2} \leq 2\sqrt{2}$, for $-1 \leq p \leq 1$. Then

$$G(1, A) < 1 - (1 + A\sqrt{2}) \frac{g(1)}{\sqrt{2}(1 + Ag(1))} < 0.$$

So, the value ξ_0 is finite. $\square$

Remark. a) It is clear that when $A = 0$ we recover the classical parabolic motion and with our approach we can recover the parabola of safety. Indeed, in this case (2) becomes

$$x = \frac{u+\xi}{1+\xi^2} \qquad \text{and} \qquad y = -\frac{1}{2}\frac{u^2-p^2}{1+\xi^2},$$

with $\xi = \tan\alpha$, and from condition (7) we have $u = 1/\xi$ for all $\xi \in (0,\infty)$. Then the points of the envelope are

$$\left(\frac{\cos\alpha}{\sin\alpha}, -\frac{\cos 2\alpha}{2\sin^2\alpha}\right),$$

which belong to the parabola of safety $y(x) = (1-x^2)/2$.

b) A similar result to Theorem 1 could be obtained for drag forces of the form $\mathbf{F}_d = -h(v)\mathbf{v}$, with h a positive and increasing function. To this end it would be enough to have an expression available for the trajectories similar to the one given in (1).

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