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Note

A note on extreme cases of Sobolev embeddings ☆

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Abstract

We study the spaces of functions on $\mathbb{R}^n$ for which the generalized partial derivatives $D_k^{r_k} f$ exist and belong to different Lorentz spaces L^{p_k,s_k}. For this kind of functions we prove a sharp version of the extreme case of the Sobolev embedding theorem using $L(\infty, s)$ spaces.

Keywords: Anisotropic spaces; Embeddings; Sobolev spaces

1. Introduction

In this paper we consider functions f on $\mathbb{R}^n$ with generalized partial derivatives

$$D_k^{r_k} f \equiv \frac{\partial^{r_k} f}{\partial x_k^{r_k}} \quad (r_k \in \mathbb{N}).$$

Our main objective is to obtain an extreme case of a Sobolev type inequality for these functions. More precisely, we want to generalize the embedding

$$W_{n/r}^r(\mathbb{R}^n) \hookrightarrow L(\infty, n/r)(\mathbb{R}^n) \quad (r, n \in \mathbb{N};\ r \leqslant n)$$

(Milman–Pustylnik [16], Bastero–Milman–Ruiz [2] for $r = 1$) to the case where the partial derivatives $D_k^{r_k} f$ of different orders belong to different Lorentz spaces L^{p_k,s_k}.

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In order to introduce the problem, we recall some basic facts and review the literature. Let $n, r \in \mathbb{N}$, $1 \leqslant p < \infty$. The *Sobolev space* $W_p^r(\mathbb{R}^n)$ is the class of functions $f \in L^p(\mathbb{R}^n)$ with all the generalized derivatives of order r belonging to $L^p(\mathbb{R}^n)$.

The classical Sobolev embedding theorem says that if $1 \leqslant p < n/r$ then

$$W_p^r(\mathbb{R}^n) \hookrightarrow L^{q^*}(\mathbb{R}^n), \quad q^* = \frac{np}{n - rp}.$$

This theorem is well known and has been extensively considered in the literature. *In this paper we deal with the extreme case* $p = n/r$ (*or equivalently*, $q^* = \infty$). If $1 = p = n/r$ it is known that (see [4, §10], [19])

$$W_1^n(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n).$$

However, it is easy to see that for $1 < p = n/r$, the functions in $W_{n/r}^r(\mathbb{R}^n)$ need not to be bounded. Many authors have studied which kind of embedding holds in this case. Hansson [6], and independently and by different methods Brézis and Wainger [5], proved that if Ω is an open domain in $\mathbb{R}^n$ $(n > 1)$ with $|\Omega| < \infty$,

$$\widetilde{W}_n^1(\Omega) \hookrightarrow H_n(\Omega), \tag{1}$$

where $\widetilde{W}_n^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in W_n^1 and

$$H_n(\Omega) = \left\{ f\colon \|f\|_{H_n(\Omega)} = \left[\int_0^{|\Omega|} \left(\frac{f^{**}(s)}{1 + \log \frac{|\Omega|}{s}} \right)^n \frac{ds}{s} \right]^{1/n} < \infty \right\}.$$

Moreover, Hansson [6] showed that $H_n(\Omega)$ is the optimal target space in the class of rearrangement invariant spaces.

However, this result can be improved in the following sense. Kolyada [10, Lemma 5.1] (see also [9, p. 7]) proved the inequality

$$f^*(t) - f^*(2t) \leqslant ct^{1/n} \big(|\nabla f|\big)^{**}(t), \quad t > 0. \tag{2}$$

Bastero, Milman and Ruiz (see [2, Remark (2.3)]) showed that

$$f^{**}(t) - f^*(t) \leqslant ct^{1/n} \big(|\nabla f|\big)^{**}(t), \quad t > 0. \tag{3}$$

Inequalities (2) and (3) are equivalent (see Remark 5).

In [1,2,15] spaces related to inequality (3) were introduced and studied. It follows immediately from (3) that the Sobolev space

$$w_{n,\infty}^1(\mathbb{R}^n) = \big\{ f\colon \nabla f \in \text{weak-}L^n(\mathbb{R}^n) \big\}$$

is contained in the Bennett–De Vore–Sharpley space[1]

$$\text{weak-}L^\infty(\mathbb{R}^n) = \left\{ f\colon \|f\|_{\text{weak-}L^\infty(\mathbb{R}^n)} = \sup_{t>0} \big\{ f^{**}(t) - f^*(t) \big\} < \infty \right\}.$$

That is (cf. [1]),

$$w_{n,\infty}^1(\mathbb{R}^n) \subset \text{weak-}L^\infty(\mathbb{R}^n).$$

[1] Weak-L^∞ is not a linear space and $\|.\|_{\text{weak-}L^\infty}$ is not a norm.

In [2], for $q > 0$, the (non-linear) spaces $L(\infty, q)(\mathbb{R}^n)$ are defined as the set of functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L(\infty,q)} = \left(\int_0^\infty \left[f^{**}(t) - f^*(t) \right]^q \frac{dt}{t} \right)^{1/q} < \infty.$$

The following strict inclusions hold for $1 < p < q < \infty$:

$$L^\infty = L(\infty, 1) \subset L(\infty, p) \subset L(\infty, q) \subset \text{weak-}L^\infty.$$

It follows from (3) that (see [2])

$$W_n^1(\mathbb{R}^n) \hookrightarrow L(\infty, n)(\mathbb{R}^n). \tag{4}$$

Equivalent statements had been proved with different methods in [21, Eq. (3.22)] and in [15]. In [2] it is shown how (4) improves (1).

From the recent results in [16], the embedding for derivatives of higher order

$$W_{n/r}^r(\mathbb{R}^n) \hookrightarrow L(\infty, n/r)(\mathbb{R}^n) \tag{5}$$

is derived.[2]

Now we can specify our objective: *find an embedding of type* (5) *for functions with partial derivatives of different orders. Existence of mixed derivatives is not assumed.*

Let us explain more specifically which is the form of the embedding we are looking for. We consider the space of functions f such that the generalized partial derivatives $D_k^{r_k} f$ $(k = 1, \ldots, n)$ belong to different spaces L^{p_k}. The corresponding classes of functions naturally appear in the embedding theory as well as in applications. The most extended theory of these classes is contained in the monograph [4]. Furthermore, in this paper we allow the derivatives to belong to different Lorentz spaces $L^{p_k,s_k}(\mathbb{R}^n)$ (where $1 \leqslant p_k, s_k < \infty$ and $s_k = 1$, if $p_k = 1$). The use of Lorentz type limitations on the derivatives can be crucial in the estimates of Fourier transforms [11,13,18], conditions for differentiability [20], and embedding theorems [21].

Then our main problem is to find an embedding of type (5) for functions with the derivatives $D_k^{r_k} f \in L^{p_k,s_k}(\mathbb{R}^n)$ $(k = 1, \ldots, n)$.

The answer is given at the following inequality, proved in Theorem 4 below

$$\|f\|_{L(\infty,s)(\mathbb{R}^n)} \leqslant c \sum_{k=1}^n \left\| D_k^{r_k} f \right\|_{p_k,s_k}, \quad 1 \leqslant p = n/r,$$

where r, p and s are suitable averages of the r_k's, p_k's and s_k's to be defined later, that are frequently used in this context.

Note that the methods from [2,15,21] cannot be used in our case since they work for $r_1 = \cdots = r_n = 1$ only. Moreover, the reasoning in [16] is not applicable because our r_k's can be different, and so, the existence of mixed derivatives is not assumed. Thus, no induction over the order of the derivatives is possible. Instead, our approach is based on embeddings of Besov spaces and the transitivity of embeddings, together with results from [14].

[2] Note that if $f \in W_{n/r}^r(\mathbb{R}^n)$, then $\nabla f \in W_{n/r}^{r-1}(\mathbb{R}^n)$, and, by the well-known embedding of Sobolev spaces into Lorentz spaces, $\nabla f \in L^{n,n/r}(\mathbb{R}^n)$. From this and (3), the embedding (5) follows also.

2. Some definitions

Let $S_0(\mathbb{R}^n)$ be the class of all measurable, almost everywhere finite functions f on $\mathbb{R}^n$, such that for each $y > 0$,

$$\lambda_f(y) \equiv \left|\left\{x \in \mathbb{R}^n\colon \left|f(x)\right| > y\right\}\right| < \infty.$$

The non-increasing rearrangement of $f \in S_0(\mathbb{R}^n)$ is a non-increasing function f^* on $\mathbb{R}_+ \equiv (0, +\infty)$ that is equimeasurable with $|f|$. The rearrangement f^* can be defined by the equality

$$f^*(t) = \sup_{|E|=t} \inf_{x\in E} \left|f(x)\right|, \quad 0 < t < \infty.$$

The following relation holds [3, Chapter 2]:

$$\sup_{|E|=t} \int_E \left|f(x)\right| dx = \int_0^t f^*(u)\, du.$$

In what follows we set

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(u)\, du.$$

Assume that $0 < q, p < \infty$. A function $f \in S_0(\mathbb{R}^n)$ belongs to the Lorentz space $L^{q,p}(\mathbb{R}^n)$ if

$$\|f\|_{q,p} \equiv \left(\int_0^\infty \left(t^{1/q} f^*(t)\right)^p \frac{dt}{t}\right)^{1/p} < \infty.$$

We have the inequality [3, p. 217]

$$\|f\|_{q,s} \leqslant c\|f\|_{q,p} \quad (0 < p < s < \infty),$$

so that $L^{q,p} \subset L^{q,s}$ for $p < s$. In particular, for $0 < p \leqslant q$,

$$L^{q,p} \subset L^{q,q} \equiv L^q.$$

Let f be a measurable function on $\mathbb{R}^n$. Let $j \in \{1, \dots, n\}$. We define the difference

$$\Delta_j(h)f(x) \equiv f(x + he_j) - f(x), \quad h \in \mathbb{R},$$

where e_j is the unit coordinate vector. If $r > 1$, inductively,

$$\Delta_j^r(h)f(x) \equiv \Delta_j(h)\left[\Delta_j^{r-1}(h)f\right](x).$$

Let $1 \leqslant q < \infty$. The function

$$\omega_j(f;\delta)_q = \sup_{0<h<\delta} \left\|\Delta_j(h)f\right\|_q, \quad \delta > 0,$$

is called the modulus of continuity of f with respect to the variable x_j in the metric L^q.

For $1 \leqslant p < \infty$ we denote $\mathcal{L}^p \equiv L^p(\mathbb{R}_+, du/u)$; set also $\mathcal{L}^\infty \equiv L^\infty(\mathbb{R}_+)$ (see [7]).

3. Auxiliary lemmas

Lemma 1. *Let $\alpha > 0$, $\theta \geqslant 1$. Let $\psi(t)$ be a function on $\mathbb{R}_+$, non-negative, non-decreasing such that $t^{-\alpha}\psi(t) \in \mathcal{L}^\theta$. Then, for any $\delta > 0$ there exists a function φ on $\mathbb{R}_+$ continuously differentiable such that*:

(i) $\psi(t) \leqslant \varphi(t)$,
(ii) $\varphi(t)t^{-\alpha-\delta}$ *decreases and* $\varphi(t)t^{-\alpha+\delta}$ *increases,*
(iii) $\|t^{-\alpha}\varphi(t)\|_{\mathcal{L}^\theta} \leqslant c\|t^{-\alpha}\psi(t)\|_{\mathcal{L}^\theta}$ *where c is a constant that only depends on δ and α.*

The proof follows the scheme of [14, Lemma 2.1], so we do not include it here.

Let $0 < \alpha_j < \infty$, $1 \leqslant \theta_j \leqslant \infty$ for $j = 1, \dots, n$. Denote

$$\alpha = n\left(\sum_{j=1}^{n} \frac{1}{\alpha_j}\right)^{-1}; \qquad \theta = \frac{n}{\alpha}\left(\sum_{j=1}^{n} \frac{1}{\alpha_j\theta_j}\right)^{-1}. \tag{6}$$

Lemma 2. *Let $n \in \mathbb{N}$, $0 < \alpha_j < \infty$ and $1 \leqslant \theta_j \leqslant \infty$ for $j = 1, \dots, n$. Set α and θ as in* (6)*. Set also*

$$0 < \delta \leqslant \frac{1}{2} \min_{1\leqslant j\leqslant n} \{\alpha_j\}.$$

For $j = 1, \dots, n$, let φ_j be positive and continuously differentiable functions on $\mathbb{R}_+$, satisfying $\varphi_j(t)t^{-\alpha_j} \in \mathcal{L}^{\theta_j}$. Suppose in addition that $\varphi_j(t)t^{-\alpha_j+\delta}$ increases and $\varphi_j(t)t^{-\alpha_j-\delta}$ decreases.

Then there exist positive functions $\delta_1, \dots, \delta_n$ on $\mathbb{R}_+$ such that

$$\prod_{j=1}^{n} \delta_j(t) = t \quad (t > 0);$$

and for $\sigma(t) \equiv \sum_{j=1}^{n} \varphi_j(\delta_j(t))$ it holds that

$$\left(\int_0^\infty t^{-\frac{\alpha\theta}{n}-1}\sigma(t)^\theta\, dt\right)^{1/\theta} \leqslant c\prod_{j=1}^{n} \left[\left\|t^{-\alpha_j}\varphi_j(t)\right\|_{\mathcal{L}^{\theta_j}}\right]^{\alpha/n\alpha_j},$$

where c is a constant that only depends on δ, r_j and n.

Proof. Let $0 < a < b$ be two positive constants. A positive function g on $\mathbb{R}_+$ is said to be of power type (a, b) if $g(t)t^{-a} \uparrow$ and $g(t)t^{-b} \downarrow$.

It is easy to see that if g is of power type (a, b), then its inverse g^{-1} exists on $\mathbb{R}_+$, and it is of power type $(1/b, 1/a)$.

Also, if g_1 is of power type (a_1, b_1) and g_2 is of power type (a_2, b_2), then g_1g_2 is of power type $(a_1 + a_2, b_1 + b_2)$ and $g_1 \circ g_2$ is of power type (a_1a_2, b_1b_2).

Note that the functions φ_j are of power type $(\alpha_j - \delta, \alpha_j + \delta)$.

Set now

$$\Phi(s) = s\prod_{j=1}^{n-1}\varphi_j^{-1}\big(\varphi_n(s)\big), \quad s>0. \tag{7}$$

Define for $t>0$,

$$\delta_n(t) = \Phi^{-1}(t), \quad \delta_j(t) = \varphi_j^{-1}\big(\varphi_n\big(\delta_n(t)\big)\big), \quad j=1,\dots,n-1. \tag{8}$$

Of course, for $j=1,\dots,n$, the functions δ_j are of power type for some (a_j,b_j). From this it follows that

$$\frac{a_j}{t} \leqslant \frac{\delta_j'(t)}{\delta_j(t)} \leqslant \frac{b_j}{t}. \tag{9}$$

Moreover, by (7),

$$\prod_{j=1}^{n}\delta_j(t) = \Phi\big(\delta_n(t)\big) = t.$$

And by (8) $(1\leqslant i, j\leqslant n)$,

$$\varphi_i\big(\delta_i(t)\big) = \varphi_j\big(\delta_j(t)\big), \tag{10}$$

which implies that $\sigma(t) = n\varphi_j(\delta_j(t))$ $(j=1,\dots,n)$.

Finally, using (10), Hölder's inequality with exponents $\frac{n\alpha_j\theta_j}{\theta\alpha}$, (9), and the change of variable $\delta_j(t)=z$, we get

$$\begin{aligned}\left(\int_0^\infty t^{-\frac{\alpha\theta}{n}-1}\sigma(t)^\theta\,dt\right)^{1/\theta} &= n\left(\int_0^\infty\prod_{j=1}^{n}\left[\frac{\varphi_j(\delta_j(t))}{\delta_j(t)^{\alpha_j}}\right]^{\theta\alpha/n\alpha_j}\frac{dt}{t}\right)^{1/\theta}\\ &\leqslant n\prod_{j=1}^{n}\left(\int_0^\infty\left[\frac{\varphi_j(\delta_j(t))}{\delta_j(t)^{\alpha_j}}\right]^{\theta_j}\frac{dt}{t}\right)^{\frac{1}{\theta_j}\frac{\alpha}{n\alpha_j}}\\ &\leqslant c\prod_{j=1}^{n}\big(\big\|t^{-\alpha_j}\varphi_j(t)\big\|_{\mathcal{L}^{\theta_j}}\big)^{\alpha/n\alpha_j}. \qquad \square\end{aligned}$$

Lemma 3. *Let $n\in\mathbb{N}$, $\alpha_1,\dots,\alpha_n>0$, $1\leqslant\theta_1,\dots,\theta_n\leqslant\infty$. Set α and θ as in* (6). *Then, for any $1\leqslant q<\infty$ and any $f\in S_0(\mathbb{R}^n)$, there exists a non-negative function $\sigma(t)$ on $\mathbb{R}_+$ such that*

$$f^*(t) \leqslant f^*(2t) + t^{-1/q}\sigma(t) \quad (t>0) \tag{11}$$

and

$$\left(\int_0^\infty t^{-\alpha\theta/n}\sigma(t)^\theta\frac{dt}{t}\right)^{1/\theta} \leqslant c\sum_{j=1}^{n}\big\|t^{-\alpha_j}\omega_j(f;t)_q\big\|_{\mathcal{L}^{\theta_j}}, \tag{12}$$

where c is a constant that does not depend on f.

Proof. Without loss of generality we can suppose that the right-hand side of (12) is finite. As $f \in S_0(\mathbb{R}^n)$, then the $\omega_j(f;\cdot)_q$ are positive functions.[3] Applying Lemma 1 to the above mentioned modulus with $\delta = \frac{1}{2}\min\{\alpha_j\}$, we conclude that there exist continuously differentiable functions $\varphi_j(t)$ on $\mathbb{R}_+$ such that

$$\begin{aligned} &0 < \omega_j(f;t)_q \leqslant \varphi_j(t) \quad (t>0), \\ &\varphi_j(t)t^{-\alpha_j-\delta}\downarrow, \qquad \varphi_j(t)t^{-\alpha_j+\delta}\uparrow, \end{aligned} \tag{13}$$

and

$$\left\| t^{-\alpha_j}\varphi_j(t)\right\|_{\mathcal{L}^{\theta_j}} \leqslant c\left\| t^{-\alpha_j}\omega_j(f;t)_q\right\|_{\mathcal{L}^{\theta_j}}. \tag{14}$$

Now, note that the functions φ_j satisfy the conditions of Lemma 2. Hence, there exist positive functions $\delta_1,\dots,\delta_n$ on $\mathbb{R}_+$ such that $\prod_{j=1}^n \delta_j(t) = t$ and for $\sigma(t) \equiv \sum_{j=1}^n \varphi_j(\delta_j(t))$ the following inequality holds:

$$\left(\int_0^\infty t^{-\alpha\theta/n}\sigma(t)^\theta \frac{dt}{t}\right)^{1/\theta} \leqslant c\prod_{j=1}^n \left(\left\| t^{-\alpha_j}\varphi_j(t)\right\|_{\mathcal{L}^{\theta_j}}\right)^{\alpha/n\alpha_j}. \tag{15}$$

Last, using [12, Lemma 10.3], we have

$$f^*(t) \leqslant f^*(2t) + ct^{-1/q}\sum_{j=1}^n \omega_j\big(f;\delta_j(t)\big)_q.$$

From this and (13) we get (11). The estimate (12) is the consequence of (15), (14) and the inequality between arithmetic and geometric averages. □

4. Embedding theorem

Theorem 4. *Let $n \geqslant 2$, $r_j \in \mathbb{N}$, $1 \leqslant p_j, s_j < \infty$ for $j = 1,\dots,n$ and $s_j = 1$ if $p_j = 1$. Set*

$$r = n\left(\sum_{j=1}^n \frac{1}{r_j}\right)^{-1}, \qquad p = \frac{n}{r}\left(\sum_{j=1}^n \frac{1}{p_j r_j}\right)^{-1}, \qquad s = \frac{n}{r}\left(\sum_{j=1}^n \frac{1}{s_j r_j}\right)^{-1}.$$

Assume that $p = n/r$. Then, for all $f \in S_0(\mathbb{R}^n)$ that possess weak derivatives $D_j^{r_j} f \in L^{p_j,s_j}(\mathbb{R}^n)$ $(j = 1,\dots,n)$, it holds that

$$\left(\int_0^\infty \left[f^{**}(t) - f^*(t)\right]^s \frac{dt}{t}\right)^{1/s} \leqslant c\sum_{j=1}^n \left\| D_j^{r_j} f\right\|_{p_j,s_j}.$$

Proof. We fix $q > \max_{1\leqslant j\leqslant n}\{p_j r_j\}$. Now we apply [14, Theorem 3.1] with the parameters q_j that are chosen in the said theorem taking the value of the q that we have just fixed.

[3] Otherwise, since $f \in S_0(\mathbb{R}^n)$, we have that $f \equiv 0$ and the result is obvious.

By this fact (i.e., $q_j = q,\ j = 1, \dots, n$) and the assumption $p = n/r$, it follows that the parameters $\rho_j, \varkappa_j, \alpha_j$ and θ_j appearing in that theorem are

$$\rho_j = \frac{1}{p_j}, \qquad \varkappa_j = \frac{p_j}{q}, \qquad \alpha_j = \frac{p_j r_j}{q}, \qquad \frac{1}{\theta_j} = \frac{1-\varkappa_j}{s} + \frac{\varkappa_j}{s_j}. \tag{16}$$

Thus we get

$$\sum_{j=1}^{n}\left(\int_0^\infty \left[h^{-\alpha_j}\left\|\Delta_j^{r_j}(h)f\right\|_q\right]^{\theta_j}\frac{dh}{h}\right)^{1/\theta_j} \leqslant c\sum_{k=1}^{n}\left\|D_k^{r_k}f\right\|_{p_k,s_k}. \tag{17}$$

Note that the left-hand side of (17) is a sum of Besov type seminorms. Then, [17, Chapter 4] as $0 < \alpha_j < 1$,

$$\left\|t^{-\alpha_j}\omega_j(f;t)_q\right\|_{\mathcal{L}^{\theta_j}} \leqslant c\left(\int_0^\infty \left[h^{-\alpha_j}\left\|\Delta_j^{r_j}(h)f\right\|_q\right]^{\theta_j}\frac{dh}{h}\right)^{1/\theta_j}. \tag{18}$$

By Lemma 3, we have

$$\left(\int_0^\infty \left[f^*(t) - f^*(2t)\right]^\theta \frac{dt}{t}\right)^{1/\theta} \leqslant \left(\int_0^\infty t^{-\theta/q}\sigma(t)^\theta\frac{dt}{t}\right)^{1/\theta} \tag{19}$$

and

$$\left(\int_0^\infty t^{-\alpha\theta/n}\sigma(t)^\theta\frac{dt}{t}\right)^{1/\theta} \leqslant c\sum_{j=1}^{n}\left\|t^{-\alpha_j}\omega_j(f;t)_q\right\|_{\mathcal{L}^{\theta_j}}, \tag{20}$$

where (by (6), (16) and $p = n/r$ [4]). The value of α is

$$\alpha = n\left(\sum_{i=1}^{n}\frac{1}{\alpha_i}\right)^{-1} = n\left(\sum_{i=1}^{n}\frac{q}{p_i r_i}\right)^{-1} = \frac{n}{q}.$$

So, the right-hand side of (19) and the left-hand side of (20) coincide. Moreover, from (6) and (16), we have

$$\theta = \frac{n}{\alpha}\left(\sum_{j=1}^{n}\frac{1}{\theta_j\alpha_j}\right)^{-1} = q\left(\sum_{j=1}^{n}\left[\frac{1-\varkappa_j}{s\alpha_j} + \frac{\varkappa_j}{s_j\alpha_j}\right]\right)^{-1} = s.$$

Finally,

$$\begin{aligned} f^{**}(t) - f^*(t) &\leqslant \frac{1}{t}\int_0^t f^*(u)\,du - \frac{2}{t}\int_{t/2}^t f^*(2u)\,du \\ &= \frac{2}{t}\int_0^t \big(f^*(u) - f^*(2u)\big)\,du. \end{aligned} \tag{21}$$

[4] Which is the same as $\sum_{j=1}^{n}\frac{1}{p_j r_j} = 1$.

And from this and Hardy's inequality [3, p. 124],

$$\left(\int_0^\infty \left[f^{**}(t) - f^*(t)\right]^s \frac{dt}{t}\right)^{1/s} \leqslant c\left(\int_0^\infty \left[f^*(t) - f^*(2t)\right]^s \frac{dt}{t}\right)^{1/s}. \tag{22}$$

Putting together (22), (19), (20), (18), (17), we obtain the result. □

Remark 5. In this paragraph we show that estimates (3) and (2) are equivalent. It is easy to see that

$$f^*(t/2) - f^*(t) \leqslant 2\big(f^{**}(t) - f^*(t)\big). \tag{23}$$

So, (3) implies (2). Note that (21) is easily proved too. From (2), using (21) and the fact that for any $g \in S_0(\mathbb{R}^n)$ $tg^{**}(t)$ increases in t, the estimate (3) follows.

Inequality (23) appears in [2, Theorem 4.1]. Note also that inequalities equivalent to (21) are used in [8, Lemma 5] and [2, Theorem 4.1].

Remark 6. In the case $r_j = r_1$, $s_j = p_j = p_1$ $(1 \leqslant j \leqslant n)$, Theorem 4 implies the embedding (5).

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