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Estimates of difference norms for functions in anisotropic Sobolev spaces

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We investigate the spaces of functions on $\mathbb{R}^n$ for which the generalized partial derivatives $D_k^{r_k} f$ exist and belong to different Lorentz spaces L^{p_k,s_k}. For the functions in these spaces, the sharp estimates of the Besov type norms are found. The methods used in the paper are based on estimates of non-increasing rearrangements. These methods enable us to cover also the case when some of the p_k's are equal to 1.

1 Introduction

In this paper we study the spaces of functions f on $\mathbb{R}^n$ which possess the generalized partial derivatives

$$D_k^{r_k} f \equiv \frac{\partial^{r_k} f}{\partial x_k^{r_k}} \quad (r_k \in \mathbb{N}). \tag{1.1}$$

Our main goal is to obtain sharp estimates for the norms of the differences

$$\Delta_k^{r_k}(h) f(x) \equiv \sum_{j=0}^{r_k} (-1)^{r_k - j} \binom{r_k}{j} f(x + jhe_k) \quad (h \in \mathbb{R}) \tag{1.2}$$

(e_k is the unit coordinate vector). We will specify this problem below; here we only note that it was completely solved in the case when all derivatives (1.1) belong to the same space $L^p(\mathbb{R}^n)$. Nevertheless, it is reasonable to suppose that the derivatives $D_k^{r_k}$ $(k = 1, \dots, n)$ belong to different spaces L^{p_k}. The corresponding classes of functions naturally appear in the embedding theory as well as in applications. The most extended theory of these classes is contained in the monography [2]. Furthermore, many authors have studied Sobolev and Nikol'skii-Besov spaces whose construction involves, instead of L^p-norms, norms in more general spaces (see [12]). In this paper we suppose that derivatives belong to different Lorentz spaces $L^{p_k,s_k}(\mathbb{R}^n)$ (where $1 \le p_k, s_k < \infty$ and $s_k = 1$, if $p_k = 1$). Note that very interesting comments and results concerning this type of Sobolev spaces can be found in [19]. There are many important problems in Analysis which lead to these spaces. For instance, it was proved by E. M. Stein [17] that the sharp condition for the differentiability a.e. for a function $f \in W_1^1$ is that $\bigtriangledown f$ belongs to the Lorentz space $L^{n,1}$. The use of Lorentz type limitations on the derivatives can be crucial in the estimates of Fourier transforms (as it can be deduced from [9, 11, 15]). That is, if we look for a sharp conditions on the derivatives to guarantee a given integrability property of the Fourier transform, then these conditions generally will be expressed in terms of Lorentz norms.

Let us return to the our main problem: estimates for the norms of the differences (1.2). As it was mentioned above, estimates of this type are already known. In particular, they give a refinement of the classical Sobolev

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embedding theorem with the limiting exponent. The simplest version of this theorem asserts that for any function f in the Sobolev space $W_p^1(\mathbb{R}^n)$ $(1 \le p < n)$

$$\|f\|_{q^*} \le c \sum_{k=1}^{n} \left\| \frac{\partial f}{\partial x_k} \right\|_p, \quad q^* = \frac{np}{n-p}. \tag{1.3}$$

Sobolev proved this inequality in 1938 for $p > 1$; his method, based on integral representations, did not work in the case $p = 1$. Only at the end of the fifties Gagliardo and Nirenberg gave simple proofs of the inequality (1.3) for all $1 \le p < n$.

The inequality (1.3) has been generalized and developed in various directions (see [2, 10, 12, 13, 20, 21] for details and references). It was proved that the left-hand side in (1.3) can be replaced by the stronger Lorentz norm; that is, there holds the inequality

$$\|f\|_{q^*,p} \le c \sum_{k=1}^{n} \left\| \frac{\partial f}{\partial x_k} \right\|_p, \quad 1 \le p < n. \tag{1.4}$$

For $p > 1$ this result follows by interpolation (see [14, 18]). In the case $p = 1$ some geometric inequalities were used to prove (1.4) (see [3, 4, 7, 8, 16]).

An elementary approach to the study of Sobolev type inequalities, based on estimates of non-increasing rearrangements, has been worked out in [8]. In [8] there was proved an extension of the inequality (1.4) to the anisotropic Sobolev spaces $W_p^{r_1,\dots,r_n}(\mathbb{R}^n)$ $(p \ge 1,\ r_k \in \mathbb{N})$ defined by the conditions $f, D_k^{r_k} f \in L^p(\mathbb{R}^n)$. Afterwards, it was shown in [10] that the same methods give an analogous result in the case when the derivatives $D_k^{r_k} f$ belong to different spaces L^{p_k}. Observe that this approach has been still further simplified in the work [11], where the iterative rearrangements were used.

The sharp estimates of the norms of differences for the functions in Sobolev spaces firstly have been proved by V. P. Il'in [2, Vol. 2, pp. 72]. For the space $W_p^1(\mathbb{R}^n)$ Il'in's result reads as follows: if $n \in \mathbb{N}$, $1 < p < q < \infty$ and $\alpha \equiv 1 - n(1/p - 1/q) > 0$, then

$$\sum_{k=1}^{n} \left(\int_0^\infty \left[h^{-\alpha} \left\| \Delta_k^1(h) f \right\|_q \right]^p \frac{dh}{h} \right)^{1/p} \le c \sum_{k=1}^{n} \left\| \frac{\partial f}{\partial x_k} \right\|_p. \tag{1.5}$$

Actually, this means that there holds the continuous embedding to the Besov space

$$W_p^1(\mathbb{R}^n) \hookrightarrow B_{q,p}^{\alpha}(\mathbb{R}^n).$$

It is easy to see that the inequality (1.5) fails to hold for $p = n = 1$. Nevertheless, it was proved in [6] that (1.5) is true in the case $p = 1$, $n \ge 2$.

The inequality (1.5) for $p = 1$, $n \ge 2$ was used to prove some estimates of Fourier transforms of functions in Sobolev spaces (see [15], [9]). In particular, using these results, we can compare the inequalities (1.3) and (1.5). Let us consider the case $p = 1$, $n = 2$. The inequality (1.3) means that for any function $f \in W_1^1(\mathbb{R}^2)$ its Fourier transform $\hat{f}$ belongs to $L^2(\mathbb{R}^2)$. At the same time, as it was shown in [9], the stronger result can be easily derived from (1.5); that is, if $f \in W_1^1(\mathbb{R}^2)$, then $\hat{f} \in L^{2,1}(\mathbb{R}^2)$. Note that this assertion does not follow from (1.4).

The extension of the inequality (1.5) to the spaces $W_p^{r_1,\dots,r_n}$ was given in [8]. This is the following inequality

$$\sum_{k=1}^{n} \left(\int_0^\infty \left[h^{-\alpha_k} \left\| \Delta_k^{r_k}(h) f \right\|_{q,p} \right]^p \frac{dh}{h} \right)^{1/p} \le c \sum_{k=1}^{n} \left\| D_k^{r_k} f \right\|_p, \tag{1.6}$$

where $0 < 1/p - 1/q < r/n$, $r \equiv n \left(\sum_{i=1}^n r_i^{-1} \right)^{-1}$ and $\alpha_k = r_k \left[1 - \frac{n}{r} \left(\frac{1}{p} - \frac{1}{q} \right) \right]$; the inequality is valid if $p > 1$, $n \ge 1$ or $p = 1$, $n \ge 2$. Using (1.6), we get the following continuous embedding

$$W_p^{r_1,\dots,r_n}(\mathbb{R}^n) \hookrightarrow B_{q,p}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n).$$

For $p > 1$ this embedding was proved by Il'in [2, Vol. 2, pp. 72]. The main result in [8] is the proof of (1.6) for $p = 1$, $n \ge 2$. This result was applied in [9] to obtain Fourier transforms estimates for functions in $W_1^{r_1,\dots,r_n}$.

Now we can specify our main problem: find the sharp estimates of the type (1.6) for the case when the derivatives $D_k^{r_k} f$ belong to *different* Lorentz spaces L^{p_k,s_k}. The main result of the paper is the following inequality (see Theorem 3.1 below)

$$\left(\int_0^\infty \left[h^{-\alpha_j} \left\|\Delta_j^{r_j}(h) f\right\|_{q_j,1}\right]^{\theta_j} \frac{dh}{h}\right)^{1/\theta_j} \le c \sum_{k=1}^n \left\|D_k^{r_k} f\right\|_{p_k,s_k}. \qquad (1.7)$$

We shall not specify here the conditions on the parameters. Technically, the most complicated case is one when some of the p_k's are equal to 1 and some of them are greater than 1. The basic difficulty is to find the *sharp* values of the parameters θ_j; let us emphasize that it is exactly the main result of the work. In this connection observe that an inequality similar to (1.7) was proved by Il'in [2, Vol. 2, pp. 72] in the case $p_k = s_k > 1$ $(k = 1, \dots, n)$, but with the value of the parameter $\theta = \max_{1 \le k \le n} p_k$, which is not sharp when p_k are different.

The general base of our approach is contained in the Lemmas 2.2, 2.3 and 2.4 given below. These lemmas were proved earlier by the first named author. Lemmas 2.3 and 2.4 give estimates of non-increasing rearrangement of a function in terms of its derivatives. We use also the scheme of the proof of the inequality (1.6) developed in [8]. Observe that in our case some essential modifications of this scheme are requiered.

Note also that as in the articles [9], [11], [15], the results of this paper can be applied to the study of estimates of Fourier transforms in Sobolev spaces.

2 Auxiliary propositions

Let $S_0(\mathbb{R}^n)$ be the class of all measurable and almost everywhere finite functions f on $\mathbb{R}^n$ such that for each $y > 0$,

$$\lambda_f(y) \equiv \left|\{x \in \mathbb{R}^n : |f(x)| > y\}\right| < \infty.$$

A non-increasing rearrangement of a function $f \in S_0(\mathbb{R}^n)$ is a non-increasing function f^* on $\mathbb{R}_+ \equiv (0, +\infty)$ that is equimeasurable with $|f|$. The rearrangement f^* can be defined by the equality

$$f^*(t) = \sup_{|E|=t} \inf_{x \in E} |f(x)|, \quad 0 < t < \infty.$$

The following relation holds [1, Ch. 2]

$$\sup_{|E|=t} \int_E |f(x)|\, dx = \int_0^t f^*(u)\, du.$$

In what follows we set

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(u)\, du.$$

Assume that $0 < q, p < \infty$. A function $f \in S_0(\mathbb{R}^n)$ belongs to the Lorentz space $L^{q,p}(\mathbb{R}^n)$ if

$$\|f\|_{q,p} \equiv \left(\int_0^\infty \left(t^{1/q} f^*(t)\right)^p \frac{dt}{t}\right)^{1/p} < \infty.$$

We have the inequality [1, pp. 217]

$$\|f\|_{q,s} \le c\, \|f\|_{q,p} \quad (0 < p < s < \infty),$$

so that $L^{q,p} \subset L^{q,s}$ for $p < s$. In particular, for $0 < p \le q$

$$L^{q,p} \subset L^{q,q} \equiv L^q.$$

Lemma 2.1 *Let $\psi \in L^{p,s}(\mathbb{R}_+)$ $(1 \le p, s < \infty)$ be a non-negative non-increasing function on $\mathbb{R}_+$. Then for any $\delta > 0$ there exists a continuously differentiable function φ on $\mathbb{R}_+$ such that:*

(i) $\psi(t) \le \varphi(t)\,,\ t \in \mathbb{R}_+$;

(ii) $\varphi(t)t^{1/p-\delta}$ *decreases and* $\varphi(t)t^{1/p+\delta}$ *increases on* $\mathbb{R}_+$;

(iii) $\|\varphi\|_{p,s} \le c\,\|\psi\|_{p,s}$,

where c is a constant that only depends on p and δ.

Proof. We can suppose that $\delta < 1/p$. Set

$$\varphi_1(t) \;=\; 2t^{\delta-1/p}\int_{t/2}^{\infty} u^{1/p-\delta}\psi(u)\,\frac{du}{u}\,.$$

Then $\varphi_1(t)\,t^{1/p-\delta}$ decreases and

$$\varphi_1(t) \;\ge\; 2t^{\delta-1/p}\psi(t)\int_{t/2}^{t} u^{1/p-\delta-1}\,du \;\ge\; \psi(t)\,.$$

Furthermore, applying Hardy's inequality [1, pp. 124], we easily get that

$$\|\varphi_1(t)\|_{p,s} \;\le\; c\,\|\psi\|_{p,s}\,. \tag{2.1}$$

Set now

$$\varphi(t) \;=\; (\delta+1/p)t^{-1/p-\delta}\int_0^t \varphi_1(u)u^{\delta+1/p}\,\frac{du}{u}\,. \tag{2.2}$$

Then $\varphi(t)t^{1/p+\delta}$ increases on $\mathbb{R}_+$ and

$$\varphi(t) \;\ge\; \varphi_1(t) \;\ge\; \psi(t)\,, \quad t \in \mathbb{R}_+\,.$$

Furthermore, the change of variable $v = u^{2\delta}$ in the right-hand side of (2.2) gives that

$$t^{1/p-\delta}\varphi(t) \;=\; c\,t^{-2\delta}\int_0^{t^{2\delta}} \eta\big(v^{1/(2\delta)}\big)\,dv\,,$$

where $\eta(u) = \varphi_1(u)u^{1/p-\delta}$ is a decreasing function on $\mathbb{R}_+$. Thus, $t^{1/p-\delta}\varphi(t)$ decreases. Finally, using Hardy's inequality and (2.1), we get (iii). The lemma is proved. □

Let $r_k \in \mathbb{N}$ and $1 \le p_k < \infty$ for $k = 1,\dots,n$ $(n \ge 2)$.

Denote

$$r \;=\; n\left(\sum_{j=1}^n \frac{1}{r_j}\right)^{-1}, \quad p \;=\; \frac{n}{r}\left(\sum_{j=1}^n \frac{1}{p_j r_j}\right)^{-1} \tag{2.3}$$

and

$$\gamma_k \;=\; 1-\frac{1}{r_k}\left(\frac{r}{n}+\frac{1}{p_k}-\frac{1}{p}\right). \tag{2.4}$$

Then $\gamma_k > 0$ and

$$\sum_{k=1}^n \gamma_k \;=\; n-1\,. \tag{2.5}$$

Indeed,

$$\left(\frac{r}{n}+\frac{1}{p_k}-\frac{1}{p}\right)\sum_{j=1}^n\frac{1}{r_j} \;=\; 1+\sum_{j\ne k}\left(\frac{1}{p_k}-\frac{1}{p_j}\right)\frac{1}{r_j} \;<\; 1+\sum_{j\ne k}\frac{1}{r_j} \;\le\; r_k\sum_{j=1}^n\frac{1}{r_j}\,.$$

Thus, $\gamma_k > 0$. The equality (2.5) follows immediately from (2.3).

To prove our main results we use estimates of the rearrangement of a given function in terms of its derivatives $D_k^{r_k} f$ $(k = 1, \ldots, n)$. Thus, we apply simultaneously n estimates in which upper bounds involve functions belonging to different Lorentz spaces. The following lemma enables us to find a sharp "intermediate" estimate.

We will use the notations (2.3) and (2.4).

Lemma 2.2 *Let $r_k \in \mathbb{N}$, $1 \le p_k, s_k < \infty$ for $k = 1, \ldots, n$ $(n \ge 2)$ and $s_k = 1$ if $p_k = 1$. Set*

$$s = \frac{n}{r}\left(\sum_{j=1}^{n} \frac{1}{s_j r_j}\right)^{-1}.$$

Let

$$0 < \delta \le \frac{1}{4} \min_{\gamma_j < 1} \min\left(\gamma_j, 1 - \gamma_j\right). \tag{2.6}$$

Suppose that $\varphi_k \in L^{p_k, s_k}(\mathbb{R}_+)$ $(k = 1, \ldots, n)$ are positive continuously differentiable functions with $\varphi_k'(t) < 0$ on $\mathbb{R}_+$ such that $\varphi_k(t) t^{1/p_k - \delta}$ decreases and $\varphi_k(t) t^{1/p_k + \delta}$ increases on $\mathbb{R}_+$. Set for $u, t > 0$

$$\eta_k(u,t) = \begin{cases} (t/u)^{r_k - 1} \varphi_k(u), & \text{if} \quad p_k = 1, \\ (t/u)^{r_k} \varphi_k(t), & \text{if} \quad p_k > 1, \end{cases}$$

and

$$\sigma(t) = \sup\left\{ \min_{1 \le k \le n} \eta_k(u_k, t) : \prod_{k=1}^{n} u_k = t^{n-1},\ u_k > 0 \right\}. \tag{2.7}$$

Then:

(i) *there holds the inequality*

$$\left(\int_0^\infty t^{s(1/p - r/n) - 1} \sigma(t)^s \, dt\right)^{1/s} \le c' \prod_{k=1}^{n} \|\varphi_k\|_{p_k, s_k}^{r/(n r_k)}; \tag{2.8}$$

(ii) *there exist positive continuously differentiable functions $u_k(t)$ on $\mathbb{R}_+$ such that*

$$\prod_{k=1}^{n} u_k(t) = t^{n-1} \tag{2.9}$$

and

$$\sigma(t) = \eta_k(u_k(t), t) \quad (t \in \mathbb{R}_+,\ k = 1, \ldots, n); \tag{2.10}$$

(iii) *for any k such that*

$$\frac{1}{p_k} > \frac{1}{p} - \frac{r}{n} \tag{2.11}$$

the function $u_k(t) t^{\delta - 1}$ decreases on $\mathbb{R}_+$;

(iv) *if $p_k = 1$, then*

$$\int_0^\infty \frac{u_k(t)}{t} \varphi_k(u_k(t)) \, dt \le c \|\varphi_k\|_1. \tag{2.12}$$

Proof. Fix $t > 0$ and denote

$$\mu_t(u) = \min_{1 \le k \le n} \eta_k(u_k, t), \quad u = (u_1, \ldots, u_n) \in \mathbb{R}_+^n.$$

This is a continuous function on $\mathbb{R}_+^n$. Observe that every function $\eta_k(s,t)$ is strictly decreasing and continuous with respect to s on $\mathbb{R}_+$. Furthermore, $\eta_k(s,t) \to 0$ as $s \to +\infty$. Thus,

$$\mu_t(u) \longrightarrow 0 \quad \text{as} \quad \max u_k \longrightarrow +\infty.$$

This implies the existence of a point $u^* \in \mathbb{R}^n_+$ such that

$$\mu_t(u^*) \;=\; \sigma(t) \quad \text{and} \quad \prod_{k=1}^n u_k^* \;=\; t^{n-1}\,.$$

For any $k = 1, \ldots, n$ there exists a unique point $u_k(t) > 0$ such that $\eta_k(u_k(t), t) = \sigma(t)$. It is clear that $u_k^* \le u_k(t)$ for all k (otherwise we would have that $\mu_t(u^*) < \sigma(t)$). Suppose that $u_j^* < u_j(t)$ for some j. Take $u'_j \in \big(u_j^*, u_j(t)\big)$ and choose $u'_k \in (0, u_k^*)$ $(k \neq j)$ such that $\prod_{k=1}^n u'_k = t^{n-1}$. Then we obtain that $\mu_t(u') > \sigma(t)$, in contradiction with the definition of $\sigma(t)$. Thus, $u_k^* = u_k(t)$ $(k = 1, \ldots, n)$, and we get that the functions $u_k(t)$ satisfy both equalities (2.9) and (2.10).

Further, for any $j = 1, \ldots, n$

$$\eta_j\big(u_j(t), t\big) \;=\; \eta_n(u_n(t), t)\,. \tag{2.13}$$

It follows that there exist functions $\psi_j(s, t) \in C^1\big(\mathbb{R}^2_+\big)$ $(j = 1, \ldots, n-1)$ such that

$$\frac{\partial \psi_j}{\partial s}(s, t) \;>\; 0\,, \quad (s, t) \in \mathbb{R}^2_+\,, \tag{2.14}$$

and

$$u_j(t) \;=\; \psi_j(u_n(t), t) \quad (j = 1, \ldots, n-1)\,. \tag{2.15}$$

Indeed, if $p_j = 1$, then (2.13) implies that

$$\lambda_j(u_j(t)) \;=\; t^{1-r_j}\eta_n(u_n(t), t)\,,$$

where $\lambda_j(s) \equiv s^{1-r_j}\varphi_j(s)$ is a continuously differentiable function with $\lambda'_j(s) < 0$ $(s > 0)$. Thus, (2.15) holds with

$$\psi_j(s, t) \;=\; \lambda_j^{-1}\big(t^{1-r_j}\eta_n(s, t)\big)\,;$$

clearly, $\psi_j \in C^1\big(\mathbb{R}^2_+\big)$ and satisfies (2.14). If $p_j > 1$, then (2.15) holds with the function

$$\psi_j(s, t) \;=\; t\big[\varphi_j(t)/\eta_n(s, t)\big]^{1/r_j}\,,$$

which also belongs to $C^1\big(\mathbb{R}^2_+\big)$ and satisfies (2.14).

It follows from (2.9) and (2.15) that for any $t > 0$

$$\Phi(u_n(t), t) \;=\; t^{n-1}\,,$$

where

$$\Phi(s, t) \;=\; s\prod_{j=1}^{n-1}\psi_j(s, t)\,.$$

Since $\Phi'_s(s, t) > 0$, we get that $u_n \in C^1(\mathbb{R}_+)$ and therefore, by (2.15), $u_j \in C^1(\mathbb{R}_+)$ for any $j = 1, \ldots, n$. The statement (ii) is proved. Note also that by (2.10) the function σ is continuously differentiable in $\mathbb{R}_+$.

Now we will prove that for all $t > 0$

$$\frac{r/n - 1/p - \delta}{t} \;\le\; \frac{\sigma'(t)}{\sigma(t)} \;\le\; \frac{r/n - 1/p + \delta}{t}\,. \tag{2.16}$$

Our conditions on φ_k imply that for any $k = 1, \ldots, n$

$$\left(\frac{1}{p_k} - \delta\right)\frac{1}{t} \;\le\; -\frac{\varphi'_k(t)}{\varphi_k(t)} \;\le\; \left(\frac{1}{p_k} + \delta\right)\frac{1}{t}\,. \tag{2.17}$$

Further, if $p_k > 1$, then by (2.10)

$$\frac{\sigma'(t)}{\sigma(t)} = \frac{r_k}{t} - r_k \frac{u_k'(t)}{u_k(t)} + \frac{\varphi_k'(t)}{\varphi_k(t)} \tag{2.18}$$

and by (2.17)

$$\frac{r_k - 1/p_k - \delta}{t} - r_k \frac{u_k'(t)}{u_k(t)} \le \frac{\sigma'(t)}{\sigma(t)} \le \frac{r_k - 1/p_k + \delta}{t} - r_k \frac{u_k'(t)}{u_k(t)} . \tag{2.19}$$

If $p_k = 1$, then we have by (2.10)

$$\frac{\sigma'(t)}{\sigma(t)} = \frac{r_k - 1}{t} - (r_k + \alpha_k(t)) \frac{u_k'(t)}{u_k(t)} , \tag{2.20}$$

where

$$\alpha_k(t) = -\left[1 + u_k(t) \frac{\varphi_k'(u_k(t))}{\varphi_k(u_k(t))}\right] .$$

By (2.17) ($p_k = 1$),

$$-\delta \le \alpha_k(t) \le \delta . \tag{2.21}$$

Now, differentiating (2.9) and taking into account (2.5), we get that for any $t > 0$ there exists $m \equiv m(t)$ such that

$$\frac{u_m'(t)}{u_m(t)} \le \frac{\gamma_m}{t} .$$

If $p_m > 1$, then by the first of the inequalities (2.19),

$$\frac{\sigma'(t)}{\sigma(t)} \ge \frac{r_m - 1/p_m - r_m\gamma_m - \delta}{t} = \frac{r/n - 1/p - \delta}{t} .$$

If $p_m = 1$ (in this case $\gamma_m < 1$), then by (2.20) and (2.21)

$$\frac{\sigma'(t)}{\sigma(t)} \ge \frac{r_m - 1 - \gamma_m(r_m + \delta)}{t} \ge \frac{r/n - 1/p - \delta}{t} .$$

Thus, we have the first inequality in (2.16). To prove the second inequality observe that by (2.5) and (2.9) for any $t > 0$ there exists $l \equiv l(t)$ such that

$$\frac{u_l'(t)}{u_l(t)} \ge \frac{\gamma_l}{t} .$$

As above, it remains to apply the right-hand side inequality of (2.19) in the case $p_l > 1$ or (2.20) and (2.21) in the case $p_l = 1$.

To prove (iii) assume that k satisfies the condition (2.11) (that is, $\gamma_k < 1$). Let $p_k > 1$. By (2.18), (2.17) and (2.16),

$$r_k \frac{u_k'(t)}{u_k(t)} = \frac{r_k}{t} - \frac{\sigma'(t)}{\sigma(t)} + \frac{\varphi_k'(t)}{\varphi_k(t)} \le \frac{r_k + 1/p - r/n - 1/p_k + 2\delta}{t} = \frac{r_k\gamma_k + 2\delta}{t} .$$

Thus, by (2.6),

$$\frac{u_k'(t)}{u_k(t)} \le \frac{1 - \delta}{t} ,$$

which implies (iii) (in the case $p_k > 1$). If $p_k = 1$, then by (2.20) and (2.16)

$$(r_k + \alpha_k(t))\,\frac{u_k'(t)}{u_k(t)} \;\le\; \frac{r_k - 1 + 1/p - r/n + \delta}{t} \;=\; \frac{r_k\gamma_k + \delta}{t}\,.$$

From here (see (2.6)),

$$\frac{u_k'(t)}{u_k(t)} \;\le\; \frac{r_k\gamma_k + \delta}{(r_k + \alpha_k(t))t} \;\le\; \frac{r_k\gamma_k + \delta}{(r_k - \delta)t} \;\le\; \frac{1-\delta}{t}\,.$$

This implies (iii).

To prove (iv) assume that $p_k = 1$. By (2.20) and (2.16)

$$(r_k + \alpha_k(t))\,\frac{u_k'(t)}{u_k(t)} \;\ge\; \frac{r_k - 1 + 1/p - r/n - \delta}{t} \;=\; \frac{r_k\gamma_k - \delta}{t}\,.$$

From here (see (2.6)),

$$\frac{u_k'(t)}{u_k(t)} \;\ge\; \frac{r_k\gamma_k - \delta}{(r_k + \alpha_k(t))t} \;\ge\; \frac{r_k\gamma_k - \delta}{(r_k + \delta)t} \;>\; \frac{\delta}{t}\,.$$

It follows that

$$\int_0^\infty \frac{u_k(t)}{t}\,\varphi_k(u_k(t))\,dt \;\le\; \frac{1}{\delta}\int_0^\infty u_k'(t)\varphi_k(u_k(t))\,dt \;=\; \frac{1}{\delta}\,\|\varphi_k\|_1\,.$$

Thus, we obtain (2.12).

It remains to prove the inequality (2.8). By (2.10), we have

$$\sigma(t)^{r/(nr_k)} \;=\; \left(\frac{t}{u_k(t)}\right)^{r/n}\left[\varphi_k(u_k(t))\frac{u_k(t)}{t}\right]^{r/(nr_k)}, \quad \text{if} \quad p_k \;=\; 1\,,$$

and

$$\sigma(t)^{r/(nr_k)} \;=\; \left(\frac{t}{u_k(t)}\right)^{r/n}\varphi_k(t)^{r/(nr_k)}\,, \quad \text{if} \quad p_k \;>\; 1\,.$$

Multiplying these equalities and using (2.9), we get

$$\sigma(t) \;=\; t^{r/n}\prod_{p_k=1}\left[\frac{u_k(t)}{t}\,\varphi_k(u_k(t))\right]^{r/(nr_k)}\prod_{p_k>1}(\varphi_k(t))^{r/(nr_k)}\,. \tag{2.22}$$

Denote

$$q_k \;=\; \frac{nr_ks_k}{rs}\,.$$

Then

$$\sum_{k=1}^n \frac{1}{q_k} \;=\; 1 \quad \text{and} \quad \sum_{k=1}^n \frac{s_k}{p_kq_k} \;=\; \frac{s}{p}\,.$$

Therefore, applying Hölder's inequality with the exponents q_k and using (2.12), we get from (2.22)

$$\int_0^\infty t^{s(1/p-r/n)-1}\sigma(t)^s\,dt \;\le\; c\prod_{p_k=1}\|\varphi_k\|_1^{rs/(nr_k)}\prod_{p_k>1}\|\varphi_k\|_{p_k,s_k}^{rs/(nr_k)}\,.$$

The proof is now complete. □

The Lebesgue measure of a measurable set $A \subset \mathbb{R}^k$ will be denoted by $\mathrm{mes}_k\,A$.

For any F_σ-set $E \subset \mathbb{R}^n$ denote by E^j the orthogonal projection of E onto the coordinate hyperplane $x_j = 0$. By the Loomis-Whitney inequality [5, 4.4.2],

$$(\operatorname{mes}_n E)^{n-1} \leq \prod_{j=1}^{n} \operatorname{mes}_{n-1} E^j . \tag{2.23}$$

As usual, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we denote by $\hat{x}_k$ the $(n-1)$-dimensional vector obtained from x by removal of its k-th coordinate.

Let $f \in S_0(\mathbb{R}^n),\ t > 0$ and let E_t be a set of type F_σ and measure t such that

$$|f(x)| \geq f^*(t) \quad \text{for all} \quad x \in E_t .$$

Denote by $\lambda_j(t)$ the $(n-1)$-dimensional measure of the projection E_t^j $(j = 1, \ldots, n)$. By (2.23), we have that

$$\prod_{j=1}^{n} \lambda_j(t) \geq t^{n-1} . \tag{2.24}$$

The following lemma was proved in [8] (see also [10]).

Lemma 2.3 *Let $n \geq 2$, $r_k \in \mathbb{N}$ $(k = 1, \ldots, n)$. Assume that a locally integrable function $f \in S_0(\mathbb{R}^n)$ has weak derivatives $D_k^{r_k} f \in L_{loc}(\mathbb{R}^n)$ $(k = 1, \ldots, n)$. Then for all $0 < t < \tau < \infty$ and $k = 1, \ldots, n$ we have*

$$f^*(t) \leq K\left[f^*(\tau) + \left(\frac{\tau}{\lambda_k(t)}\right)^{r_k} \left(D_k^{r_k} f\right)^{**}(\tau)\right] \tag{2.25}$$

and

$$f^*(t) \leq K\left[f^*(\tau) + \left(\frac{\tau}{\lambda_k(t)}\right)^{r_k - 1} \psi_k^*\left(\frac{\lambda_k(t)}{2}\right)\right], \tag{2.26}$$

where K is a constant depending only on $r_1, \ldots, r_n$ and

$$\psi_k(\hat{x}_k) = \int_{\mathbb{R}} |D_k^{r_k} f(x)|\, dx_k , \quad \hat{x}_k \in \mathbb{R}^{n-1} . \tag{2.27}$$

Lemma 2.4 *Let $n \geq 2$, $r_k \in \mathbb{N}$, $1 \leq p_k, s_k < \infty$ for $k = 1, \ldots, n$ and $s_k = 1$ if $p_k = 1$. Set*

$$r = n\left(\sum_{k=1}^{n} \frac{1}{r_k}\right)^{-1}, \quad p = \frac{n}{r}\left(\sum_{k=1}^{n} \frac{1}{p_k r_k}\right)^{-1}, \tag{2.28}$$

and

$$s = \frac{n}{r}\left(\sum_{k=1}^{n} \frac{1}{s_k r_k}\right)^{-1}. \tag{2.29}$$

Assume that a locally integrable function $f \in S_0(\mathbb{R}^n)$ has weak derivatives $D_k^{r_k} f \in L^{p_k, s_k}(\mathbb{R}^n)$ $(k = 1, \ldots, n)$. Then for any $\xi > 1$

$$f^*(t) \leq K\left[f^*(\xi t) + \xi^{\bar{r}} \sigma(t)\right], \tag{2.30}$$

where $\bar{r} = \max r_k$, the constant K depends only on $r_1, \ldots, r_n$ and

$$\left(\int_0^\infty t^{s(1/p - r/n) - 1} \sigma(t)^s\, dt\right)^{1/s} \leq c \prod_{k=1}^{n} \left\| D_k^{r_k} f \right\|_{p_k, s_k}^{r/(n r_k)} . \tag{2.31}$$

P r o o f. For every fixed $k = 1, \dots, n$ we take (see (2.27))

$$\psi(t) \equiv \psi_k(t) = \begin{cases} \psi_k^*(t/2), & \text{if} \quad p_k = 1, \\ (D_k^{r_k} f)^{**}(t), & \text{if} \quad p_k > 1. \end{cases}$$

Then $\|\psi_k\|_1 = 2 \left\|D_k^{r_k}\right\|_1$, if $p_k = 1$, and by Hardy's inequality [1, pp. 124]

$$||\psi_k||_{p_k,s_k} \le c \left\|D_k^{r_k} f\right\|_{p_k,s_k},$$

if $p_k > 1$. Next we apply Lemma 2.1 with δ defined as in Lemma 2.2. This way we obtain the functions which we denote by $\varphi_k(t)$ $(k = 1, \dots, n)$. Further, with these functions φ_k we define the function $\sigma(t)$ by (2.7). By Lemma 2.2, we have the inequality (2.31). Using Lemma 2.3 with $\tau = \xi t$, we obtain

$$f^*(t) \le K\left[f^*(\xi t) + \xi^{\bar r}\left(\frac{t}{\lambda_k(t)}\right)^{r_k} \varphi_k(t)\right],$$

if $p_k > 1$, and

$$f^*(t) \le K\left[f^*(\xi t) + \xi^{\bar r}\left(\frac{t}{\lambda_k(t)}\right)^{r_k - 1} \varphi_k(\lambda_k(t))\right],$$

if $p_k = 1$. Taking into account (2.7) and (2.24), we immediately get (2.30). □

Note that in the case $p_1 = \ldots = p_n,\ s_1 = \ldots = s_n$ Lemma 2.4 actually is contained in [10] (see Lemmas 7 and 8 in [8]).

Corollary 2.5 *Assume that a function f satisfies the conditions of Lemma* 2.4 *and $f \in L^1(\mathbb{R}^n) + L^{p_0}(\mathbb{R}^n)$ for some $p_0 > 0$ such that*

$$\frac{1}{p_0} > \frac{1}{p} - \frac{r}{n}.$$

Let $\max(1, p_0) < q < \infty$ *and*

$$\frac{1}{q} > \frac{1}{p} - \frac{r}{n}. \tag{2.32}$$

Then for any $\theta > 0$ $f \in L^{q,\theta}(\mathbb{R}^n)$ and

$$\|f\|_{q,\theta} \le c\left[\|f\|_{L^1 + L^{p_0}} + \prod_{k=1}^{n} \left\|D_k^{r_k} f\right\|_{p_k,s_k}^{r/(n r_k)}\right]. \tag{2.33}$$

P r o o f. We can assume that $\theta < \min(1, p_0, s)$. Let $f = g + h$, with $g \in L^1(\mathbb{R}^n)$ and $h \in L^{p_0}(\mathbb{R}^n)$. Applying Hölder inequality, we obtain

$$\begin{aligned} J_1 \equiv \int_1^\infty \left[t^{1/q} f^*(t)\right]^\theta \frac{dt}{t} &\le 2^\theta \left[\int_1^\infty \left[t^{1/q} g^*(t/2)\right]^\theta \frac{dt}{t} + \int_1^\infty \left[t^{1/q} h^*(t/2)\right]^\theta \frac{dt}{t}\right] \\ &\le c\left[\left(\int_0^\infty g^*(t)\, dt\right)^\theta + \left(\int_0^\infty h^*(t)^{p_0}\, dt\right)^{\theta/p_0}\right]. \end{aligned}$$

It follows that

$$J_1 \le c' \|f\|_{L^1 + L^{p_0}}^\theta. \tag{2.34}$$

Let $0 < \delta < 1$. Using (2.30) with $\xi = \left(2^{1/\theta}K\right)^q$, we get by Hölder inequality and (2.32):

$$J_\delta \;\equiv\; \int_\delta^\infty \left[t^{1/q}f^*(t)\right]^\theta \frac{dt}{t} \;\le\; J_1 + K^\theta \int_\delta^1 \left[t^{1/q}f^*(\xi t)\right]^\theta \frac{dt}{t} + c\int_0^1 t^{\theta/q-1}\sigma(t)^\theta\,dt$$
$$\le\; J_1 + \frac{1}{2}J_\delta + c'\left(\int_0^1 t^{s(1/p-r/n)-1}\sigma(t)^s\,dt\right)^{\theta/s}.$$

By (2.34), $J_\delta < \infty$. The inequality (2.33) follows now from (2.31) and (2.34). □

Remark 2.6 Let $r_k \in \mathbb{N}$, $1 \le p_k, s_k < \infty$ for $k = 1,\dots,n$ $(n \ge 2)$ and $s_k = 1$, if $p_k = 1$. Let r, p and s be the numbers defined by (2.28) and (2.29). Assume that $p < n/r$ and set $q^* = np/(n-rp)$. Then for any function $f \in C^\infty(\mathbb{R}^n)$ with compact support we have

$$\|f\|_{q^*,s} \;\le\; c\prod_{k=1}^n \left\|D_k^{r_k}f\right\|_{p_k,s_k}^{r/(nr_k)}. \tag{2.35}$$

This statement follows immediately from the Lemma 2.4. The inequality (2.35) gives a generalization of the classical Sobolev's inequality with limiting exponent. A slightly different scheme of the proof of (2.35) was given in [10, Theorem 13.1]. In the case $p_k = s_k > 1$ $(k = 1,\dots,n)$ the inequality (2.35) contains in [2, Ch. 4]. For $r_1 = \ldots = r_n = 1$ the proof of (2.35) was given in [19]. One can find a detailed description of the preceding results in [10] (see also [19]).

3 The main theorem

Theorem 3.1 *Let $n \ge 2$, $r_k \in \mathbb{N}$, $1 \le p_k, s_k < \infty$ for $k = 1,\dots,n$ and $s_k = 1$ if $p_k = 1$. Let r, p and s be the numbers defined by* (2.28) *and* (2.29). *For every p_j $(1 \le j \le n)$ satisfying the condition*

$$\rho_j \;\equiv\; \frac{r}{n} + \frac{1}{p_j} - \frac{1}{p} \;>\; 0\,,$$

take arbitrary $q_j > p_j$ such that

$$\frac{1}{q_j} \;>\; \frac{1}{p} - \frac{r}{n}$$

and denote

$$\varkappa_j \;=\; 1 - \frac{1}{\rho_j}\left(\frac{1}{p_j} - \frac{1}{q_j}\right), \quad \alpha_j \;=\; \varkappa_j r_j\,, \quad \frac{1}{\theta_j} \;=\; \frac{1-\varkappa_j}{s} + \frac{\varkappa_j}{s_j}\,.$$

Then for any function $f \in S_0(\mathbb{R}^n)$ which has the weak derivatives $D_k^{r_k}f \in L^{p_k,s_k}(\mathbb{R}^n)$ $(k = 1,\dots,n)$ there holds the inequality

$$\left(\int_0^\infty \left[h^{-\alpha_j}\left\|\Delta_j^{r_j}(h)f\right\|_{q_j,1}\right]^{\theta_j}\frac{dh}{h}\right)^{1/\theta_j} \;\le\; c\sum_{k=1}^n \left\|D_k^{r_k}f\right\|_{p_k,s_k}, \tag{3.1}$$

where c is a constant that does not depend on f.

P r o o f. First observe that by our conditions $0 < \varkappa_j < 1$. Denote

$$g_k(x) \;=\; \left|D_k^{r_k}f(x)\right|.$$

Further, assume that $j = 1$ and set for $h > 0$

$$f_h(x) \;=\; |\Delta_1^{r_1}(h)f(x)|\,.$$

For almost all $x \in \mathbb{R}^n$ we have (see [2, Vol. 1, pp. 101])

$$f_h(x) \le \int_0^h \dots \int_0^h g_1(x + (u_1 + \dots + u_{r_1})e_1)\, du_1 \dots du_{r_1} . \tag{3.2}$$

From here,

$$f_h^*(t) \le h^{r_1} g_1^{**}(t) . \tag{3.3}$$

Indeed, for any subset $A \subset \mathbb{R}^n$ with $|A| = t$

$$\int_A f_h(x)\, dx \le h^{r_1} \sup_{B\subset\mathbb{R}^n,\, |B|=t} \int_B g_1(y)\, dy = h^{r_1} t g_1^{**}(t) .$$

From this, it follows (3.3).

If $p_1 = 1$ (in this case $s_1 = 1$), then it follows from (3.2) that $f_h \in L^1(\mathbb{R}^n)$. If $p_1 > 1$, then (3.3) implies that $f_h \in L^{p_1,s_1}(\mathbb{R}^n)$. Thus, by Corollary 2.5 we have that $f_h \in L^{q_1,1}(\mathbb{R}^n)$.

Denote for $h > 0$

$$J(h) \equiv \|f_h\|_{q_1,1} = \int_0^\infty t^{1/q_1 - 1} f_h^*(t)\, dt .$$

Set $\xi_0 = (4K)^{q_1}$ and

$$Q(h) = \{t > 0 : f_h^*(t) \ge 2K f_h^*(\xi_0 t)\} , \tag{3.4}$$

where K is the constant in Lemma 2.3. Then

$$\begin{aligned}\int_{\mathbb{R}_+\setminus Q(h)} t^{1/q_1 - 1} f_h^*(t)\, dt &\le 2K \int_0^\infty t^{1/q_1 - 1} f_h^*(\xi_0 t)\, dt \\ &= 2K\xi_0^{-1/q_1} \int_0^\infty t^{1/q_1 - 1} f_h^*(t)\, dt = \frac{1}{2} J(h) .\end{aligned}$$

Therefore,

$$J(h) \le 2 \int_{Q(h)} t^{1/q_1 - 1} f_h^*(t)\, dt \equiv 2J'(h) . \tag{3.5}$$

Denote

$$\psi_k(\hat{x}_k) = \int_{\mathbb{R}} g_k(x)\, dx_k , \quad \text{if} \quad p_k = 1 .$$

Let $\varepsilon = (1 - \varkappa_1)/2$ and

$$0 < \delta < \varepsilon \min\left(\left(\frac{r_1 n}{r} - 1\right)^{-1}, \frac{1}{2} \min_{\gamma_j<1} \min\left(\gamma_j, 1 - \gamma_j\right)\right) . \tag{3.6}$$

Now for every $k = 1, \dots, n$ we apply Lemma 2.1 with $\psi(t) = \psi_k^*(t/2)$ in the case $p_k = 1$ and $\psi(t) = g_k^{**}(t)$ in the case $p_k > 1$. We obtain that there exist functions $\varphi_k(t)$ $(k = 1, \dots, n)$ on $\mathbb{R}_+$ such that

$$\varphi_k(t)\, t^{1/p_k - \delta} \downarrow , \quad \varphi_k(t)\, t^{1/p_k + \delta} \uparrow , \tag{3.7}$$

$$\psi_k^*(t/2) \le \varphi_k(t) , \quad \text{if} \quad p_k = 1 , \tag{3.8}$$

$$g_k^{**}(t) \le \varphi_k(t) , \quad \text{if} \quad p_k > 1 , \tag{3.9}$$

and

$$\|\varphi_k\|_{p_k,s_k} \le c \left\|D_k^{r_k} f\right\|_{p_k,s_k} . \tag{3.10}$$

We shall estimate $f_h^*(t)$ for fixed $h > 0$ and $t \in Q(h)$. Let $E(t,h)$ be a set of type F_σ and measure t such that

$$f_h(x) \ \geq \ f_h^*(t) \quad \text{for all} \quad x \in E(t,h)\,. \tag{3.11}$$

Denote by $\lambda_k(t,h)$ the $(n-1)$-dimensional measure of the orthogonal projection of $E(t,h)$ onto the coordinate hyperplane $x_k = 0$. By Lemma 2.3, (3.8) and (3.9), we have that for each $t \in Q(h)$

$$f_h^*(t) \ \leq \ c\left(\frac{t}{\lambda_k(t,h)}\right)^{r_k-1}\varphi_k(\lambda_k(t,h))\,, \quad \text{if} \quad p_k \ = \ 1\,, \tag{3.12}$$

and

$$f_h^*(t) \ \leq \ c\left(\frac{t}{\lambda_k(t,h)}\right)^{r_k}\varphi_k(t)\,, \quad \text{if} \quad p_k \ > \ 1\,. \tag{3.13}$$

Applying inequality (2.23) and Lemma 2.2, we obtain that there exist a non-negative function $\sigma(t)$ and positive continuously differentiable functions $u_k(t)$ $(k = 1,\ldots,n)$ on $\mathbb{R}_+$ satisfying the following conditions:

$$f_h^*(t) \ \leq \ c\,\sigma(t)\,, \quad t \in Q(h)\,, \tag{3.14}$$

$$\left(\int_0^\infty t^{s(1/p\,-r/n)-1}\sigma(t)^s\,dt\right)^{1/s} \ \leq \ c\prod_{k=1}^n \left\|D_k^{r_k}f\right\|_{p_k,s_k}^{r/(nr_k)}\,, \tag{3.15}$$

$$\sigma(t) \ = \ \begin{cases} (t/u_k(t))^{r_k-1}\,\varphi_k(u_k(t))\,, & \text{if} \quad p_k \ = \ 1\,, \\ (t/u_k(t))^{r_k}\,\varphi_k(t)\,, & \text{if} \quad p_k \ > \ 1\,, \end{cases} \tag{3.16}$$

$$\prod_{k=1}^n u_k(t) \ = \ t^{n-1}\,, \tag{3.17}$$

$$u_1(t)t^{\delta-1} \text{ decreases}\,, \tag{3.18}$$

$$\int_0^\infty \frac{u_k(t)}{t}\,\varphi_k(u_k(t))\,dt \ \leq \ c\left\|D_k^{r_k}f\right\|_1\,, \quad \text{if} \quad p_k \ = \ 1\,. \tag{3.19}$$

The estimate (3.14) can be used for t "sufficiently small". For "large" t we need different estimates, involving h.

First, we have the estimate (3.3). Nevertheless, this estimate does not work in the case $p_1 = 1$ (the operator $g \to g^{**}$ is unbounded in L^1).

We shall prove an estimate which can be applied for all values of $p_1 \geq 1$. Denote

$$\beta(t) \ = \ t/u_1(t)\,. \tag{3.20}$$

We shall prove that for any $h > 0$ and any $t \in Q(h)$

$$f_h^*(t) \ \leq \ c\,h^{r_1-\varepsilon}\beta(t)^\varepsilon\chi(t)\,, \tag{3.21}$$

where $\varepsilon = (1-\varkappa_1)/2$ and

$$\chi(t) \ \equiv \ \sigma(t)\beta(t)^{-r_1} \ = \ \begin{cases} u_1(t)\varphi_1(u_1(t))/t\,, & \text{if} \quad p_1 \ = \ 1\,, \\ \varphi_1(t)\,, & \text{if} \quad p_1 \ > \ 1 \end{cases} \tag{3.22}$$

(see (3.16)). By (3.10) and (3.19),

$$\|\chi\|_{p_1,s_1} \ \leq \ c\left\|D_1^{r_1}f\right\|_{p_1,s_1}\,. \tag{3.23}$$

For $h \geq \beta(t)$ $(t \in Q(h))$ the inequality (3.21) follows directly from (3.14) and (3.22). Assume that $0 < h \leq \beta(t)$, $t \in Q(h)$. If $p_1 > 1$, then (3.21) is the immediate consequence of (3.3), (3.9) and (3.22).

Let $p_1 = 1$. First suppose that there exists $1 \leq j \leq n$ such that

$$\lambda_j(t,h) \ \geq \ \frac{1}{2}\,u_j(t)\left(\frac{\beta(t)}{h}\right)^{r_1/r_j}\,.$$

If $p_j > 1$, then by (3.13) and (3.16)

$$f_h^*(t) \le c\left(\frac{t}{u_j(t)}\right)^{r_j} \varphi_j(t)\left(\frac{h}{\beta(t)}\right)^{r_1} = c\,\sigma(t)\left(\frac{h}{\beta(t)}\right)^{r_1}.$$

If $p_j = 1$, then we apply (3.12). Notice that

$$\lambda_j(t,h) \ge \frac{1}{2}\,u_j(t)\,.$$

Taking into account that for $\delta_j = \varepsilon\, r_j/r_1$ the function $\varphi_j(u)\,u^{1-\delta_j}$ decreases and the function $\varphi_j(u)\,u^{1+\delta_j}$ increases (see (3.6) and (3.7)), we get that

$$\big(\lambda_j(t,h)\big)^{1-\delta_j}\varphi_j\big(\lambda_j(t,h)\big) \le \left(\frac{1}{2}\,u_j(t)\right)^{1-\delta_j}\varphi_j\left(\frac{u_j(t)}{2}\right) \le c\big(u_j(t)\big)^{1-\delta_j}\varphi_j\big(u_j(t)\big)\,.$$

Thus, by (3.12)

$$f_h^*(t) \le c\,h^{r_1-\varepsilon}\beta(t)^{\varepsilon-r_1}\left(\frac{t}{u_j(t)}\right)^{r_j-1}\varphi_j(u_j(t)).$$

From these estimates and (3.16) it follows the inequality (3.21), where $\chi(t)$ is defined by (3.22).

Now assume that for each $j = 1,\ldots,n$.

$$\lambda_j(t,h) < \frac{1}{2}\,u_j(t)\,(\beta(t)/h)^{r_1/r_j}\,. \tag{3.24}$$

First of all, it follows that

$$\lambda_1(t,h) < \frac{t}{2h}\,. \tag{3.25}$$

Further, for any F_σ-set $A \subset E \equiv E(t,h)$ denote by A_j the orthogonal projection of A onto the hyperplane $x_j = 0$. If

$$\mathrm{mes}_{n-1} A_1 \le \frac{1}{2}\,u_1(t)\left(\frac{h}{\beta(t)}\right)^{\frac{r_1 n}{r}-1} \equiv \frac{1}{2}\,\gamma(t,h)\,, \tag{3.26}$$

then

$$\mathrm{mes}_n A \le \frac{t}{2}\,.$$

Indeed, otherwise we would have by (3.26) and (2.23)

$$\prod_{j=2}^{n}\mathrm{mes}_{n-1} A_j \ge \frac{t^{n-1}}{2^{n-2}\gamma(t,h)} = \frac{1}{2^{n-2}}\left(\frac{\beta(t)}{h}\right)^{r_1\sum_{j=2}^{n} r_j^{-1}}\prod_{j=2}^{n} u_j(t)\,,$$

contrary to the assumption (3.24).

Using Lemma 3 of [8], we decompose the projection $E_1(t,h)$ into measurable disjoint subsets P and S such that

$$\mathrm{mes}_{n-1} S = \frac{1}{2}\,\gamma(t,h)$$

and

$$\int_P \psi_1(\hat{x}_1)\,d\hat{x}_1 \le \int_{\gamma(t,h)/2}^{t/(2h)} \psi_1^*(u)\,du\,. \tag{3.27}$$

It follows from the observation given above that the measure of the set

$$E' = \{x \in E(t,h) : \hat{x}_1 \in P\}$$

is at least $t/2$. For $\hat{x}_1 \in E_1(t,h)$ we denote by $T(\hat{x}_1)$ the section of the set $E(t,h)$ by the line that passes through $\hat{x}_1$ and is perpendicular to the hyperplane $x_1 = 0$ (note that $T(\hat{x}_1)$ is a set of type F_σ). For almost all $\hat{x}_1 \in E_1(t,h)$ we have (see (3.2))

$$f_h^*(t)\,\mathrm{mes}_1\, T(\hat{x}_1) \;\le\; \int_{T(\hat{x}_1)} f_h(x)\,dx_1 \;\le\; h^{r_1}\int_{\mathbb{R}} |D_1^{r_1} f(x)|\,dx_1 \;=\; h^{r_1}\psi_1(\hat{x}_1)\,.$$

Integrating this inequality with respect to $\hat{x}_1$ over P and taking into account (3.27) and the inequality

$$\int_P \mathrm{mes}_1\, T(\hat{x}_1)\,d\hat{x}_1 \;=\; |E'| \;\ge\; \frac{t}{2}\,,$$

we get (see also (3.8))

$$f_h^*(t) \;\le\; \frac{h^{r_1}}{t}\int_{\gamma(t,h)}^{t/h} \varphi_1(u)\,du\,. \tag{3.28}$$

For $0 < h \le \beta(t)$ we have

$$\gamma(t,h) \;\le\; u_1(t) \;\le\; t/h\,.$$

Furthermore, let $\eta = \varepsilon/\left(\frac{r_1 n}{r} - 1\right)$. By (3.7), $\varphi_1(u)\,u^{1+\eta}$ increases and $\varphi_1(u)\,u^{1-\varepsilon}$ decreases on $(0,\infty)$. Thus, we have

$$\begin{aligned}\int_{\gamma(t,h)}^{t/h} \varphi_1(u)\,du \;&=\; \int_{\gamma(t,h)}^{u_1(t)} \varphi_1(u)\,du + \int_{u_1(t)}^{t/h} \varphi_1(u)\,du\\ &\le\; \varphi_1(u_1(t))u_1(t)^{1+\eta}\gamma(t,h)^{-\eta}/\eta + \varphi_1(u_1(t))u_1(t)^{1-\varepsilon}(t/h)^{\varepsilon}/\varepsilon\\ &=\; c\,h^{-\varepsilon}\beta(t)^{\varepsilon}u_1(t)\varphi_1(u_1(t))\,.\end{aligned}$$

From here and (3.28) it follows (3.21).

Finally, taking into account (3.14) and (3.21), we obtain that for any $h > 0$ and any $t \in Q(h)$

$$f_h^*(t) \;\le\; c\,\Phi(t,h)\,, \tag{3.29}$$

where

$$\Phi(t,h) \;=\; \min\left(\sigma(t), h^{r_1-\varepsilon}\beta(t)^{\varepsilon}\chi(t)\right) \tag{3.30}$$

and $\chi(t)$ is defined by (3.22).

Further, we have (see (3.5))

$$J'(h) \;\le\; c\int_0^\infty t^{1/q_1 - 1}\Phi(t,h)\,dt$$

and

$$J \;\equiv\; \int_0^\infty h^{-\alpha_1\theta_1 - 1}J(h)^{\theta_1}\,dh \;\le\; c\int_0^\infty h^{-\alpha_1\theta_1-1}\,dh\left(\int_0^\infty t^{1/q_1 - 1}\Phi(t,h)\,dt\right)^{\theta_1}.$$

By (3.18), the function $\beta(t)t^{-\delta}$ increases on $\mathbb{R}_+$. It easily follows that the inverse function β^{-1} exists on $\mathbb{R}_+$ and satisfies the condition

$$\beta^{-1}(2z) \;\le\; 2^{1/\delta}\beta^{-1}(z)\,. \tag{3.31}$$

Furthermore, we have

$$J \le c\left[\int_0^\infty h^{-\alpha_1\theta_1-1}\,dh\left(\int_0^{\beta^{-1}(h)} t^{1/q_1-1}\Phi(t,h)\,dt\right)^{\theta_1}\right.$$
$$\left.+\int_0^\infty h^{-\alpha_1\theta_1-1}\,dh\left(\int_{\beta^{-1}(h)}^\infty t^{1/q_1-1}\Phi(t,h)\,dt\right)^{\theta_1}\right] \equiv c\,(J_1+J_2)\,.$$

Applying Minkowsi's inequality, we obtain

$$\begin{aligned} J_1^{1/\theta_1} &= \left(\int_0^\infty h^{-\alpha_1\theta_1-1}\,dh\left(\sum_{k=0}^\infty\int_{\beta^{-1}(2^{-k-1}h)}^{\beta^{-1}(2^{-k}h)} t^{1/q_1-1}\sigma(t)\,dt\right)^{\theta_1}\right)^{1/\theta_1}\\ &\le \sum_{k=0}^\infty\left(\int_0^\infty h^{-\alpha_1\theta_1-1}\,dh\left(\int_{\beta^{-1}(2^{-k-1}h)}^{\beta^{-1}(2^{-k}h)} t^{1/q_1-1}\sigma(t)\,dt\right)^{\theta_1}\right)^{1/\theta_1}\\ &= \sum_{k=0}^\infty 2^{-k\alpha_1}\left(\int_0^\infty z^{-\alpha_1\theta_1-1}\,dz\left(\int_{\beta^{-1}(z/2)}^{\beta^{-1}(z)} t^{1/q_1-1}\sigma(t)\,dt\right)^{\theta_1}\right)^{1/\theta_1}\,. \end{aligned}$$

Further, using the Hölder inequality and (3.31), we get

$$\int_{\beta^{-1}(z/2)}^{\beta^{-1}(z)} t^{1/q_1-1}\sigma(t)\,dt \le c\left(\int_0^{\beta^{-1}(z)} t^{\theta_1/q_1-1}\sigma(t)^{\theta_1}\,dt\right)^{1/\theta_1}$$

Thus, by Fubini's theorem and (3.22)

$$\begin{aligned} J_1 &\le c\int_0^\infty z^{-\alpha_1\theta_1-1}\,dz\int_0^{\beta^{-1}(z)} t^{\theta_1/q_1-1}\sigma(t)^{\theta_1}\,dt\\ &= c\int_0^\infty t^{\theta_1/q_1-1}\sigma(t)^{\theta_1}\,dt\int_{\beta(t)}^\infty z^{-\alpha_1\theta_1-1}\,dz\\ &= c'\int_0^\infty t^{\theta_1/q_1-1}\sigma(t)^{\theta_1}\beta(t)^{-\alpha_1\theta_1}\,dt\\ &= c'\int_0^\infty t^{\theta_1/q_1-1}\chi(t)^{\varkappa_1\theta_1}\sigma(t)^{(1-\varkappa_1)\theta_1}\,dt\,. \end{aligned} \tag{3.32}$$

The same reasonings give that

$$\begin{aligned} J_2 &\le c\int_0^\infty z^{[r_1(1-\varkappa_1)-\varepsilon]\theta_1}\,\frac{dz}{z}\int_{\beta^{-1}(z)}^\infty t^{\theta_1/q_1-1}\beta(t)^{\theta_1\varepsilon}\chi(t)^{\theta_1}\,dt\\ &= c\int_0^\infty t^{\theta_1/q_1-1}\beta(t)^{\theta_1\varepsilon}\chi(t)^{\theta_1}\,dt\int_0^{\beta(t)} z^{[r_1(1-\varkappa_1)-\varepsilon]\theta_1-1}\,dz\\ &= c'\int_0^\infty t^{\theta_1/q_1-1}\chi(t)^{\theta_1}\beta(t)^{r_1(1-\varkappa_1)\theta_1}\,dt\,. \end{aligned}$$

By (3.22) the last integral is the same as one on the right-hand side of (3.32). Therefore, we have that

$$J \le c\int_0^\infty t^{\theta_1/q_1-1}\chi(t)^{\varkappa_1\theta_1}\sigma(t)^{(1-\varkappa_1)\theta_1}\,dt\,.$$

Now we apply Hölder inequality with the exponents $u = s_1/(\varkappa_1\theta_1)$ and $u' = s_1/(s_1 - \varkappa_1\theta_1)$. Observe that

$$(1-\varkappa_1)\theta_1 u' \;=\; s\,, \quad \left(\frac{\theta_1}{q_1} - \frac{s_1}{p_1 u}\right) u' \;=\; s\left(\frac{1}{p} - \frac{r}{n}\right).$$

Thus, we obtain, using (3.15) and (3.23):

$$\begin{aligned} J^{1/\theta_1} \;&\le\; c\left(\int_0^\infty t^{s(1/p\,-r/n)-1}\sigma(t)^s\,dt\right)^{(1-\varkappa_1)/s} \|D_1^{r_1} f\|_{p_1,s_1}^{\varkappa_1} \\ &\le\; c\left(\prod_{j=1}^n \left\|D_j^{r_j} f\right\|_{p_j,s_j}^{r/(nr_j)}\right)^{1-\varkappa_1} \left\|D_1^{r_1} f\right\|_{p_1,s_1}^{\varkappa_1}. \end{aligned}$$

Since

$$\sum_{j=1}^n \frac{r}{nr_j} \;=\; 1\,,$$

we obtain the inequality (3.1). The theorem is proved. □

Remark 3.2 First we recall the definition of the Besov space in the direction of the coordinate axis x_j (see [13, Ch. 4]).

Let $\alpha > 0$, $1 \le p, \theta < \infty$ and $1 \le j \le n$. Define the space $B^\alpha_{p,\theta;j}(\mathbb{R}^n)$ as the class of all functions $f \in L^p(\mathbb{R}^n)$ for which

$$\|f\|_{B^\alpha_{p,\theta;j}} \;\equiv\; \|f\|_p + \left(\int_0^\infty \left[h^{-\alpha}\left\|\Delta_j^r(h) f\right\|_p\right]^\theta \frac{dh}{h}\right)^{1/\theta} \;<\; \infty \tag{3.33}$$

for any integer $r > \alpha$. Of course, the right-hand side in (3.33) depends on r, but every choice of the integer $r > \alpha$ leads to equivalent norms [13, Ch. 4].

Now observe that the conditions of Theorem 3.1 do not imply the belongness of the function f to some $L^\nu(\mathbb{R}^n)$. However, if we assume in addition that $f \in L^{p_0}(\mathbb{R}^n)$ for some $p_0 \ge 1$ and that $q_j > p_0$, then by Corollary 2.5 we get $f \in L^{q_j,1}(\mathbb{R}^n)$. Thus, with these additional conditions Theorem 3.1 implies that $f \in B^{\alpha_j}_{q_j,\theta_j;j}(\mathbb{R}^n)$ and

$$\|f\|_{B^{\alpha_j}_{q_j,\theta_j;j}} \;\le\; c\left[\|f\|_{p_0} + \sum_{k=1}^n \left\|D_k^{r_k} f\right\|_{p_k,s_k}\right].$$

Remark 3.3 It is important to emphasize that the values of the parameters θ_k found in the Theorem 3.1 are sharp. To verify this statement we shall consider the following simple example.

Assume that $n = 2$, $r_1 = r_2 = 1$, $1 \le p_1, p_2 < \infty$ and $s_1 = p_1$, $s_2 = p_2$. Furthermore, suppose that

$$p \;\equiv\; 2\left(\frac{1}{p_1} + \frac{1}{p_2}\right)^{-1} \;<\; 2\,.$$

Then

$$\frac{1}{p_i} \;>\; \frac{1}{p} - \frac{1}{2} \quad (i = 1, 2)\,.$$

Let $q_1 > p_1$ be such that

$$\frac{1}{q_1} \;>\; \frac{1}{p} - \frac{1}{2}\,.$$

As in Theorem 3.1, set

$$\varkappa_1 \;=\; 1 - \frac{1/p_1 - 1/q_1}{1/p_1 - 1/p + 1/2}\,, \quad \alpha_1 \;=\; \varkappa_1\,, \quad \frac{1}{\theta_1} \;=\; \frac{1-\varkappa_1}{p} + \frac{\varkappa_1}{p_1}\,.$$

Let $0 < \varepsilon < \theta_1$; define the following numbers

$$\alpha = \frac{2/p-1}{1+2(1/p_1-1/p)}\,, \qquad \beta = \frac{2/p-1}{1+2(1/p_2-1/p)}\,,$$
$$\delta = \frac{1}{p_1[1+2(1/p_1-1/p)]}\frac{\theta_1}{\theta_1-\varepsilon}\,, \quad \gamma = \frac{1}{p_2[1+2(1/p_2-1/p)]}\frac{\theta_1}{\theta_1-\varepsilon}\,.$$

Further, denote for $(x,y) \in [-1,1]^2$

$$\varphi_0(x,y) = |x|^\alpha \left(\log\frac{e}{|x|}\right)^\delta + |y|^\beta \left(\log\frac{e}{|y|}\right)^\gamma .$$

Set

$$D = \left\{(x,y) \in [-1,1]^2 : \varphi_0(x,y) \le 1\right\}$$

and

$$f(x,y) \equiv f_\varepsilon(x,y) = \begin{cases} [\varphi_0(x,y)]^{-1} - 1\,, & \text{if} \quad (x,y) \in D\,, \\ 0\,, & \text{if} \quad (x,y) \notin D\,. \end{cases}$$

Carrying out routine calculations, one can show that the function f has the following properties:

(i) $f \in L^\nu(\mathbb{R}^2)$ for any $1 \le \nu \le 2p/(2-p)$;

(ii) $\dfrac{\partial f}{\partial x} \in L^{p_1}(\mathbb{R}^2)$, $\dfrac{\partial f}{\partial y} \in L^{p_2}(\mathbb{R}^2)$;

(iii) $\displaystyle\int_0^\infty \left[h^{-\alpha_1}\left\|\Delta_1^1(h)f\right\|_{q_1}\right]^{\theta_1-\varepsilon}\frac{dh}{h} = +\infty\,.$

This implies that the values of θ_k in Theorem 3.1 cannot be reduced.

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