Frobenius-Padé approximants of $|x|$

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Abstract

We construct the Frobenius–Padé approximants of the function $|x|$ in $(-1,1)$ for the Chebyshev weight. These rational functions are linked with the Frobenius-Padé approximants of the function $\sqrt{x}$ in $(0,1)$. We prove geometric convergence of the approximants of $|x|$ in $\mathbb{C} \setminus \{z : \Re(z) = 0\}$.

Key words: Frobenius–Padé approximation, Fourier–Padé approximation, rational approximation, orthogonal polynomials, varying measures

1. Introduction

We know that the Fourier expansion of a piecewise analytic function converges slowly in the L^2-norm but in general we do not have convergence in the uniform norm. It is also well known that around the discontinuities of a function we find the Gibbs phenomenon. To overcome these problems, many different approaches have been developed (see, for example, [1] and references therein). Many problems in approximation theory can be connected with the problem of approximating the piecewise analytic function $|x|$ on a set having the origin as an inner point. One of the main reasons is the fact that $|x|$ can be seen as the prototype of a $1-$Lipschitz function (see [5]). In [9] and [12] it is proved that $|x|$ can be approximated uniformly on $[-1,1]$ by rational functions of degree n with an error $O(\exp(-\pi\sqrt{n}))$. For polynomials of degree n the error is $O(1/n)$.

In this paper, we study Frobenius-Padé approximation of $|x|$ with respect to the Chebyshev weight. We obtain an explicit formula for these rational functions. The rate of convergence of such approximants is $O(1/n^{3/2})$ at zero

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and for compact subset of $\mathbb{C}\setminus\{z\in\mathbb{C}:\Re(z)=0\}$ is $O(q^n)$ for some $q\in(0,1)$ which depends on the compact subset. This problem is related to Frobenius-Padé approximation of $\sqrt{x}$ in $[0,1]$ and the proof uses techniques from Padé approximation and potential theory.

Let μ be a finite positive Borel measure on the real line $\mathbb{R}$ and let $\{\varphi_j\}_{j\ge0}$ denote the sequence of orthonormal polinomial with respecto to μ. Given $f\in L^1(\mu)$ its *Frobenius-Padé approximant* of order (n,m) with respect to μ is a rational function $\Pi_{n,m}=P_{n,m}/Q_{n,m}$ where $P_{n,m}$ and $Q_{n,m}$ are polynomials such that $\deg P_{n,m}\le n$, $\deg Q_{n,m}\le m$, $Q_{n,m}\not\equiv 0$, and

$$\int (Q_{n,m}(x)f(x)-P_{n,m}(x))\varphi_j(x)\,d\mu(x)=0,\quad j=0,1,\ldots,n+m. \tag{1.1}$$

It means that the Fourier expansion of $Q_{n,m}(x)f(x)-P_{n,m}(x)$ with respect to $\{\varphi_j : j\ge 0\}$ starts at least in the term of index $n+m+1$. Thus, $P_{n,m}(x)$ is the Fourier partial sum of order n of $Q_{n,m}f$ and

$$\int Q_{n,m}(x)f(x)\varphi_j(x)\,d\mu(x)=0,\quad j=n+1,n+2,\ldots,n+m,$$

which is a linear system in the coefficients of $Q_{n,m}$ of $m+1$ equations. Hence, a Frobenius-Padé approximant of f always exists. In general it is not unique. However, if the denominators of the approximants of order (n,m) are always of degree exactly m, then the approximant of this order is unique.

In [7], it is studied Frobenius-Padé approximants for Markov functions. The measure defining Markov functions are supported on an interval disjoint with the support of the measure which defines the expansion in the Frobenius-Padé approximants. This is the typical situation considered up to now (see also [3], [4] and [8]).

We obtain the following formulas for the numerators and denominators of Frobenius-Padé approximants of $|x|$ with respect to the Chebyshev weight $\frac{dx}{\pi\sqrt{1-x^2}}$, $x\in(-1,1)$.

Theorem 1.1. *The denominator of the Frobenius-Padé approximant of order* $(2n+1,2m+1)$ *of* $|x|$ *with respect to the Chebyshev weight is given by*

$$\begin{aligned} Q_{2n+1,2m+1}(x) &= x\,{}_3F_2(-m,-n+1/2,n+m+5/2;2,3/2;x^2)\\ &=\sum_{j=0}^{m}\frac{(-m)_j(-n+1/2)_j(n+m+5/2)_j}{j!(j+1)!(3/2)_j}x^{2j+1}. \end{aligned}\tag{1.2}$$

For the order $(2n,2m)$ *we have*

$$\begin{aligned} Q_{2n,2m}(x) &= {}_3F_2(-m,-n+1/2,n+m+3/2;3/2,1;x^2)\\ &=\sum_{j=0}^{m}\frac{(-m)_j(-n+1/2)_j(n+m+3/2)_j}{(j!)^2(3/2)_j}x^{2j}. \end{aligned}\tag{1.3}$$

As usual, $(a)_n$ *denotes the Pochhammer symbol, i.e.*

$$(a)_n := a(a+1)\cdots(a+n-2)(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Theorem 1.2. *The numerator of the Frobenius-Padé approximant of order* $(2n+1, 2m+1)$ *of* $|x|$ *with respect to the Chebyshev weight is given by*

$$\begin{aligned} P_{2n+1,2m+1}(x) &= xB_{n,m}\,{}_3F_2(-n, -m-1/2, n+m+2; 1/2, 3/2; x^2) \\ &= B_{n,m} \sum_{j=0}^{n} \frac{(-n)_j(-m-1/2)_j(n+m+2)_j}{j!(1/2)_j(3/2)_j} x^{2j+1}, \end{aligned}$$

where $B_{n,m} = \frac{m!(n+m+1)!(1/2)_n}{\sqrt{\pi}n!(1/2)_{m+1}\Gamma(n+m+5/2)}$. *For the order* $(2n, 2m)$ *we have*

$$\begin{aligned} P_{2n,2m}(x) &= C_{n,m}\,{}_3F_2(-n, -m-1/2, n+m+1; 1/2, 1/2; x^2) \\ &= C_{n,m} \sum_{j=0}^{n} \frac{(-n)_j(-m-1/2)_j(n+m+1)_j}{j!(1/2)_j^2} x^{2j}, \end{aligned}$$

where $C_{n,m} = \frac{m!(n+m)!(1/2)_n}{\sqrt{\pi}n!(3/2)_m\Gamma(n+m+3/2)}$.

A formula for Frobenius-Padé approximants of $\mathrm{sgn}(x)$ with respect to the Chebyshev weight is given in [10]. Theorems 1.1 and 1.2 are proved in Section 3. We also obtain the rate of convergence of these approximants and results of overconvergence.

Theorem 1.3. 1. *We have*

$$\lim_n \Pi_{n,n}(z) = \begin{cases} z, & \text{for } \Re(z) > 0, \\ -z, & \text{for } \Re(z) < 0, \end{cases}$$

uniformly on each compact subset of the mentioned region and for all $a > 0$, $\lim_n \Pi_{n,n}(x) = |x|$ *uniformly on* $[-a, a]$.

2. *The rate of convergence of* $\Pi_{n,n}$ *is geometric in each compact subset of* $\mathbb{C} \setminus \{\Re(z) = 0\}$*; for example, if* $K \subset \{z : \Re z > 0\}$*, there exists* $q = q(K) \in (0, 1)$ *such that,*

$$\limsup_n \sup_{z \in K} |\Pi_{n,n}(z) - z|^{1/n} \leq q.$$

The proof of Theorem 1.3 is reduced to the study of Frobenius-Padé approximants of square root function in $(0, 1)$. Its proof requires some auxiliary results from potential theory which are obtained in Section 4.

2. Auxiliary results

Let $\Pi_n := P_n/Q_n$ denote a Frobenius-Padé approximant of order (n, n) of $|x|$ with respect to the Chebyshev weight $d\mu(x) = \frac{dx}{\pi\sqrt{1-x^2}}$ in $(-1, 1)$. From the definition 1.1 of Frobenius-Padé approximant we obtain the following result.

Lemma 2.1. 1. *If* $n = 2k$, $\Pi_{2k}(z) = P_{2k}(z)/Q_{2k}(z) = p_k(z^2)/q_k(z^2)$, *where* $\pi_k = p_k/q_k$ *is the Frobenius-Padé approximant of order* (k,k) *of* $\sqrt{x}$ *with respect to the measure* $\frac{dx}{\pi\sqrt{x-x^2}}$, $x \in (0,1)$.

2. *There exist* $2k+1$ *points* $z_1, z_2, \ldots, z_{2k+1}$ *in* $(0,1)$ *such that*

$$q_k(z_j)\sqrt{z_j} - p_k(z_j) = 0, \quad j = 1, \ldots, 2k+1.$$

Let $w_{2k+1}(z) := \prod_{j=1}^{2k+1}(z - z_j)$. *Of course each* z_j *depends on* k *but for a simple notation we does not write these dependence.*

3. *The polynomial* $q_k(x)$ *has exactly degree* k *and satisfies the orthogonality relations*

$$\int_{-\infty}^{0} q_k(t)t^j \frac{\sqrt{-t}dt}{w_{2k+1}(t)} = 0, \quad j = 0, 1, \ldots, k-1. \tag{2.1}$$

All zeros of q_k *are simple, contained in* $(-\infty, 0)$*, and their total number is exactly* k*. Let* $\zeta_1 < \zeta_2 < \ldots < \zeta_k$ *denote the zeros of* q_k.

4. *The following formula holds true*

$$q_k(z)\sqrt{z} - p_k(z) = \frac{w_{2k+1}(z)}{\pi h_k(z)} \int_{-\infty}^{0} \frac{h_k(x)q_k(x)}{x-z} \frac{\sqrt{-x}dx}{w_{2k+1}(x)}, \tag{2.2}$$

where $z \in \mathbb{C} \setminus (-\infty, 0)$ *and* h_k *denotes any no null polynomial of degree* $\leq k$*. In particular, if* $h_k = q_k$ *we have*

$$\sqrt{z} - \pi_k(z) = \frac{w_{2k+1}(z)}{\pi q_k^2(z)} \int_{-\infty}^{0} \frac{q_k^2(x)}{x-z} \frac{\sqrt{-x}dx}{w_{2k+1}(x)}. \tag{2.3}$$

Of course, we have analogous formulas for odd order. For example,

$$P_{2k+1}(x) = xp_k(x^2), \quad Q_{2k+1}(x) = xq_k(x^2),$$

where $\pi_k = p_k/q_k$ *is the Frobenius-Padé approximants of order* (k,k) *of* $\sqrt{x}$ *with respect to the measure* $\frac{\sqrt{x}\,dx}{\pi\sqrt{1-x}}$, $x \in (0,1)$.

Proof. From the symmetry of μ and $f(x) = |x|$ it follows the first statement in the lemma. The second one is a direct consequence of the orthogonality condition in the definition of Frobenius-Padé approximants.

Let C be a positively oriented Jordan curve in $\mathbb{C} \setminus (-\infty, 0]$. Since

$$(q_k(z)\sqrt{z} - p_k(z))/w_{2k+1}(z)$$

is an analytic function in $\mathbb{C} \setminus (-\infty, 0]$, by Cauchy's Theorem we have

$$0 = \int_C \frac{q_k(z)\sqrt{z} - p_k(z)}{w_{2k+1}(z)} z^j dz = \int_C \frac{q_k(z)\sqrt{z}}{w_{2k+1}(z)} z^j dz - \int_C \frac{p_k(z)}{w_{2k+1}(z)} z^j dz$$
$$= \int_C \frac{q_k(z)\sqrt{z}}{w_{2k+1}(z)} z^j dz,$$

for $j = 0, 1, \ldots, k$. We see that $p_k(z)z^j/w_{2k+1}(z)$ has a zero of order at least 2 at infinity for such values of j, and is analytic outside C. Hence, we have

$$\int_C \frac{q_k(z)\sqrt{z}}{w_{2k+1}(z)} z^j dz = 0.$$

If we deform C to the boundary of an annulus slit along the negative real axis and let its inner radius tends to 0 and its outer radius tends to ∞, then the integral above converges to (2.1).

Observe that w_{2k+1} is a polynomial whose zeros lies in $(0, 1)$ and has constant sign in $(-\infty, 0]$. From the orthogonality relation (2.1) it follows rather immediately that $\deg(q_k) = k$, and that all its zeros are simple and contained in $(-\infty, 0)$ (see [13], Chapter III).

By Cauchy's integral formula, for all z in the bounded region limited by C we deduce

$$\frac{h_k(z)(q_k(z)\sqrt{z} - p_k(z))}{w_{2k+1}(z)} = \frac{1}{2\pi i}\int_C \frac{h_k(t)(q_k(t)\sqrt{t} - p_k(t))}{w_{2k+1}(t)} \frac{dt}{t - z}$$

and for all h_k polynomial of degree $\leq k$. Letting again curves C deform to $(-\infty, 0]$ as before yields (2.2). We note that the integral in (2.2) exists for all $z \in \mathbb{C} \setminus (-\infty, 0)$ and is a continuous function. Further, we note that the factor $2i$ arises from the analytic continuation of $\sqrt{z}$ to $(-\infty, 0)$ from both sides. $\square$

Remark 2.2. The poles and the zeros of the Frobenius-Padé approximants of $|x|$ are located in $\Re(z) = 0$ and they are strictly interlace.

For the rational approximant $\pi_k = p_k/q_k$ given in Lemma 2.1 we have the following properties.

Lemma 2.3. 1. *We have*

$$\pi_k(z) = \frac{p_k(z)}{q_k(z)} = \sum_{j=1}^{k} \frac{\lambda_j}{z - \zeta_j} + A_k. \tag{2.4}$$

Moreover, the following inequalities hold

$$0 < \pi_k(0) < \pi_k(1) < 1, \quad A_k > 0, \quad \text{and} \quad \lambda_j < 0, \quad j = 1, \ldots, k. \tag{2.5}$$

2. *The zeros of* $\sqrt{z} - \pi_k(z)$ *in* $\mathbb{C} \setminus (-\infty, 0]$ *are precisely* $2k+1$ *points in* $(0, 1)$.
3. *The polynomial* p_k *has exactly* k *zeros,* $\eta_1, \eta_2, \ldots, \eta_k$*, which alternate with* $\{\zeta_j\}$*, i.e.*

$$-\infty < \zeta_1 < \eta_1 < \zeta_2 < \ldots < \zeta_{k-1} < \eta_{k-1} < \zeta_k < \eta_k < 0.$$

4. *The derivative*

$$\pi_k^{(j)}(z) \text{ has the same sign as } (\sqrt{z})^{(j)} \text{ in } (0, \infty), \quad j \geq 0.$$

Therefore, π_k *is strictly increasing in* $[0, \infty)$.

Proof. The formula (2.4) follows immediately because of the zeros of q_k are simple. Setting $h_k(z) = \frac{q_k(z)}{z-\zeta_j}$ in (2.2), we obtain

$$\sqrt{z} - \pi_k(z) = \frac{w_{2k+1}(z)}{\pi \frac{q_k(z)}{z-\zeta_j} q_k(z)} \int_{-\infty}^{0} \frac{\frac{q_k(x)}{x-\zeta_j} q_k(x)}{x - z} \frac{\sqrt{-x}\, dx}{w_{2k+1}(x)}.$$

Multiplying this identity by $(z - \zeta_j)$, taking limit $z \to \zeta_j$, and by the partial fraction decomposition of π_k, we have

$$-\lambda_j = \frac{w_{2k+1}(\zeta_j)}{\pi(q_k'(\zeta_j))^2} \int_{-\infty}^{0} \left(\frac{q_k(x)}{x - \zeta_j} \right)^2 \frac{\sqrt{-x}\, dx}{w_{2k+1}(x)}.$$

Because w_{2k+1} is a monic polynomial of degree odd whose zeros lie in $(0, 1)$ and $\zeta_j < 0$, we get

$$\lambda_j < 0, \quad j = 1, \dots, n.$$

If $\sqrt{z} - \pi_k(z)$ had more than $2n + 1$ zeros in $(0, 1)$, by employing the same line of reasoning as in the proof of (2.1), we conclude that $q_k \equiv 0$; moreover,

$$\Im \left(\int_{-\infty}^{0} \frac{q_k^2(x)}{x - z} \frac{\sqrt{-x} dx}{w_{2k+1}(x)} \right) = \Im(z) \int_{-\infty}^{0} \frac{q_k^2(x)}{|x - z|^2} \frac{\sqrt{-x} dx}{w_{2k+1}(x)} \neq 0, \quad \text{for } \Im(z) \neq 0,$$

$$\int_{-\infty}^{0} \frac{q_k^2(x)}{x - z} \frac{\sqrt{-x} dx}{w_{2k+1}(x)} < 0, \qquad \text{for } z \geq 0.$$

Hence, the zeros of $\sqrt{z} - \pi_k(z)$ in $\mathbb{C} \setminus (-\infty, 0]$ are precisely $2k+1$ points in $(0, 1)$.

Setting $z = 0$ in (2.3), it follows that

$$-\pi_k(0) = \frac{w_{2n+1}(0)}{q_k^2(0)} \int_{-\infty}^{0} \frac{q_k^2(x)}{x} \frac{\sqrt{-x}}{w_{2k+1}(x)} dx < 0 \Rightarrow \pi_k(0) > 0.$$

This, along with

$$\lim_{z \to \zeta_j^{\pm}} \pi_k(z) = \lim_{z \to \zeta_j^{\pm}} \left(\sum_{j=1}^{k} \frac{\lambda_j}{z - \zeta_j} + A_k \right) = \mp\infty,$$

allows us to conclude that π_k (or equivalently p_k) has a simple zero in $(\zeta_k, 0)$ and in each interval (ζ_{j-1}, ζ_j). Also,

$$\pi_k(0) = A_k \prod_{j=1}^{k} \frac{\eta_j}{\zeta_j},$$

so we get $A_k > 0$.

Of course, the ith derivative of π_k according to the formula (2.4) equals

$$\pi_k^{(i)}(z) = (-1)^i \sum_{j=1}^{k} \frac{\lambda_j}{(z - \zeta_j)^i},$$

which has the same sign as $(\sqrt{z})^{(i)}$.

Observe that $\sqrt{z} - \pi_k(z)$ is negative at zero and alternate the sign at $2k+1$ points $z_1 < z_2 < \ldots < z_{2k+1} < 1$. So $\sqrt{z} - \pi_k(z) > 0$ for $z \in (z_{2k+1}, 1)$ and $\pi_k(1) < 1$.

□

Remark 2.4. In Figure 1 we have showed the error function of the Frobenius-Padé approximants of order $(1,1)$ of $\sqrt{x}$ for the measure $\frac{dx}{\pi\sqrt{x-x^2}}$, $x \in (0,1)$, in the interval $(0,2)$, respectively.

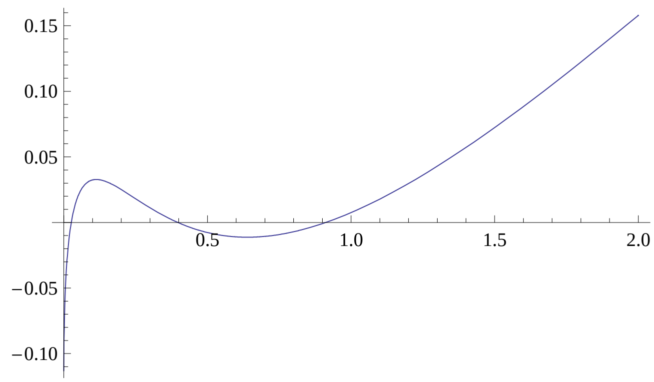

Figure 1: Error function $\sqrt{x} - \pi_1$ in the interval $(0,2)$.

Now we consider the function

$$N_k(z) := \frac{\sqrt{z} - \pi_k(z)}{\sqrt{z} + \pi_k(z)} = \frac{1 - z^{-1/2}\pi_k(z)}{1 + z^{-1/2}\pi_k(z)}, \quad z \in \mathbb{C} \setminus (-\infty, 0], \quad k \geq 1.$$

It is equivalent to

$$\pi_k(z) = \sqrt{z}\left(1 - 2\frac{N_k(z)}{1 + N_k(z)}\right), \quad \forall z \in \mathbb{C} \setminus (-\infty, 0]. \tag{2.6}$$

Lemma 2.5. *Each function N_k, $k \in \mathbb{N}$, is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and has zeros precisely at $z_1, z_2, \ldots, z_{2k+1}$. The sequence $\{N_k\}$ is uniformly bounded on each compact subset of $\mathbb{C} \setminus (-\infty, 0]$ and*

$$|N_k(x \pm i0)| = 1, \quad x \in (-\infty, 0],$$

where $N_k(x \pm i0)$ is the limit of $N_k(z)$ as $z \to x$ with $\Im(z) > 0$ or $\Im(z) < 0$, respectively. Moreover, the sequence $\{\pi_k\}$ is uniformly bounded on each compact subset of $\mathbb{C} \setminus (-\infty, 0]$.

Proof. The zeros of $\sqrt{z} - \pi_k(z)$ in $\mathbb{C} \setminus (-\infty, 0]$ are $z_1, z_2, \ldots, z_{2k+1}$ which lies in $(0,1)$ and the poles of $\sqrt{z} + \pi_k(z)$ are the zeros of q_k which lies in $(-\infty, 0)$. Thus, the zeros of N_k in $\mathbb{C} \setminus (-\infty, 0]$ are $z_1, z_2, \ldots, z_{2k+1}$.

We know that all coefficients λ_j, $j = 1, 2, \ldots, k$, in the partial fraction representation (2.4) have identical signs, thus the value $\pi_k(z)$ runs through the extended real line $\overline{\mathbb{R}}$ when z is moved along the interval (ζ_j, ζ_{j+1}) with ζ_j and ζ_{j+1} two adjacent poles. Also, we have that

$$\lim_{z \to 0 \pm i0} N_k(z) = -1, \qquad \lim_{z \to \infty \pm i0} N_k(z) = 1. \tag{2.7}$$

Therefore, from the definition of the function N_k, the bijectivity of the mapping $\tau(x) = \frac{\zeta - x}{\zeta + x}$, and for $x \in \mathbb{R}$, $\arg(\zeta) = \pm\frac{\pi}{2}$, i.e. $\Re(\zeta) = 0$,

$$\left| \frac{\zeta - x}{\zeta + x} \right| = 1,$$

it follows that $\arg(N_k(z))$ grows exactly by 2π if z is moved from ζ_j to ζ_{j+1} on $\mathbb{R}_- + i0$. Correspondingly, $\arg(N_k(z))$ grows by 2π if z is moved in the opposite direction from ζ_{j+1} to ζ_j on the other bank $\mathbb{R}_- + i0$ of $\mathbb{R}_-$. Because of (2.7) the same conclusions hold for the intervals $(-\infty, \zeta_1) + i0 \cup (-\infty, \zeta_1) - i0$, $((\zeta_k, 0) - i0)$ and $((0, \zeta_k) + i0)$.

Thus $\arg(N_k(z))$ grows by $2\pi(2k+1)$ if z moves once around the boundary of the domain $\mathbb{C} \setminus (-\infty, 0]$. This boundary consists of the two branches $\mathbb{R}_- + i0$ and $\mathbb{R}_- + i0$ of $(-\infty, 0]$. We know that N_k has exactly $2k+1$ simple zeros in $\mathbb{C} \setminus (-\infty, 0]$. These are the zeros of the polynomial w_{2k+1}. Since the growth of $\arg(N_k(z))$ along the boundary of $\mathbb{C} \setminus (-\infty, 0]$ is $2\pi(2k+1)$, it follows from the argument principle that the function $N_k(z)$ has no poles in $\mathbb{C} \setminus (-\infty, 0]$ and given a compact set $K \subset \mathbb{C} \setminus (-\infty, 0]$ there exists a constant $q < 1$ such that

$$|N_k(z)| < q, \quad z \in K.$$

From this inequality and (2.6) follows immediately that the sequence $\{\pi_k\}$ is normal in $\mathbb{C} \setminus (-\infty, 0]$. □

3. Proof of Theorems 1.1 and 1.2

In this section we prove the formulas for the Frobenius-Padé approximants of $|x|$ with respect to the Chebyshev weight. Let us only give details in the case of odd indexes because the other is similar. To avoid many indexes let us set $P_{2n+1} := P_{2n+1,2m+1}$ and $Q_{2m+1} := Q_{2n+1,2m+1}$.

Proof of Theorem 1.1. Let $T_k(x) = \cos k\theta$, $x = \cos\theta$, $k = 0, 1, \ldots$. We have

$$x^{2k+1}|x| = 2\,(k+1)!(\frac{1}{2})_{(k+1)} \sum_{j=0}^{\infty} \frac{T_{2j+1}(x)}{\Gamma(k+j+\frac{5}{2})\Gamma(k-j+\frac{3}{2})}, \tag{3.1}$$

since

$$\frac{\frac{1}{\pi}\int_{-1}^{1} x^{2k+1}|x|\, T_{2j+1}(x)\, \frac{dx}{\sqrt{1-x^2}}}{\|T_{2j+1}\|^2} = \frac{4}{\pi}\int_0^{\pi/2} \cos^{2k+2}\theta \cos(2j+1)\theta\, d\theta$$
$$= \frac{2\,(k+1)!(\frac{1}{2})_{(k+1)}}{\Gamma(k+j+\frac{5}{2})\Gamma(k-j+\frac{3}{2})}, \quad j = 0, 1, \ldots. \tag{3.2}$$

According to the definition of $Q_{2m+1}(x) = \sum_{k=0}^{m} \alpha_k x^{2k+1}$ and (3.1) it should be satisfied

$$\sum_{k=0}^{m} \alpha_k \frac{(k+1)!(\frac{1}{2})_{(k+1)}}{\Gamma(k+i+\frac{5}{2})\Gamma(k-i+\frac{3}{2})} = 0, \quad i = n+1, n+2, \ldots, n+m.$$

Since

$$\Gamma(k+i+\frac{5}{2}) = (i+5/2)_k \Gamma(i+5/2), \quad \Gamma(k-i+\frac{3}{2}) = (3/2-i)_k \Gamma(3/2-i),$$

we have

$$\sum_{k=0}^{m} \alpha_k \frac{(k+1)!(\frac{1}{2})_{(k+1)}}{(i+5/2)_k (3/2-i)_k} = 0, \quad i = n+1, n+2, \ldots, n+m.$$

A solution to this system is

$$\alpha_k = \frac{(-m)_k (-n+1/2)_k (n+m+5/2)}{(k+1)!k!(1/2)_{k+1}}.$$

In fact, the Saalschütz formula (see [2, Theorem 2.2.6])

$$\sum_{k=0}^{m} \frac{(-m)_k (a)_k (b)_k}{(c)_k (1+a+b-c-m)_k k!} = \frac{(c-a)_m (c-b)_m}{(c)_m (c-a-b)_m} \tag{3.3}$$

for $a = m+n+5/2$, $b = -n+1/2$, $c = 5/2+i$, yields

$$\begin{aligned} \sum_{k=0}^{m} \alpha_k \frac{(k+1)!(\frac{1}{2})_{(k+1)}}{(i+5/2)_k (3/2-i)_k} &= \sum_{k=0}^{m} \frac{(-m)_k (n+m+5/2)_k (-n+1/2)_k}{(i+5/2)_k (3/2-i)_k k!} \\ &= \frac{(i-m-n)_m (i+n+2)_m}{(5/2+i)_m (i-m-1/2)_m}, \end{aligned} \tag{3.4}$$

which is obviously zero for $i = n+1, n+2, \ldots, n+m$.

□

Proof of Theorem 1.2. According to the definition of P_{2n+1}, (1.2) and (3.1), we have

$$P_{2n+1}(x) = \sum_{j=0}^{n} \sum_{k=0}^{m} \frac{(-m)_k (-n+1/2)_k (n+m+5/2)_k}{k!\Gamma(k+j+\frac{5}{2})\Gamma(k-j+\frac{3}{2})} T_{2j+1}(x)$$

Since

$$\Gamma(k+j+5/2) = (j+5/2)_k \Gamma(j+5/2),$$

$$\Gamma(k-j+3/2) = (-j+3/2)_k (-j+1/2)(-j-1/2)(-j-3/2)\Gamma(1-(j+5/2)),$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad \sin(\frac{5}{2}+j)\pi = (-1)^j,$$

we get

$$P_{2n+1}(x) = \frac{1}{\pi} \sum_{j=0}^{n} \sum_{k=0}^{m} \frac{(-1)^j (-m)_k (-n+1/2)_k (n+m+5/2)_k T_{2j+1}(x)}{k!(j+\frac{5}{2})_k (-j+\frac{3}{2})_k (-j+\frac{1}{2})(j+\frac{1}{2})(j+\frac{3}{2})}.$$

By Pfaff-Saalschütz formula (3.3)

$$\sum_{k=0}^{m} \frac{(-m)_k (-n+1/2)_k (n+m+5/2)_k}{k!(j+5/2)_k (-j+3/2)_k} = \frac{(n-j+1)_m (-j-m-n-1)_m}{(-j+3/2)_m (-j-m-3/2)_m}$$

and since (see [2, Definition 2.5.1 and p. 101] or [13, p. 62])

$$T_{2j+1}(x) = (-1)^j (2j+1) x \, {}_2F_1(-j, j+1; 3/2; x^2),$$

we obtain

$$P_{2n+1}(x) = \frac{2}{\pi} x \sum_{j=0}^{n} \frac{(-1-j-m-n)_m (n-j+1)_m \, {}_2F_1(-j, j+1; 3/2; x^2)}{(-j+1/2)(j+3/2)(3/2-j)_m (-3/2-j-m)_m}.$$

Now we use

$$(-j-m-n-1)_m = (-1)^m \frac{(n+m+2)_j (n+m+1)!}{(n+2)_j (n+1)!},$$

$$(n-j+1)_m = \frac{(m+n)!(-n)_j}{(-n-m)_j n!},$$

$$(-j+3/2)_m = \frac{(1/2)_{m+1} (1/2)_j}{(-m-1/2)_j (1/2-j)},$$

$$(-j-m-3/2)_m = (-1)^m \frac{(m+5/2)_j (5/2)_m}{(5/2)_j},$$

then with $A_{n,m} = \frac{(n+m)!(n+m+1)!}{n!(n+1)!(1/2)_{m+1}(5/2)_m}$

$$P_{2n+1}(x) = \frac{2}{\pi} x A_{n,m} \sum_{j=0}^{n} \frac{(n+m+2)_j (-n)_j (-m-\frac{1}{2})_j (-j+\frac{1}{2})_j (\frac{5}{2})_j \, {}_2F_1(-j, j+1; \frac{3}{2}; x^2)}{(-j+1/2)(j+3/2)(n+2)_j (-n-m)_j (1/2)_j (m+5/2)_j}.$$

Changing the order of summation

$$\sum_{j=0}^{n} \sum_{i=0}^{j} a_{i,j} x^{2i} = \sum_{i=0}^{n} x^{2i} \sum_{j=0}^{n-i} a_{i,j+i}$$

with $(x)_{j+i} = (x+i)_j (x)_i$ and

$$\frac{(j+\frac{5}{2})_i (\frac{5}{2})_j}{(\frac{3}{2})_j (i+j+\frac{3}{2})} = \frac{2}{3}(j+\frac{3}{2})_i, \quad \frac{(-i-j)_j (i+j+1)_j}{(1/2)_j j!} = \frac{(-1)^j 2^{2j} (2j+1)_i}{i!},$$

we have

$$P_{2n+1}(x) = \frac{4x}{3\pi} A_{n,m} \sum_{i=0}^{n} (-4x^2)^i \frac{(-n)_i (n+m+2)_i (-1/2-m)_i}{(-n-m)_i (n+2)_i (m+5/2)_i} W,$$

where

$$\begin{aligned} W &= \sum_{j=0}^{n-i} \frac{(2i+1)_j (-n+i)_j (n+m+2+i)_j (-m-1/2+i)_j (3/2+i)_j}{j!(-n-m+i)_j (n+2+i)_j (m+5/2+i)_j (1/2+i)_j} \\ &= \frac{(-1)^n \Gamma(n+i+2)\Gamma(-i+1/2)\Gamma(m+i+5/2)\Gamma(m+1)(-n-m)_i}{\Gamma(2i+2)\Gamma(-n+1/2)\Gamma(n+m+5/2)\Gamma(n+m+1)}. \end{aligned}$$

In the second identity above for W we have used the Dougall formula (see [2, formula (2.2.10)]),

$$\begin{aligned} &{}_5F_4\left(a, a/2+1, -b, -c, -d; a/2, a+b+1, a+c+1, a+d+1; 1\right) \\ &\qquad = \frac{\Gamma(a+b+1)\Gamma(a+c+1)\Gamma(a+d+1)\Gamma(a+b+c+d+1)}{\Gamma(a+1)\Gamma(a+b+c+1)\Gamma(a+b+d+1)\Gamma(a+c+d+1)} \end{aligned}$$

with

$$a = 2i+1,\ -b = n+m+i+2,\ -c = 1/2-n+i,\ -d = -m+i.$$

Simplifying $(-n-m)_i$ and using that $\Gamma(n+m+1) = (n+m)!$, $(n+2)_i = \frac{\Gamma(n+2+i)}{\Gamma(n+2)}$, $(m+5/2)_i = \frac{\Gamma(m+5/2+i)}{\Gamma(m+5/2)}$, $\Gamma(-i+1/2) = \frac{\Gamma(1/2)}{(-1)^i(1/2)_i}$, $\Gamma(2i+2) = (3/2)_i 4^i i!$ and $(5/2)_m = \frac{\Gamma(5/2+m)}{(3/4)\Gamma(1/2)}$, the proof is finished. □

By Stirling's asymptotic formula is straightforward the following results.

Corollary 3.1. *Let α_n denote the leading coefficient of q_n. We have*

$$\lim_n \alpha_n^{1/n} = \frac{3\sqrt{3}}{2},$$

and as $n \to \infty$,

$$\Pi_{n,n}(0) = \pi_n(0) \sim \begin{cases} \frac{2}{\sqrt{\pi}} n^{-3/2}, & n \text{ odd}, \\ \frac{1}{\sqrt{\pi}} n^{-3/2}, & n \text{ even}. \end{cases}$$

Moreover, all the coefficients of q_n have $n-$th root asymptotic behavior uniformly on the index; it means that if $\alpha_{j,n}$ denotes the coefficient of z^j in $q_n(z)$, then $\sup_j \limsup_n |\alpha_{j,n}|^{1/n} < \infty$ *and this also holds for the coefficients of p_n.*

Corollary 3.2.

$$\lim_n \sup_{x\in[-1,1]} |Q_n(x)|x| - P_n(x)| = 0.$$

Proof. Again we only give details in the case of odd indexes. According to the definition of Frobenius-Padé approximants

$$Q_{2n+1}(x)|x| - P_{2n+1}(x) = \sum_{j\geq 2n+1} f_{n,2j+1} T_{2j+1}(x),$$

where

$$\begin{aligned} f_{n,2j+1} &= \frac{1}{\pi}\int_{-1}^{1} (Q_{2n+1}(x)|x| - P_{2n+1}(x)) \frac{T_{2j+1}(x)}{\|T_{2j+1}\|^2} \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{1}{\pi}\int_{-1}^{1} Q_{2n+1}(x)|x| \frac{T_{2j+1}(x)}{\|T_{2j+1}\|^2} \frac{dx}{\sqrt{1-x^2}}. \end{aligned}$$

By (1.2), (3.2), and (3.4), we have, for a constant α which is independent of n and j,

$$f_{n,2j+1} = \alpha \frac{(j-2n)_n (j+n+2)_n}{(j+5/2)_n (j-n-1/2)_n} \frac{1}{\Gamma(j+5/2)\Gamma(3/2-j)}.$$

Since

$$\frac{(j-n-1)(j+2n+1)}{(j+n+3/2)(j-3/2)} < 1,$$

$$\begin{aligned} 0 < \frac{(j-2n)_n (j+n+2)_n}{(j+5/2)_n (j-n-1/2)_n} &= \frac{(j-n-1)(j+2n+1)}{(j+n+3/2)(j-3/2)} \\ &\times \frac{(j^2-(n+2)^2)(j^2-(n+3)^2)\cdots(j^2-(2n)^2)}{(j^2-(5/2)^2)(j^2-(7/2)^2)\cdots(j^2-(n+1/2)^2)} < 1, \end{aligned}$$

$$\frac{1}{\Gamma(j+5/2)\Gamma(3/2-j)} = \frac{(-1)^j}{\Gamma(1/2)^2 (j+1/2)(j+3/2)}.$$

These relations immediately imply the corollary. □

Remark 3.3. By Corollary 3.2 and since $q_n(0) = 1$, there exists

$$h_n \in \text{span}\{1, x, x^{3/2}, \ldots, x^n, x^{n+1/2}\}$$

such that

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |x^{1/2} - h_n(x)| = 0$$

and

$$\int_0^1 (x^{1/2} - h_n(x))\, x^j \, d\mu(x) = 0, \quad j = 0, \ldots, 2n,$$

An analogous conclusion holds changing $x^{1/2}$ by any power $x^{k/2}$ with k even.

Remark 3.4. The hypergeometric functions $Q_{2n+1,2m+1}$ and $Q_{2n,2m}$ are contigous (see [2, p. 94]). The same conclusion holds for $P_{2n+1,2m+1}$ and $P_{2n,2m}$. A five-term recurrence relation for Frobenius-Padé approximats is proved in [14].

4. Proof of Theorem 1.3

Let π_n be the Frobenius-Padé approximant of order (n,n) of $\sqrt{x}$ with respect to the measure μ with $d\mu = \frac{dx}{\pi\sqrt{x-x^2}}$ or $d\mu = \frac{\sqrt{x}\,dx}{\pi\sqrt{1-x}}$, $x \in (0,1)$. By Lemma 2.1 the proof of Theorem 1.3 is equivalent to prove the corresponding convergence of $\{\pi_n\}_{n\in\mathbb{N}}$ to $\sqrt{x}$. If we check this convergence uniformly on compact subset of $(0,1]$, since $\sqrt{x}$ is continuous in $[0,1]$, and $\pi_n(x)$ is a nondecreasing positive function of $x \in [0,1]$, it holds $\lim_n \pi_n = \sqrt{x}$ uniformly on $[0,1]$. Therefore, it is enough to prove the following result.

Theorem 4.1. *For every compact subset of $\mathbb{C}\setminus(-\infty,0]$ there exists a constant $q \in (0,1)$ such that*

$$\limsup_n \sup_{z\in K} |\pi_n(z) - \sqrt{z}|^{1/n} \le q.$$

In the proof of Theorem 4.1 we use some results from potential theory. We associate with apolynomial P of degree n the normalized counting measure for its zeros

$$\nu_P = \frac{1}{n} \sum_{\zeta:P(\zeta)=0} \delta_\zeta,$$

where δ_ζ is the Dirac measure concentrated at ζ and each zero is counted according to its multiplicity. The convergence of a sequence of measure will be understood in the sense of weak-* topology. For a positive Borel measure β in $\mathbb{C}$ its logarithmic potential is defined by

$$V(\beta,z) := -\int \log|z-t|\,d\beta(t),$$

where we assume that $\log|z-t| \in L^1(\beta)$ or the function $\log|z-t|$ is bounded from above in the support of β.

Lemma 4.2. *Let $g_n(z) := \frac{1}{n}\log|q_n(z)|$. The sequence $\{g_n\}$ of harmonic functions in $\mathbb{C}\setminus(-\infty,0]$ is uniformly bounded on each compact subset of $\mathbb{C}\setminus(-\infty,0]$. Thus, for each sequence of indexes $\Lambda \subset \mathbb{N}$ there is a subsequence $\Lambda' \subset \Lambda$ such that $\{g_n(z) : n \in \Lambda'\}$ converges uniformly on each compact subset of $\mathbb{C}\setminus(-\infty,0]$ and the following limit exists*

$$\lim_{n\in\Lambda'} \nu_{q_n} =: \sigma. \tag{4.1}$$

Moreover, the probability measure σ is supported in $[-\infty,0]$, its logarithmic potential is a convex strictly decreasing function with $V(\sigma,x) < 0$, $t \in (0,\infty)$, and $\lim_{t\to 0-} V(\sigma,x) \le 0$ exists.

Proof. Let $K \subset \mathbb{C}\setminus(-\infty,0]$ be a compact set. Since the zeros of q_n are in $(-\infty,0)$ and according to Corollary 3.1, we have

$$g_n(z) \ge \frac{1}{n}\log c_n + \log d(K,(-\infty,0]) \ge b,$$

$$g_n(z) \le \frac{1}{n}\log q_n(|z|) \le \begin{cases} \frac{1}{n}\log q_n(1), & \text{for } |z|\le 1, \\ \frac{1}{n}\log|z| + \frac{1}{n}\log q_n(1), & \text{in other case} \end{cases} \le B,$$

for all $z \in K$, where b, B are real numbers which depend on the compact K and $d(K,(-\infty,0]) := \inf_{z\in K,\, x\in(-\infty,0]} |z-x|$. Therefore, the sequence $\{g_n\}$ is a bounded harmonic sequence in $\mathbb{C}\setminus(-\infty,0]$, and any sequence $\{g_n\}_{n\in\Lambda}$ have a subsequence $\{g_n\}_{n\in\Lambda'}$, $\Lambda'\subset\Lambda$, that converges uniformly on each compact subset of $\mathbb{C}\setminus(-\infty,0]$. Hence

$$\lim_{n\in\Lambda'}\int f\,d\nu_{q_n}$$

exists for all $f\in C(\mathbb{C}\setminus(-\infty,0])$, so $\{\nu_{q_n}\}_{n\in\Lambda'}$ converges. Let σ be its limit.

Observe that $g_n(0) = 0$ and from Corollary 3.1 $\lim g_n(1) > 0$, $g_n(z) \le \frac{1}{n}\log|z| + g_n(1)$ for $|z|>1$. As $g_n(x)$ is a concave monotone function $x\in[0,\infty)$, there are not points $x_1, x_2 \in (0,\infty)$ with $V(\sigma,x_1) = 0$ or $V(\sigma,x_1) = V(\sigma,x_2)$, because the uniform limit of concave function is a concave function.

□

Let

$$I_n(z) := \int_{-\infty}^{0} \frac{q_k^2(x)}{x-z}\frac{(-x)^{1/2}dx}{w_{2n+1}(x)}, \quad z\in\mathbb{C}\setminus(-\infty,0).$$

By Lemma 2.1 we have $I_n(z) > 0$ for $z\in[0,\infty)$,

$$q_n(z)\sqrt{z} - p_n(z) = \frac{w_{2n+1}(z)I_n(z)}{q_n(z)}, \quad z\in\mathbb{C}\setminus(-\infty,0) \tag{4.2}$$

and

$$\int_0^1 w_{2n+1}(x)x^j\frac{I_n(x)}{q_n(x)}\frac{dx}{\sqrt{1-x}} = 0, \quad j = 0,1,\ldots,2n. \tag{4.3}$$

Lemma 4.3. *Given any sequence of indexes $\Lambda\subset\mathbb{N}$, there exists a subsequence $\Lambda'\subset\Lambda$ such that the following limit exists*

$$\lim_{n\in\Lambda'}\nu_{w_{2n+1}} =: \omega. \tag{4.4}$$

The measure ω is a probability measure supported on $[0,1]$. There exists a constant $I\in\mathbb{R}$ such that

$$\lim_{n\in\Lambda'}\frac{1}{n}\log|I_n(z)| = I, \tag{4.5}$$

uniformly on each compact subset of $\mathbb{C}\setminus(-\infty,0]$.

Proof. Because $\{\nu_{w_{2n+1}} : n\in\Lambda\}$ is a sequence of probability measures supported on $[0,1]$ (see Lemma 2.1), the statement (4.4) follows at once from Banach-Aloglou's theorem. This is equivalent to

$$\lim_{n\in\Lambda'} V(\nu_{w_{2n+1}}, z) = V(\omega, z), \tag{4.6}$$

uniformly on compact subset of $\mathbb{C}\setminus[0,1]$. We run through $n\in\Lambda'$ and let $z_0\in(0,\infty)$ be fixed, then

$$I_n(z)=\int_{-\infty}^{0}\frac{x-z_0}{x-z}\frac{1}{x-z_0}\frac{q_n^2(x)(-x)^{1/2}dx}{w_{2n+1}(x)\sqrt{1-x}},$$

so for each compact set $K\subset\mathbb{C}\setminus(-\infty,0]$, there exist positive constants b,B such that

$$b|I_n(z_0)|\le|I_n(z)|\le B|I_n(z_0)|,\quad z\in K.$$

Further,

$$|q_n(z)|^2|\sqrt{z}-\pi_n(z)|=|w_{2n+1}(z)||I_n(z)|$$

$$\Rightarrow|I_n(z)|\le\frac{|w_{2n+1}(z)|}{|q_n(z)|^2},\quad|\sqrt{z}-\pi_n(z)|=\frac{|w_{2n+1}(z)||I_n(z)|}{|q_n(z)|^2}$$

so there exist $I\in[-\infty,\infty)$ such that (4.5) holds. Since $\sqrt{z}$ is a branch of a multiple-valued function, the value $I=-\infty$ is rejected because in such case from (4.6) and Lemma 4.2 taking subsequence if necessary we have

$$\lim_{n\in\Lambda'}|\sqrt{z}-\pi_n(z)|^{1/n}=0,$$

uniformly on compact subset of $\mathbb{C}\setminus(-\infty,1]$ which is impossible (see [6]).

□

Proof of Theorem 4.1. Let $e_n(z):=q_n(z)\sqrt{z}-p_n(z)$, $z\in\mathbb{C}\setminus(-\infty,0]$. Then $\sqrt{z}-\pi_n(z)=\frac{e_n(z)}{q_n(z)}$. According to Corollary 3.2 for each compact subset K of $(0,1]$ we have

$$\limsup_n\sup_{z\in K}|e_n(z)|^{1/n}\le 1$$

and by Lemma 4.2

$$\liminf_n\inf_{z\in K}|q_n(z)|^{1/n}>1,$$

then theorem follows for compact subset of $(0,1]$.

Now let us prove that $\lim_n\pi_n(z)=\sqrt{z}$ uniformly on compact subset of $\mathbb{C}\setminus(-\infty,0]$. By Lemma 2.5 $\{\pi_n\}_{n\in\mathbb{N}}$ is a normal family in $\mathbb{C}\setminus(-\infty,0]$, thus by Montel's theorem it is enough to prove that each convergent subsequence of $\{\pi_n\}_{n\in\mathbb{N}}$ converges to $\sqrt{z}$. This fact happens in $(0,1]$ according to the first paragraph of this proof. The uniqueness principle for analytic function completes our aim.

Let K be a compact subset of $\mathbb{C}\setminus(-\infty,0]$. Let Λ denote a sequence of indexes such that such that

$$\lim_{n\in\Lambda}\sup_{z\in K}|\pi_n(z)-\sqrt{z}|^{1/n}=\limsup_n\sup_{z\in K}|\pi_n(z)-\sqrt{z}|^{1/n}.$$

By Lemmas 4.2 and 4.3 there exists a subsequence $\Lambda'\subset\Lambda$ such that (4.1), (4.4) and (4.5) hold. Then according to (2.3) and $\lim_n\pi_n(z)=\sqrt{z}$ in $\mathbb{C}\setminus(-\infty,0]$, we obtain

$$-\lim_{n\in\Lambda'}\frac{1}{n}\log|\pi_n(z)-\sqrt{z}|=V(\omega,z)-V(\nu,z)+I\ge 0,\tag{4.7}$$

uniformly on compact subset of $\mathbb{C} \setminus (-\infty, 0]$. Therefore, the function $V(\omega, z) - V(\nu, z) + I$, $z \in \mathbb{C} \setminus (-\infty, 0]$, is a nonnegative superharmonic function in $\mathbb{C} \setminus (-\infty, 0]$ which is greater than 0 in $(0, 1]$. By the minimum principle for superharmonic function (see [11, Theorem 0.5.2])

$$V(\omega, z) - V(\nu, z) + I > 0, \quad z \in \mathbb{C} \setminus (-\infty, 0]$$

and for each compact subset $K \subset \mathbb{C} \setminus (-\infty, 0]$ we get

$$\min_{z \in K} (V(\omega, z) - V(\nu, z) + I) > 0. \tag{4.8}$$

Combining (4.7) and (4.8) the proof is completed.

□

References

[1] **B. Adcock, A. C. Hansen**, Generalized sampling and the stable and accurate reconstruction of piecewise analytic functions from their Fourier coefficients. Math. Comp. 84 (2015), no. 291, 237–270.

[2] **G. E. Andrews, R. Askey, R. Roy**, Special Functions, Encyclopedia of Mathematics and its Applications, 71, Cambridge: Cambridge University Press, 1999.

[3] **B. Beckermann, V. Kalyagin, A. C. Matos, F. Wielonsky**, How well the Hermité-Padé approximation smooth the Gibbs phenomenon? Math. Comp. 80 (2011), no. 274, 931–958.

[4] **M. Bello Hernández, G. López Lagomasino, J. Mínguez Ceniceros**, Fourier–Padé approximants for Angelesco systems. Constr. Approx. (2007) 26, 339–359.

[5] **R. A. DeVore**, Approximation by rational functions. Proc. Amer. Math. Soc. 98 (1986), no. 4, 601–604.

[6] **A. A. Gonchar**, A local condition of single-valuedness of analytic functions. Math. USSR Sbornik, v. 18, n. 1, 1972.

[7] **A. A. Gonchar, E. A. Rakhmanov, S. P.Suetin**, On the rate of convergence of Padé approximants of orthogonal expansions. Progress in approximation theory (Tampa, FL, 1990), 169–190, Springer Ser. Comput. Math., 19, Springer, New York, 1992.

[8] **G. López Lagomasino, J. Mínguez Ceniceros**, Fourier-Pad approximants for Nikishin systems. Constr. Approx. 30 (2009), no. 1, 53–69.

[9] **D. J. Newman**, Rational approximation to $|x|$, Michigan Math. J. 11 (1964), 11–14.

[10] **G. Németh, G. Paris**, The Gibbs phenomenon in generalized Pad approximation. J. Math. Phys. 26 (1985), no. 6, 11751178.

[11] **E. B. Saff, V. Totik**, Logarithmic Potentials with External Fields, Springer-Verlag Berlin Heidelberg 1997.

[12] **H. R. Stahl**, Best uniform rational approximation of x^α on $[0, 1]$. Acta Math. 190 (2003), no. 2, 241–306.

[13] **G. Szegő**, Orthogonal polynomials, five ed., *Amer. Math. Soc.,* Providence, R. I., 1975.

[14] **J. Van Iseghem**, Shohat-Favard type theorem for orthogonal series. J. Comput. Appl. Math. 219 (2008), no. 2, 537–550.