WEIGHTED INEQUALITIES FOR THE RIESZ POTENTIAL ON THE SPHERE

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Abstract. We prove a version of the Stein-Weiss inequality for the Riesz potential of the conformal Laplacian on the sphere. Moreover, we show that the result can be improved for functions invariant under the action of the group $SO(d-1)$. This last result will be a consequence of a more general one for ultraspherical expansions.

1. Introduction and main results

The result (see [1, Theorem 7]) due to G. H. Hardy and J. E. Littlewood for the boundedness of the fractional integral operator (or Riesz potential)

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\sigma}}\, dy, \qquad 0<\sigma<d,$$

was generalized by E. M. Stein and G. Weiss in [2, Theorem B*] including weights in the following way.

Theorem 1. *Let $d \geq 1$, $0<\sigma<d$, $1<p\leq q<\infty$, $a<d/q$, $b<d/p'$,[1] $a+b\geq 0$, and*

$$\frac{1}{q} = \frac{1}{p} + \frac{a+b-\sigma}{d}.$$

Then the inequality

$$\||x|^{-a} T_\sigma f\|_{L^q(\mathbb{R}^d)} \leq C \||x|^b f\|_{L^p(\mathbb{R}^d)}$$

holds for any function f such that $|x|^b f \in L^p(\mathbb{R}^d)$, where C is a constant independent of f.

This theorem can not be improved for general functions in $\mathbb{R}^d$ as it can be deduced from the theory in [3]. However, if we reduce ourselves to radially symmetric functions, it is possible to improve the range of admissible power weights in Theorem 1. Partial results in this line can be found in [4, Theorem 3.1], [5, Theorem 2.1], and [6, Lemma 4]. The most general case was given by B. S. Rubin in [7] but was seemingly overlooked. Later, P. L. De Napoli, I. Drelichman, and R. A. Durán state the same result in [8, Theorem 1.2] as follows.

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[1] As usual, p' is the conjugate of p, given by $1/p+1/p'=1$.

Theorem 2. *Let $d \geq 1$, $0 < \sigma < d$, $1 < p \leq q < \infty$, $a < d/q$, $b < d/p'$, $a + b \geq (d-1)(1/q - 1/p)$, and*

$$\frac{1}{q} = \frac{1}{p} + \frac{a+b-\sigma}{d}.$$

Then the inequality

$$\||x|^{-a} T_\sigma f\|_{L^q(\mathbb{R}^d)} \leq C \||x|^b f\|_{L^p(\mathbb{R}^d)} \tag{1}$$

holds for any radially symmetric function f such that $|x|^b f \in L^p(\mathbb{R}^d)$, where C is a constant independent of f.

The key point to prove this theorem is rewrite (1) as a convolution inequality in the multiplicative group $\mathbb{R}^+$ with the Haar measure dx/x. An estimate with more general weights for T_σ when we consider radial functions can be seen in [9].

Our target in this paper is to study a Stein-Weiss inequalities in the setting of the d-dimensional sphere $\mathbb{S}^{d-1}$. We consider the operator

$$A_\sigma f(x) = \frac{\Gamma(\sqrt{-L_\lambda} + \frac{1-\sigma}{2})}{\Gamma(\sqrt{-L_\lambda} + \frac{1+\sigma}{2})} f(x), \qquad x \in \mathbb{S}^{d-1},$$

where $-L_\lambda$ is the conformal Laplacian on the sphere

$$L_\lambda = -\Delta_{\mathbb{S}^{d-1}} + \lambda^2, \qquad \lambda = \frac{d-2}{2}, \tag{2}$$

with $-\Delta_{\mathbb{S}^{d-1}}$ being the spherical Laplacian. In [10], T. P. Branson observed that the operator A_σ defined in the sphere plays the same role as the fractional integral T_σ in the Euclidean space (they are related by conformal transforms). We will show (see (14) below) that A_σ can be written in terms of the Riesz potential on the sphere, as it was defined in [11],

$$\int_{\mathbb{S}^{d-1}} \frac{f(y)}{|x-y|^{d-1-\sigma}}.$$

So we will refer to A_σ as the Riesz potential of the conformal Laplacian. An analogous of the Hardy-Littlewood-Sobolev inequality for A_σ was given by W. Beckner in [12].

To establish our results about the operator A_σ we will consider the family of weights

$$w_{r,s}(x) = |x - e_d|^r |x + e_d|^s, \qquad r, s \in \mathbb{R},$$

where e_d is the north pole of the sphere $\mathbb{S}^{d-1}$. Moreover, we have to define the conditions to obtain the boundedness of the operator. For each $d \geq 3$, we say that $(\sigma, p, q, a, b, A, B) \in S_{d-1}$ when $0 < \sigma < d-1$, $1 < p \leq q < \infty$,

$$a < (d-1)/q, \qquad A < (d-1)/p', \tag{3}$$

$$a + A > 0, \tag{4}$$

$$\frac{2}{q} - 1 = \frac{a+b-\sigma}{d-1}, \qquad 1 - \frac{2}{p} = \frac{A+B-\sigma}{d-1}, \tag{5}$$

and

$$\frac{1}{q} - \frac{1}{p} = \frac{a+A-\sigma}{d-1}. \tag{6}$$

With the previous definitions our result for general functions on $\mathbb{S}^{d-1}$ reads as follow.

Theorem 3. *Let $d \geq 3$ and $(\sigma, p, q, a, b, A, B) \in S_{d-1}$. Then the inequality*

$$\|w_{-a,-b} A_\sigma f\|_{L^q(\mathbb{S}^{d-1})} \leq C_\sigma \|w_{A,B} f\|_{L^p(\mathbb{S}^{d-1})} \tag{7}$$

holds for any function f such that $w_{A,B} f \in L^p(\mathbb{S}^{d-1})$, where C_σ is a constant independent of f. Moreover

$$C_\sigma = \frac{D_{\sigma,d-1} E_\sigma^{1/r}}{2^{(a+b+A+B)/2}},$$

with

$$r = \frac{d-1}{a+A+d-1-\sigma}, \qquad D_{\sigma,d-1} = \frac{1}{2\pi^{d/2}} \frac{\Gamma\left(\frac{1+\sigma}{2}\right)\Gamma\left(\frac{d-1-\sigma}{2}\right)}{\Gamma(\sigma)}, \tag{8}$$

and

$$E_\sigma = \pi^{(d-1)/2} \frac{\Gamma\left(r(a+A)\right)\Gamma\left(r\left(\frac{d-1}{p'} - A\right)\right)\Gamma\left(r\left(\frac{d-1}{q} - a\right)\right)}{\Gamma\left(r\frac{d-1-\sigma}{2}\right)\Gamma\left(r\left(\frac{d-1}{q} + A\right)\right)\Gamma\left(r\left(\frac{d-1}{p'} + a\right)\right)}. \tag{9}$$

The previous theorem for the case $p = q$ can be found in [13]. Our proof follows the lines in [14]: we write the operator A_σ as a proper convolution to apply a Young type inequality. Therefore the constant C_σ appearing in Theorem 3 is sharp but not attained. We have to note that inequality (7) it is verified with a different constant for any pair (p^*, q^*) such that $p \leq p^*$ and $q^* \leq q$ when $(\sigma, p, q, a, b, A, B) \in S_{d-1}$. This fact is due to $L^r(\mathbb{S}^{d-1}) \subseteq L^s(\mathbb{S}^{d-1})$, for $s \leq r$. For the limit case $a = A = b = B = 0$ it is known that A_σ satisfies a $L^p(\mathbb{S}^{d-1}) - L^q(\mathbb{S}^{d-1})$ inequality for $\frac{1}{p} - \frac{1}{q} \leq \frac{\sigma}{d-1}$, unfortunately this case is not cover in Theorem 3.

The analogous role on the sphere of radially symmetric functions is played by functions which are invariant under the action of $SO(d-1)$. By $SO(d-1)$-invariance we mean that f is invariant under the action of the group $SO(d-1)$ on $\mathbb{S}^{d-1}$ whenever $SO(d-1)$ is embedded into $SO(d)$ in a suitable way. Each function f of this kind can be written as $f(x) = g(\langle x, e_d\rangle)$, for a certain function g defined in $(-1, 1)$. In this case we consider a new kind of conditions to obtain the estimate of A_σ. For $d \geq 3$, we say that $(\sigma, p, q, a, b, A, B) \in S_{d-1,\text{rad}}$ when $(\sigma, p, q, a, b, A, B)$ satisfy $0 < \sigma < d-1$, $1 < p \leq q < \infty$, (3), (5), (6), and

$$a + A \geq -(d-2)\sigma. \tag{10}$$

With the previous notation, our next result gives the boundedness of the Riesz potential of the conformal Laplacian for $SO(d-1)$-invariant functions on the sphere; i. e., the analogous result of Theorem 2 in this setting.

Theorem 4. *Let $d \geq 3$ and $1 < p^*, q^* < \infty$. If there exists p and q such that $p \leq p^*$, $q^* \leq q$ and $(\sigma, p, q, a, b, A, B) \in S_{d-1,rad}$, then the inequality*

$$\|w_{-a,-b} A_\sigma f\|_{L^{q^*}(\mathbb{S}^{d-1})} \leq C \|w_{A,B} f\|_{L^{p^*}(\mathbb{S}^{d-1})}$$

holds for any $SO(d-1)$-invariant function f such that $w_{A,B} f \in L^{p^*}(\mathbb{S}^{d-1})$, where C is a constant independent of f.

In order to prove Theorem 4 we will use the fact that the operator $A_\sigma f$ can be expressed in terms of ultraspherical polynomials. As we have commented previously, each $SO(d-1)$-invariant function on $\mathbb{S}^{d-1}$ can be regarded as functions on the interval $(-1,1)$ and they can be expanded in terms of ultraspherical polynomials. Theorem 4 will be a consequence of a more general result (see Theorem 5 in Section 3) in the ultraspherical setting. Theorem 4 improves the range of parameters in Theorem 3 (as Theorem 2 improves Theorem 1), but in this case the sharp constant is not determined.

Next sections are structured in the following way. The proof of Theorem 3 will be given in the next section. The expression of A_σ for $SO(d-1)$-invariant functions in terms of ultraspherical polynomials and the proof of Theorem 4 are contained in Section 3.

2. The operator A_σ and proof of Theorem 3

Following [15, Ch. 2], for a function f in the sphere, we have

$$f(x) = \sum_{k=0}^{\infty} \operatorname{proj}_k f(x),$$

where $\operatorname{proj}_k f$ is the projection of the function f onto $\mathcal{H}_k^d$, the space of spherical harmonics of degree k, and it is given by

$$\operatorname{proj}_k f(x) = \frac{1}{s_d}\int_{\mathbb{S}^{d-1}} f(y) Z_k(x,y)\, d\omega(y),$$

with

$$Z_k(x,y) = \sum_{j=1}^{\dim \mathcal{H}_k^d} Y_j(x) Y_j(y),$$

for $\{Y_j : 1 \le j \le \dim \mathcal{H}_k^d\}$ an orthonormal basis of $\mathcal{H}_k^d$, and s_d is the surface area of $\mathbb{S}^{d-1}$; i. e.,

$$s_d = \int_{\mathbb{S}^{d-1}} d\omega = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)},$$

where $d\omega$ is the surface area measure. It is well know that Z_k is independent of the particular choice of basis of $\mathcal{H}_k^d$ and

$$Z_k(x,y) = \frac{k+\lambda}{\lambda} C_k^\lambda(\langle x,y\rangle),$$

where we denote by C_n^α the ultraspherical polynomial of degree n and order α and with λ as was defined in (2).

Let us obtain an appropriate expression for $A_\sigma f$. By using that

$$-L_\lambda Y_j(x) = (k+\lambda)^2 Y_j(x),$$

for each spherical harmonic of degree k, we have

$$A_\sigma f(x) = \sum_{k=0}^{\infty} \frac{\Gamma\left(k+\lambda+\frac{1-\sigma}{2}\right)}{\Gamma\left(k+\lambda+\frac{1+\sigma}{2}\right)} \operatorname{proj}_k f(x).$$

Now, from the identity [16, Eq. 4, p. 362]

$$\int_0^\infty e^{-ut} (\sinh \rho t)^{v-1}\, dx = \frac{1}{2^v \rho} \mathrm{B}\left(v, \frac{u}{2\rho} + \frac{1-v}{2}\right),$$

for $\operatorname{Re} v, \operatorname{Re} \rho > 0$ and $\operatorname{Re} u > \operatorname{Re} \rho(v-1)$, it is clear that

$$\int_0^\infty e^{-(k+\lambda)t}(\sinh t/2)^{\sigma-1}\,dt = \frac{\Gamma(\sigma)}{2^{\sigma-1}}\frac{\Gamma\left(k+\lambda+\frac{1-\sigma}{2}\right)}{\Gamma\left(k+\lambda+\frac{1+\sigma}{2}\right)}, \tag{11}$$

for $d \geq 3$ and $0 < \sigma < d-1$. In this way, combining (11) and the identity

$$e^{-t\sqrt{-L_\lambda}}f(x) = \sum_{k=0}^\infty e^{-(k+\lambda)t}\operatorname{proj}_k f(x),$$

we obtain

$$A_\sigma f(x) = \frac{2^{\sigma-1}}{\Gamma(\sigma)}\int_0^\infty e^{-t\sqrt{-L_\lambda}}f(x)(\sinh t/2)^{\sigma-1}\,dt. \tag{12}$$

By using that (see [17, Eq. 1.27])

$$\sum_{k=0}^\infty r^k \frac{k+\lambda}{\lambda}C_k^\lambda(t) = \frac{1-r^2}{(1-2rt+r^2)^{\lambda+1}}, \qquad |r|<1, \quad \lambda>0, \tag{13}$$

we have

$$\begin{aligned} e^{-t\sqrt{-L_\lambda}}f(x) &= \frac{1}{s_d}\int_{\mathbb{S}^{d-1}} f(y)\sum_{k=0}^\infty e^{-(k+\lambda)t}Z_k(x,y)\,d\omega(y) \\ &= \frac{1}{s_d}\int_{\mathbb{S}^{d-1}} f(y)P_t(\langle x,y\rangle)\,d\omega(y), \end{aligned}$$

where

$$P_t(s) = \frac{1}{2^\lambda}\frac{\sinh t}{(\cosh t - s)^{\lambda+1}}.$$

Then, by (12) and applying Fubini's theorem, we deduce that

$$A_\sigma f(x) = \frac{2^{\sigma-1}}{s_d\Gamma(\sigma)}\int_{\mathbb{S}^{d-1}} f(y)\int_0^\infty P_t(\langle x,y\rangle)(\sinh t/2)^{\sigma-1}\,dt\,d\omega(y).$$

With the change of variable $w = 2(\sinh t/2)^2$, we have

$$\int_0^\infty P_t(\langle x,y\rangle)(\sinh t/2)^{\sigma-1}\,dt = \frac{\Gamma\left(\frac{1+\sigma}{2}\right)\Gamma\left(\frac{d-1-\sigma}{2}\right)}{2^{\sigma-1}\Gamma(\frac{d}{2})}\frac{1}{|x-y|^{d-1-\sigma}},$$

and, therefore,

$$A_\sigma f(x) = D_{\sigma,d-1}\int_{\mathbb{S}^{d-1}}\frac{f(y)}{|x-y|^{d-1-\sigma}}\,d\omega(y). \tag{14}$$

Proof of the Theorem 3. Taking the function $g(x) = w_{A,B}(x)f(x)$, the inequality to be proved reduces to

$$\|w_{-a,-b}A_\sigma(w_{-A,-B}g)\|_{L^q(\mathbb{S}^{d-1})} \leq C_\sigma\|g\|_{L^p(\mathbb{S}^{d-1})}. \tag{15}$$

Now, taking $x = (\sqrt{1-t^2}x', t)$ and $y = (\sqrt{1-r^2}y', r)$, with $x', y' \in \mathbb{S}^{d-2}$, the q-th power of the left hand side of (15) becomes

$$(16) \quad D_{\sigma,d-1}^q 2^{-(a+b+A+B)q/2}\int_{-1}^1\int_{\mathbb{S}^{d-2}}(1-t)^{-aq/2}(1+t)^{-bq/2}$$
$$\left(\int_{-1}^1\int_{\mathbb{S}^{d-2}}\frac{g(r,y')(1-r)^{-A/2}(1+r)^{-B/2}\,d\omega(y')\,d\mu_{(d-2)/2}(r)}{(2-2(tr+\sqrt{1-t^2}\sqrt{1-r^2}\langle x',y'\rangle))^{(d-1-\sigma)/2}}\right)^q d\omega(x')\,d\mu_{(d-2)/2}(t),$$

where

$$d\mu_s(t) = (1-t^2)^{s-1/2}\,dt, \qquad s > -1/2. \tag{17}$$

With the change of variables

$$v = \sqrt{\frac{1-t}{1+t}} \qquad \text{and} \qquad u = \sqrt{\frac{1-r}{1+r}},$$

taking

$$\alpha = -\frac{a}{2} + \frac{d-1}{2q} - \frac{d-1-\sigma}{4}, \qquad \beta = -\frac{b}{2} + \frac{d-1}{2q} - \frac{d-1-\sigma}{4},$$

$$\delta = -\frac{A}{2} + \frac{d-1}{2}\left(1-\frac{1}{p}\right) - \frac{d-1-\sigma}{4}, \qquad \gamma = -\frac{B}{2} + \frac{d-1}{2}\left(1-\frac{1}{p}\right) - \frac{d-1-\sigma}{4},$$

and

$$h(u,y') = g\left(\frac{1-u^2}{1+u^2}, y'\right)\left(\frac{2u}{1+u^2}\right)^{(d-1)/p},$$

we can write (16) as

$$\begin{aligned} &D^q_{\sigma,d-1} 2^{-(a+b+A+B)q/2+(\alpha+\beta+\delta+\gamma)q} \int_0^\infty \int_{\mathbb{S}^{d-2}} \frac{v^{2q\alpha}}{(1+v^2)^{q(\alpha+\beta)}} \\ &\times \left(\int_0^\infty \int_{\mathbb{S}^{d-2}} \frac{u^{2\delta}}{(1+u^2)^{\delta+\gamma}} \frac{h(u,y')}{\left(\frac{u}{v}+\frac{v}{u}-2\langle x',y'\rangle\right)^{(d-1-\sigma)/2}}\,d\omega(y')\frac{du}{u}\right)^q d\omega(x')\frac{dv}{v}. \end{aligned} \tag{18}$$

The conditions in (5) are equivalent to $\alpha+\beta=0$ and $\delta+\gamma=0$. In addition, from (6) it is verified that $\alpha=-\delta$. Then (18) transforms into

$$\begin{aligned} &D^q_{\sigma,d-1} 2^{-(a+b+A+B)q/2} \\ &\quad \times \int_0^\infty \int_{\mathbb{S}^{d-2}} \left(\int_0^\infty \int_{\mathbb{S}^{d-2}} h(u,y') K\left(\frac{u}{v}, \langle x',y'\rangle\right) d\omega(y')\frac{du}{u}\right)^q d\omega(x')\frac{dv}{v}, \end{aligned} \tag{19}$$

where

$$K(z,w) = \frac{z^{2\delta}}{(z+z^{-1}-2w)^{(d-1-\sigma)/2}}, \qquad (z,w)\in(0,\infty)\times[-1,1].$$

From the convolution type inequality [14, formula (32)]

$$\begin{aligned} &\left\| \int_0^\infty \int_{\mathbb{S}^{d-2}} h(u,y') K\left(\frac{v}{u}, \langle x',y'\rangle\right) d\sigma(y')\frac{du}{u} \right\|_{L^q((0,\infty)\times\mathbb{S}^{d-2}, d\omega\,\frac{dv}{v})} \\ &\qquad\qquad \le \|K\|_{L^r((0,\infty)\times\mathbb{S}^{d-2}, d\omega\,\frac{dv}{v})} \|h\|_{L^p((0,\infty)\times\mathbb{S}^{d-2}, d\omega\,\frac{dv}{v})}, \end{aligned}$$

with $1/q+1=1/p+1/r$, we conclude the estimate (15). Indeed, from the previous relation and due to (6), r is given by (8). Moreover, it is easy to check that

$$\int_0^\infty \int_{\mathbb{S}^{d-2}} (K(z,w_{d-1}))^r\,d\omega(w)\frac{dz}{z} = \int_{\mathbb{R}^{d-1}} \frac{dx}{|x-e_{d-1}|^{r(d-1-\sigma)}|x|^{d-1-r(2\delta+(d-1-\sigma)/2)}},$$

with e_{d-1} the north pole of $\mathbb{S}^{d-2}$. Then, when (3) and (4) hold and using the well known identity

$$\int_{\mathbb{R}^{d-1}} \frac{dx}{|x-y|^\nu |x|^\mu} = \pi^{(d-1)/2} \frac{\Gamma\left(\frac{d-1-\nu}{2}\right)\Gamma\left(\frac{d-1-\mu}{2}\right)\Gamma\left(\frac{\nu+\mu-d+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{2d-2-\nu-\mu}{2}\right)} |y|^{d-1-\nu-\mu},$$

for $0 < \nu, \mu < d-1$ and $\nu + \mu > n$, we have

$$\int_0^\infty \int_{\mathbb{S}^{d-2}} (K(z, w_{d-1}))^r \, d\omega(w) \frac{dz}{z} = E_\sigma,$$

where the value of E_σ is the given in (9). Observing that

$$\|h\|_{L^p((0,\infty)\times\mathbb{S}^{d-2}, d\omega \frac{dv}{v})} = w_{d-1}^{-1/p} \|g\|_{L^p(\mathbb{S}^{d-1})},$$

we finish the proof. $\square$

3. The operator A_σ for ultraspherical expansions and proof of Theorem 4

We start obtaining an expression for the operator $A_\sigma f$ as a multiplier for a ultraspherical expansion if we consider a $SO(d-1)$-invariant function f. Each function of this kind, taking $x = (\sqrt{1-t^2}x', t)$ with $x' \in \mathbb{S}^{d-2}$, can be written as $f(x) = g(\langle x, e_d \rangle) = g(t)$ for some function g defined in $(-1, 1)$. From this fact, we have the projection operator can be written as

$$\begin{aligned}\operatorname{proj}_k f(x) = \frac{1}{s_d} \frac{k+\lambda}{\lambda} \int_{-1}^1 g(t) \\ \times \int_{\mathbb{S}^{d-2}} C_k^\lambda \left(tr + \sqrt{1-t^2}\sqrt{1-r^2}\langle x', y' \rangle \right) \, d\omega(y') \, d\mu_\lambda(t),\end{aligned}$$

where we have considered $y = (\sqrt{1-r^2}y', r)$, $y' \in \mathbb{S}^{d-2}$, and λ is the value in (2). To express $\operatorname{proj}_k f$ as in terms of ultraspherical polynomials, we will use Funck-Hecke formula [15, Eq. A.5.1.]

$$\int_{\mathbb{S}^{d-2}} f(\langle x, y \rangle) \, d\omega(y) = s_{d-2} \int_{-1}^1 f(z) \, d\mu_{(d-3)/2}(z), \qquad y \in \mathbb{S}^{d-2},$$

and the product formula for the ultraspherical polynomials [15, Eq. B.2.9.]

$$\frac{C_n^\lambda(t) C_n^\lambda(r)}{C_n^\lambda(1)} = c_\lambda \int_{-1}^1 C_n^\lambda (tr + z\sqrt{1-t^2}\sqrt{1-r^2}) \, d\mu_{\lambda-1/2}(z), \qquad \lambda > 0, \tag{20}$$

with c_λ a constant such that $c_\lambda \int_{-1}^1 d\mu_\lambda = 1$. In this manner, by using the identity [15, Eq. B.2.3]

$$\int_{-1}^1 (C_n^\lambda(t))^2 \, d\mu_\lambda(t) = \frac{\lambda C_n^\lambda(1)}{(n+\lambda)c_\lambda},$$

we deduce that

$$\begin{aligned}\operatorname{proj}_k f(x) &= \frac{s_{d-2}}{s_d} \frac{1}{c_\lambda^2} c_k^\lambda(t) \int_{-1}^1 g(r) c_k^\lambda(r) \, d\mu_\lambda(r) \\ &= \frac{\Gamma(\lambda+1/2)^2}{\Gamma(\lambda+1)\Gamma(\lambda)} c_k^\lambda(t) \int_{-1}^1 g(r) c_k^\lambda(y) \, d\mu_\lambda(r),\end{aligned}$$

where c_n^λ denotes orthonormal ultraspherical polynomial. Therefore, for functions f defined on the interval $(-1, 1)$ the operator $A_\sigma f$ can be written as

$$A_\sigma f(x) = \frac{\Gamma(\lambda+1/2)^2}{\Gamma(\lambda+1)\Gamma(\lambda)} \sum_{k=0}^\infty \frac{\Gamma\left(k+\lambda+\frac{1-\sigma}{2}\right)}{\Gamma\left(k+\lambda+\frac{1+\sigma}{2}\right)} a_k(g) c_k^\lambda(t), \tag{21}$$

where $a_k(g)$ is the k-th Fourier coefficient of g associated to the ultraspherical polynomials

$$a_k(g) = \int_{-1}^{1} g(r) c_k^\lambda(r)\, d\mu_\lambda(r).$$

In (21) we are using $\lambda = (d-2)/2$ but it is clear that we can consider any real value such that $\lambda > 0$. Indeed, for any $\lambda > 0$, $0 < \sigma < 2\lambda + 1$ and g defined in $(-1,1)$, we consider the operator

$$\mathbf{A}_\sigma g(t) = \sum_{k=0}^{\infty} \frac{\Gamma\left(k+\lambda+\frac{1-\sigma}{2}\right)}{\Gamma\left(k+\lambda+\frac{1+\sigma}{2}\right)} a_k(g) c_k^\lambda(t).$$

To analyze the boundedness of this operator we consider the weights

$$W_{r,s}(t) = (1-t)^{r/2}(1+t)^{s/2}, \qquad r,s \in \mathbb{R}.$$

Moreover, for $\lambda > 0$, we say that $(\sigma, p, q, a, b, A, B) \in R_{2\lambda+1}$ when $0 < \sigma < 2\lambda + 1$, $1 < p \le q < \infty$,

$$a < (2\lambda+1)/q, \qquad A < (2\lambda+1)/p', \tag{22}$$

$$a + A \ge -2\lambda\sigma, \tag{23}$$

$$\frac{2}{q} - 1 = \frac{a+b-\sigma}{2\lambda+1}, \qquad 1 - \frac{2}{p} = \frac{A+B-\sigma}{2\lambda+1}, \tag{24}$$

and

$$\frac{1}{q} - \frac{1}{p} = \frac{a+A-\sigma}{2\lambda+1}. \tag{25}$$

Theorem 5. *Let $\lambda > 0$ and $1 < p^*, q^* < \infty$. If there exists p and q such that $p \le p^*$, $q^* \le q$ and $(\sigma, p, q, a, b, A, B) \in R_{2\lambda+1}$, then the inequality*

$$\|W_{-a,-b}\mathbf{A}_\sigma g\|_{L^{q^*}((-1,1),d\mu_\lambda)} \le C \|W_{A,B} g\|_{L^{p^*}((-1,1)d\mu_\lambda)}, \tag{26}$$

holds for any function g such that $W_{A,B} g \in L^{p^*}((-1,1), d\mu_\lambda)$, where C is a constant independent of g.

Theorem 4 is a particular case of the previous one taking $2\lambda + 1 = d - 1$ because in that case $R_{d-1} = S_{d-1,\mathrm{rad}}$, so we omit the details.

The proof of Theorem 5 will be a consequence of a Young's inequality for weak type spaces state below.

Lemma 1. [18, Theorem 1.4.24] *Let G be a locally compact group with left Haar measure μ that satisfies $\mu(A) = \mu(A^{-1})$ for all measurable A of G. Let $1 < p, q, r < \infty$ satisfy*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}.$$

Then, there exists a constant $B_{pqr} > 0$ such that for all $f \in L^p(G,\mu)$ and $g \in L^{r,\infty}(G,\mu)$ we have

$$\|f * g\|_{L^q(G,\mu)} \le B_{pqr} \|g\|_{L^{r,\infty}(G,\mu)} \|f\|_{L^p(G,\mu)}.$$

Proof of Theorem 5. Because $p \le p^*$ and $q^* \le q$, we have

$$\|W_{-a,-b}\mathbf{A}_\sigma g\|_{L^{q^*}((-1,1),d\mu_\lambda)} \le C\|W_{-a,-b}\mathbf{A}_\sigma g\|_{L^{q}((-1,1),d\mu_\lambda)}$$

and

$$\|W_{A,B}g\|_{L^{p}((-1,1),d\mu_\lambda)} \le C\|W_{A,B}g\|_{L^{p^*}((-1,1),d\mu_\lambda)},$$

then we can prove (26) for $p^* = p$ and $q^* = q$ with $(\sigma, p, q, a, b, A, B) \in R_{2\lambda+1}$.

Proceeding as in the previous section to prove (14), by using (11), (13), and (20), we obtain

$$\tag{27} \begin{aligned}\mathbf{A}_\sigma g(t) &= \frac{c_\lambda^2 D_{\sigma,2\lambda+1}}{\Gamma(\lambda+1)}\\ &\times \frac{1}{2^{\lambda+1/2-\sigma/2}} \int_{-1}^{1} g(r) \int_{-1}^{1} \frac{d\mu_{\lambda-1/2}(w)}{(1-tr-w\sqrt{1-t^2}\sqrt{1-r^2})^{\lambda+1/2-\sigma/2}}\, d\mu_\lambda(r).\end{aligned}$$

The inequality (26) is equivalent to

$$\tag{28} \|W_{-a,-b}\mathbf{A}_\sigma(W_{-A,-B}g)\|_{L^q((-1,1),d\mu_\lambda)} \le C\|g\|_{L^p((-1,1),d\mu_\lambda)}.$$

Then, by (27), with the changes of variables

$$v = \sqrt{\frac{1-t}{1+t}} \qquad \text{and} \qquad u = \sqrt{\frac{1-r}{1+r}},$$

taking δ as in the proof of Theorem 3 but with d changes by $2\lambda+2$, and using (24) and (25), we deduce that

$$\begin{aligned}\|W_{-a,-b}\mathbf{A}_\sigma(W_{-A,-B}g)\|^q_{L^q((-1,1),d\mu_\lambda)} &= \left(\frac{c_\lambda^2 D_{\sigma,2\lambda+1}}{\Gamma(\lambda+1)}\right)^q\\ &\times \frac{1}{2^{(a+b+A+B)q/2}} \int_0^\infty \left(\int_0^\infty h(u)\mathcal{K}\left(\frac{v}{u}\right)\frac{du}{u}\right)^q \frac{dv}{v},\end{aligned}$$

with

$$\mathcal{K}(z) = \int_{-1}^{1} \frac{z^{2\delta}}{(z+z^{-1}-2w)^{\lambda+1/2-\sigma/2}}\, d\mu_{\lambda-1/2}(w)$$

and

$$h(u) = g\left(\frac{1-u^2}{1+u^2}\right)\left(\frac{u}{1+u^2}\right)^{\frac{2\lambda+1}{p}}.$$

In this way, with the identity

$$\|h\|_{L^p((0,\infty),du/u)} = \|g\|_{L^p((-1,1),\,d\mu_\lambda)}$$

and using Lemma 1, we obtain (28) because

$$\tag{29} \|\mathcal{K}\|_{L^{r,\infty}((0,\infty),du/u)} < \infty,$$

due to the conditions (22) and (23).

The proof will be completed checking (29). It is clear that

$$I = \int_{-1}^{1} \frac{d\mu_{\lambda-1/2}(w)}{(u+u^{-1}-2w)^{\lambda+1/2-\sigma/2}} \le C\frac{(1-u)^{\sigma-1}}{u^{(\sigma-1)/2}} \int_0^{4u/(1-u)^2} \frac{x^{\lambda-1}}{(1+x)^{\lambda+1/2-\sigma/2}}\, dx.$$

Then

$$I \le Cu^{\lambda+1/2-\sigma/2}, \qquad \text{for } 0 < u < 1/2,$$

and, for $1/2 \le u \le 1$,

$$I \le C \begin{cases} 1, & \sigma > 1, \\ 1 + \log\left(\frac{4u}{(1-u)^2}\right), & \sigma = 1, \\ (1-u)^{\sigma-1}, & \sigma < 1. \end{cases}$$

Taking into account that

$$\|\mathcal{K}\|_{L^{r,\infty}((0,\infty),\, du/u)} \le (\|u^{2\delta} I\|_{L^{r,\infty}((0,1),du/u)} + \|u^{-2\delta} I\|_{L^{r,\infty}((0,1),\, du/u)}),$$

it is enough to analyze the norms

$$\|(u^{2\delta} + u^{-2\delta}) u^{\lambda+1/2-\sigma/2}\|_{L^{r,\infty}((0,1/2),du/u)}, \tag{30}$$

$$\left\| \log\left(\frac{4u}{(1-u)^2}\right) \right\|_{L^{r,\infty}((1/2,1),du)}, \qquad \text{for } \sigma = 1, \tag{31}$$

and

$$\|(1-z)^{\sigma-1}\|_{L^{r,\infty}((1/2,1),du)}, \qquad \text{for } \sigma < 1. \tag{32}$$

It is easy to check that for $0 < r < \infty$, $t \in \mathbb{R}$, $a > 0$ it is verified $\chi_{(0,a)}(x)x^t \in L^{r,\infty}(x^s\,dx)$ if and only if $rt + s + 1 \ge 0$, $(t,s) \ne (0,1)$. So, the norm in (30) will be finite for

$$2\delta + \lambda + \frac{1-\sigma}{2} > 0 \qquad \text{and} \qquad -2\delta + \lambda + \frac{1-\sigma}{2} > 0,$$

but these conditions are equivalent to (22). With an elementary change of variable we see that the norm in (32) is finite for $r(\sigma-1)+1 \ge 0$ and this is condition (23), where we have used that

$$r = \frac{2\lambda+1}{a + A - \sigma + 2\lambda + 1}.$$

Finally, the norm in (31) is convergent because $r > 0$. □

References

[1] Hardy GH, Littlewood JE. Some properties of fractional integrals, I. Math. Z. 1928;27:565–606.

[2] Stein EM, Weiss G. Fractional integrals on n-dimensional Euclidean space. J. Math. Mech. 1958;7:503–514.

[3] Sawyer E, Wheeden RL. Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. Amer. J. Math. 1992;114:813–874.

[4] Gasper G, Stempak K, Trebels W. Fractional integration for Laguerre expansions. Methods Appl. Anal. 1995;2:67–75.

[5] Hidano K, Kurokawa Y. Weighted HLS inequalities for radial functions and Strichartz estimates for wave and Schrödinger equations. Illinois J. Math. 2008;52:365–388.

[6] Vilela MC. Regularirty solutions to the free Schrödinger equation with radial initial data. Illinois J. Math. 2001;45:361–370.

[7] Rubin BS. One-dimensional representation, inversion and certain properties of Riesz potentials of radial functions (Russian). Mat. Zametki. 1983;34:521–533. English translation: Math. Notes. 1983;34:751–757.

[8] De Napoli PL, Drelichman I, Durán RG. On weighted inequalities for fractional integrals of radial functions. Illinois J. Math. 2011;55:575–587.

[9] Duoandikoetxea J. Fractional integrals on radial functions with applications to weighted inequalities. Ann. Mat. Pura Appl. 2013;192:553–568.

[10] Branson TP. Sharp inequalities, the functional determinant, and the complementary series. Trans. Amer. Math. Soc. 1995;347:3671–3742.

[11] Rubin BS. The inversion of fractional integrals on a sphere. Israel J. Math. 1992;79:47–81.

[12] Beckner W. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. Ann. of Math. (2). 1993;138:213–242.
[13] Samko S. Best constant in the weighted Hardy inequality: the spatial and spherical version. Fract. Calc. Appl. Anal. 2005;8:39–52.
[14] Beckner W. Pitt's inequality with sharp convolution estimates. Proc. Amer. Math. Soc. 2008;136:1871–1885.
[15] Dai F, Xu Y. Approximation theory and Harmonic Analysis on spheres and balls. New York: Springer; 2013.
[16] Prudnikov AP, Brychkov YA, Marichev OI. Integrals and series. Vol. 1. Elementary functions. Translated from the Russian and with a preface by N. M. Queen. New York: Gordon and Breach Science Publishers, 1986.
[17] Askey R. Orthogonal Polynomials and Special Functions. Philadelphia: Society for Industrial and Applied Mathematics, 1975.
[18] Grafakos L. Classical Fourier Analysis, second edition. New York: Springer, 2008.

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