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New identities in the Catalan triangle

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Abstract

In this paper we prove new identities in the Catalan triangle whose (n, p) entry is defined by

$$B_{n,p} := \frac{p}{n}\binom{2n}{n-p}, \quad n, p \in \mathbb{N}, \ p \leqslant n.$$

In fact, we show some new identities involving the well-known Catalan numbers, and specially the identity

$$\sum_{p=1}^{i} B_{n,p} B_{n,n+p-i}(n+2p-i) = (n+1)C_n \binom{2(n-1)}{i-1}, \quad i \leqslant n,$$

that appears in a problem related with the dynamical behavior of a family of iterative methods applied to quadratic polynomials.

Keywords: Catalan numbers; Combinatorial identities; Binomial coefficients

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0. Introduction

In [10] Shapiro introduced the following triangle of numbers:

$n \backslash p$	1	2	3	4	5	6	$\cdots$
1	1						
2	2	1					
3	5	4	1				
4	14	14	6	1			
5	42	48	27	8	1		
6	132	165	110	44	10	1	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

where the entries are given by

$$B_{n,p} := \frac{p}{n}\binom{2n}{n-p}, \quad n, p \in \mathbb{N}, \ p \leqslant n. \tag{1}$$

The above triangle is called Catalan triangle because the well-known Catalan numbers are the entries in the first column. Catalan numbers are defined recursively by $C_0 = 1$ and $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$, $n \geqslant 1$. The general term in the Catalan sequence is given by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geqslant 1.$$

Catalan numbers appear in a wide range of problems and they have a lot of different interpretations. For instance, they give the solution of the famous Euler's Polygon Division Problem, that consists in finding how many ways a plane convex polygon of $n+2$ sides can be divided into triangles by diagonals [4]. They also give the number of binary bracketings of $n+1$ letters (Catalan's problem); the solution to the ballot problem [7]; the number of trivalent planted planar trees [3]; the number of states possible in an n-flexagon; the number of different diagonals possible in a frieze pattern with $n+1$ rows and so many others, see more in [13].

Although the numbers $B_{n,p}$ given by (1) are not as famous as Catalan numbers, they have also several applications. As a sample, we cite now some of them:

(i) $B_{n,p}$ is the number of leaves at level $p+1$ in all ordered trees with $n+1$ edges.
(ii) $B_{n,p}$ is the number of walks of n steps, each in direction N, S, W or E, starting at the origin, remaining in the upper half-plane and ending at height p.
(iii) $B_{n,p}$ denote the number of pairs of non-intersecting paths of length n and distance p.

See [2,10] and [11] for more information. Other generalized Catalan numbers are considered in [7].

In addition, the recurrence relation

$$B_{n,p} = B_{n-1,p-1} + 2B_{n-1,p} + B_{n-1,p+1}, \quad p \geqslant 2,$$

is satisfied by the numbers $B_{n,p}$. A relation between $B_{n,p}$ and C_n (see [10]) is given by

$$B_{n,p} = \sum_{i_1+\cdots+i_p=n} C_{i_1} \cdots C_{i_p}. \tag{2}$$

In this paper we establish new identities for the numbers $B_{n,p}$, and, in particular, the identity

$$\sum_{p=1}^{i} B_{n,p} B_{n,n+p-i}(n+2p-i) = (n+1)C_n \binom{2(n-1)}{i-1}, \quad i \leqslant n, \tag{3}$$

that appears in a problem related with the dynamical behavior of a family of iterative processes. Furthermore, we derive new relations that involve the Catalan numbers. For instance, observe that if $i = n$ in (3), then we have

$$\sum_{p=1}^{n} p(B_{n,p})^2 = \frac{n(n+1)}{2} C_n C_{n-1}, \quad n \in \mathbb{N}.$$

We also use our techniques to give an alternative proof of the identity

$$\sum_{p=1}^{n}(B_{n,p})^2 = C_{2n-1}, \quad n \in \mathbb{N}$$

[10, Corollary 3.5], and to prove the unknown identity,

$$\sum_{p=1}^{n}(pB_{n,p})^2 = (3n-2)C_{2(n-1)}, \quad n \in \mathbb{N},$$

see Theorem 3. In the last section we give some comments, open questions and conclusions about the numbers $B_{n,p}$.

As we have said before, the formula (3) appears in the study of the general convergence of a family of one-point iterative processes with high order of convergence to approximate the roots of a complex quadratic polynomial f [9]. We are interested in the dynamical study of the rational Newton-type root-finding algorithms defined in the extended complex plane by

$$\begin{cases} z_{m+1} = R_n(z_m) = z_m - H_n\big(L_f(z_m)\big)\dfrac{f(z_m)}{f'(z_m)}, \quad m \geqslant 0, \\ H_n(z) = \displaystyle\sum_{j=0}^{n} \frac{1}{2^j} C_j z^j, \quad n \geqslant 0, \end{cases} \tag{4}$$

where $L_f(z) = \frac{f(z)f''(z)}{f'(z)^2}$. Notice that the Catalan numbers are involved in the coefficients that define the rational iteration function R_n in (4). In [6] it is established the monotonous convergence of methods (4) and proved that they are convergent with order of convergence at least $n+2$.

The idea of general convergence was introduced by Smale [12] and later generalized by McMullen [8]. It is said that a rational root-finding method $f \to R_f$ is generally convergent for polynomials f of degree d if the sequence $z_{m+1} = R_f(z_m)$ converges to a root of f for almost every starting point z_0 and for almost every polynomial of degree d. We use the Lebesgue measure in the complex plane and on the space of coefficients of polynomials.

In [9] we wonder if the methods defined in (4) are generally convergent for quadratic polynomials. Thus let $f(z) = (z-a)(z-b)$ be a quadratic complex polynomial with $a \neq b$. Then the conjugate map of R_n, $S_n := MR_nM^{-1}$, via the Möbius map $M(z) = (z-a)/(z-b)$ is given by

$$S_n(z) = z^{n+2}\frac{P_n(z)}{\widehat{P}_n(z)}, \tag{5}$$

with $P_n(z) = \sum_{p=0}^{n} B_{n+1,p+1}z^p$ and $\widehat{P}_n(z) = \sum_{p=0}^{n} B_{n+1,n+1-p}z^p$ [9]. Thus the numbers $B_{n,p}$ appear in the coefficients of the polynomials P_n and $\widehat{P}_n$ that define the conjugate map S_n. Taking into account (5), the expression of the first derivative of the conjugate map S_n is given by

$$S_n'(z) = \frac{z^{n+1}}{\widehat{P}_n(z)^2}\big((n+2)P_n(z)\widehat{P}_n(z) + z\big(P_n'(z)\widehat{P}_n(z) - P_n(z)\widehat{P}_n'(z)\big)\big). \tag{6}$$

Now we apply the new identity (3) in the formula (6) to obtain that

$$S_n'(z) = \frac{(n+2)C_{n+1}z^{n+1}(1+z)^{2n}}{\widehat{P}_n(z)^2}. \tag{7}$$

By (7) note that rational maps S_n have precisely two forward invariant Fatou components: a superattacting component where iterates converge to ∞ and a superattacting component where iterates converge to 0. Moreover all the critical points of S_n have finite forward orbits and then it is well known (see [1]) that $m(\mathcal{J}(S_n)) = 0$ where m is the Lebesgue measure on $\mathbb{C}$ and $\mathcal{J}(S_n)$ is the Julia set of S_n. Then the corresponding iteration rational map R_n is generally convergent for quadratic polynomials, see more details in [9].

Notation. Throughout this paper $\mathbb{N}$ and $\mathbb{C}$ are the set of natural and complex numbers, respectively; n, m, p, k, i are natural numbers; $n!$ is the usual factorial number and $\binom{n}{p}$ the combinatorial number; z is a complex number.

1. Combinatorial identities in the Catalan triangle

Vandermonde's convolution formula (also called Chu–Vandermonde formula)

$$\sum_{k=0}^{p}\binom{n}{k}\binom{m}{p-k}=\binom{n+m}{p},\quad 0\leqslant p\leqslant m,n, \tag{8}$$

is the main tool to prove our results in this paper. It can be found in [13]. A special case gives the identity

$$\sum_{k=0}^{n-p}\binom{n}{k}\binom{n}{p+k}=\binom{2n}{n-p},\quad 0\leqslant p\leqslant n,$$

and the most famous special case follows from taking $m=n=p$ in (8) to obtain

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n},\quad n\in\mathbb{N}, \tag{9}$$

see [13]. Vandermonde's convolution formulas are often used to manage combinatorial identities, see for example the recent paper [5]. We apply them in the next proposition which we will use in theorems of this section.

Proposition 1. *Let $n\in\mathbb{N}$. Then*

(i) $$\sum_{l=0}^{n-1}\binom{2n}{l}^2=\frac{1}{2}\left(\binom{4n}{2n}-\binom{2n}{n}^2\right).$$

(ii) $$\sum_{l=0}^{n-2}\binom{2n-1}{l}^2=\frac{1}{2}\binom{2(2n-1)}{2n-1}-\binom{2n-1}{n-1}^2\quad \textit{with } n\geqslant 2.$$

(iii) $$\sum_{l=1}^{n-1}l^2\binom{2n}{l}^2=2n^2\left(\binom{2(2n-1)}{2n-1}-2\binom{2n-1}{n}^2\right)\quad \textit{with } n\geqslant 2.$$

(iv) $$\sum_{l=1}^{n-1}l\binom{2n}{l}^2=n\binom{4n-1}{2n-1}-3n\binom{2n-1}{n}^2\quad \textit{with } n\geqslant 2.$$

(v) $$\sum_{l=1}^{n-2}l\binom{2n-1}{l}^2=\frac{2n-1}{2}\left(\binom{4n-3}{2(n-1)}-2\binom{2n-1}{n-1}\binom{2(n-1)}{n-2}-\binom{2(n-1)}{n-1}^2\right)\quad \textit{with } n\geqslant 3.$$

Proof. To check (i) we use the formula (9) to have

$$\sum_{l=0}^{n-1}\binom{2n}{l}^2=\binom{4n}{2n}-\binom{2n}{n}^2-\sum_{l=n+1}^{2n}\binom{2n}{l}^2.$$

Since $\binom{2n}{l}=\binom{2n}{2n-l}$ we conclude the proof.

The proof of item (ii) is similar. To obtain (iii) note that

$$l\binom{2n}{l}=2n\binom{2n-1}{l-1},\quad l\geqslant 1,$$

and we apply the part (ii). Parts (iv) and (v) are proved in a similar way. □

Although the next result appear in [10] we have decided to include this alternative proof to show how Proposition 1 works in a easy case.

Theorem 2. *Let $n\in\mathbb{N}$. Then* $\sum_{p=1}^{n}(B_{n,p})^2=C_{2n-1}$.

Proof. Observe that,

$$\sum_{p=1}^{n}(B_{n,p})^2=\frac{1}{n^2}\sum_{p=0}^{n-1}\bigl(n^2-2np+p^2\bigr)\binom{2n}{p}^2.$$

We apply Proposition 1(i), (iii) and (iv) to obtain

$$\sum_{p=1}^{n}(B_{n,p})^2=\frac{1}{2}\binom{4n}{2n}+2\binom{4n-2}{2n-1}-2\binom{4n-1}{2n-1}-\left(\frac{1}{2}\binom{2n}{n}^2+4\binom{2n-1}{n}^2-6\binom{2n-1}{n}^2\right).$$

Noting that

$$\frac{1}{2}\binom{2n}{n}^2+4\binom{2n-1}{n}^2-6\binom{2n-1}{n}^2=0$$

and

$$\frac{1}{2}\binom{4n}{2n}+2\binom{4n-2}{2n-1}-2\binom{4n-1}{2n-1}=\frac{(4n-2)!}{(2n-1)!(2n)!}=C_{2n-1}$$

we conclude the proof. □

Remark. These techniques may also be used to prove other known identities, for example,

$$\sum_{p=1}^{n}B_{n,p}B_{n+1,p}=C_{2n},\quad n\geqslant 1$$

[10], as well as new results, see the next theorem.

Theorem 3. *Let $n\in\mathbb{N}$. Then $\sum_{p=1}^{n}(pB_{n,p})^2=(3n-2)C_{2(n-1)}$.*

Proof. We will prove that

$$\frac{1}{n^2}\sum_{p=0}^{n-1}(n-p)^4\binom{2n}{p}^2=(3n-2)C_{2(n-1)},\quad n\geqslant 1.$$

We use the identity

$$(n-p)^4=p^2(p-1)^2+(2-4n)p^2(p-1)+p^2\bigl(6n^2-4n+1\bigr)-4n^3p+n^4$$

to get that

$$\frac{1}{n^2}\sum_{p=0}^{n-1}(n-p)^4\binom{2n}{p}^2=4(2n-1)^2\sum_{p=0}^{n-3}\binom{2n-2}{p}^2+4(2-4n)\sum_{p=0}^{n-2}p\binom{2n-1}{p}^2$$
$$+\frac{(6n^2-4n+1)}{n^2}\sum_{p=0}^{n-1}p^2\binom{2n}{p}^2-4n\sum_{p=0}^{n-1}p\binom{2n}{p}^2+n^2\sum_{p=0}^{n-1}\binom{2n}{p}^2.$$

We apply Proposition 1 to obtain

$$\frac{1}{n^2}\sum_{p=0}^{n-1}(n-p)^4\binom{2n}{p}^2=4(2n-1)^2\left(\frac{1}{2}\left(\binom{4n-4}{2n-2}-\binom{2n-2}{n-1}\right)-\binom{2n-2}{n-1}\right)$$
$$-4(2n-1)^2\left(\binom{4n-3}{2(n-1)}-2\binom{2n-1}{n-1}\binom{2(n-1)}{n-2}-\binom{2(n-1)}{n-1}^2\right)$$
$$+2\bigl(6n^2-4n+1\bigr)\left(\binom{2(2n-1)}{2n-1}-2\binom{2n-1}{n}^2\right)$$

$$- 4n^2\left(\binom{4n-1}{2n-1} - 3\binom{2n-1}{n}^2\right) + \frac{n^2}{2}\left(\binom{4n}{2n} - \binom{2n}{n}^2\right).$$

To conclude the proof it is enough to check that

$$4(2n-1)^2\left(\frac{1}{2}\binom{2n-2}{n-1} + \binom{2n-2}{n-1}\right) - 4(2n-1)^2\left(2\binom{2n-1}{n-1}\binom{2(n-1)}{n-2} + \binom{2(n-1)}{n-1}^2\right)$$
$$+ 4\left(6n^2-4n+1\right)\binom{2n-1}{n}^2 - 12n^2\binom{2n-1}{n}^2 + \frac{n^2}{2}\binom{2n}{n}^2 = 0$$

and

$$(3n-2)C_{2(n-1)} = 2(2n-1)^2\binom{4n-4}{2n-2} - 4(2n-1)^2\binom{4n-3}{2(n-1)}$$
$$+ 2\left(6n^2-4n+1\right)\binom{2(2n-1)}{2n-1} - 4n^2\binom{4n-1}{2n-1} + \frac{n^2}{2}\binom{4n}{2n},$$

which it may be done in a straightforward way. □

2. A new identity in the Catalan triangle

In this section we prove the main result of the paper, Theorem 5. This theorem allows us to conclude the general convergence of methods of family (4) applied to quadratic polynomials. We also give two corollaries. First we prove the following lemma to shorten the proof of Theorem 5.

Lemma 4. *Let $k, j \in \mathbb{N}$ where $0 \leqslant j \leqslant k$. Then,*

$$\sum_{i=j}^{k}\binom{2k+1}{i}\left(3(k+1)\binom{2k+2}{k-j+i+1} - (j+1)\binom{2k+2}{k-j+i+1}\right.$$
$$\left. + 2i\binom{2k+1}{k-j+i} - (5k+j+4)\binom{2k+1}{k-j+i}\right) = 0.$$

Proof. First note that

$$3(k+1)\binom{2k+2}{k-j+i+1} - (5k+j+4)\binom{2k+1}{k-j+i}$$
$$= 3(k+1)\left(\binom{2k+2}{k-j+i+1} - \binom{2k+1}{k-j+i}\right) - (2k+j+1)\binom{2k+1}{k-j+i}$$
$$= 3(k+1)\binom{2k+1}{k-j+i+1} - (2k+j+1)\binom{2k+1}{k-j+i}.$$

We use again $\binom{2k+2}{k-j+i+1} = \binom{2k+1}{k-j+i+1} + \binom{2k+1}{k-j+i}$ to obtain that

$$3(k+1)\binom{2k+1}{k-j+i+1} - (2k+j+1)\binom{2k+1}{k-j+i} - (j+1)\binom{2k+2}{k-j+i+1}$$
$$= (3k-j+2)\binom{2k+1}{k-j+i+1} - 2(k+j+1)\binom{2k+1}{k-j+i}.$$

Thus we have to prove that

$$0 = \sum_{i=j}^{k}\binom{2k+1}{i}\left((3k-j+2)\binom{2k+1}{k-j+i+1} - 2(k+j+1-i)\binom{2k+1}{k-j+i}\right)$$
$$= \sum_{i=j}^{k}\binom{2k+1}{i}\binom{2k+1}{k-j+i+1}(k-2i+j),$$

which it is equivalent to the identity

$$\sum_{i=j}^{k}\binom{2k+1}{i}\binom{2k+1}{k-j+i+1}(k-i)=\sum_{i=j}^{k}\binom{2k+1}{i}\binom{2k+1}{k-j+i+1}(i-j)$$

which is easily checked and we conclude the proof. □

Theorem 5. *Let $n, i \in \mathbb{N}$ where $i \leqslant n$. Then,*

$$\sum_{p=1}^{i} B_{n,p}B_{n,n+p-i}(n+2p-i)=(n+1)C_n\binom{2(n-1)}{i-1}.$$

Proof. In fact we prove the equivalent identity

$$\frac{1}{n^2}\sum_{p=0}^{i-1}\binom{2n}{p}\binom{2n}{n-i+p}(n-p)(i-p)(n+i-2p)=\binom{2(n-1)}{i-1}\binom{2n}{n}.$$

Note that

$$(n-p)(i-p)(n+i-2p)=-2p(p-1)(n-i+p)+(5n+i-2)p(n-i+p)-6n^2p+ni(n+i)$$

and using identities as

$$p(p-1)(n-i+p)\binom{2n}{p}\binom{2n}{n-i+p}=(2n)^2(2n-1)\binom{2n-2}{p-2}\binom{2n-1}{n-i+p-1},$$

we obtain

$$\begin{aligned}\binom{2(n-1)}{i-1}\binom{2n}{n}&=-8(2n-1)\sum_{p=0}^{i-3}\binom{2n-2}{p}\binom{2n-1}{n-i+p+1}+4(5n+i-2)\sum_{p=0}^{i-2}\binom{2n-1}{p}\binom{2n-1}{n-i+p}\\&\quad-12n\sum_{p=0}^{i-2}\binom{2n-1}{p}\binom{2n}{n-i+p+1}+\frac{i(n+i)}{n}\sum_{p=0}^{i-1}\binom{2n}{p}\binom{2n}{n-i+p}\\&=I+II+III+IV.\end{aligned}$$

Using the Vandermonde's convolution formula we obtain that

$$I=-4(2n-1)\left(\binom{4n-3}{n+i-2}-\binom{2n-2}{i-2}\binom{2n-2}{n-2}\right)+4(2n-1)\left(\sum_{p=i-2}^{n-1}\binom{2n-2}{p}\binom{2n-1}{n+p-i+1}\right),$$

$$II=2(5n+i-2)\left(\binom{4n-2}{n+i-1}-\sum_{p=i-1}^{n}\binom{2n-1}{p}\binom{2n-1}{n-i+p}\right),$$

$$III=-6n\left(\binom{4n-1}{n+i-1}-\binom{2n-1}{n-1}\binom{2n-1}{i-1}\right)-6n\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n}{n-i+p+1},$$

$$IV=\frac{i(n+i)}{2n}\left(\binom{4n}{n+i}-\sum_{p=i}^{n}\binom{2n}{p}\binom{2n}{n-i+p}\right).$$

Now, it is easy to see that

$$-4(2n-1)\binom{4n-3}{n+i-2}+2(5n+i-2)\binom{4n-2}{n+i-1}-6n\binom{4n-1}{n+i-1}+\frac{i(n+i)}{2n}\binom{4n}{n+i}=0.$$

Thus it is enough to prove that

$$\begin{aligned}\binom{2n-2}{i-1}\binom{2n}{n} &= 4(2n-1)\binom{2n-2}{i-2}\binom{2n-2}{n-2} + 4(2n-1)\binom{2n-2}{i-2}\binom{2n-1}{n-1}\\ &\quad + 4(2n-1)\sum_{p=i-1}^{n-1}\binom{2n-2}{p}\binom{2n-1}{n+p-i+1}\\ &\quad - 2(5n+i-2)\binom{2n-1}{n}\binom{2n-1}{i-1}\\ &\quad - 2(5n+i-2)\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n-1}{n-i+p}\\ &\quad + 6n\left(\binom{2n-1}{n-1}\binom{2n-1}{i-1} + \sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n}{n-i+p+1}\right)\\ &\quad - \frac{i(n+i)}{2n}\sum_{p=i}^{n}\binom{2n}{p}\binom{2n}{n-i+p}.\end{aligned}$$

On the other hand, it is straightforward to check that

$$\begin{aligned}&4(2n-1)\left(\binom{2n-2}{n}\binom{2n-2}{i-2} + \binom{2n-2}{n}\binom{2n-2}{i-2}\right)\\ &\quad - 2(5n+i-2)\binom{2n-1}{n}\binom{2n-1}{i-1} + 6n\binom{2n-1}{n-1}\binom{2n-1}{i-1}\\ &= \binom{2n-2}{i-1}\binom{2n}{n} - 2(n-i+1)\binom{2n}{n}\binom{2n-1}{i-1}\end{aligned}$$

and to finish the proof, we only need to prove that

$$\begin{aligned}2(n-i+1)\binom{2n}{n}\binom{2n-1}{i-1} &= 4(2n-1)\sum_{p=i-1}^{n-1}\binom{2n-2}{p}\binom{2n-1}{n+p-i+1}\\ &\quad - 2(5n+i-2)\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n-1}{n-i+p}\\ &\quad + 6n\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n}{n-i+p+1}\\ &\quad - \frac{i(n+i)}{2n}\sum_{p=i}^{n}\binom{2n}{p}\binom{2n}{n-i+p}.\end{aligned}$$

Since

$$\begin{aligned}&4(2n-1)\sum_{p=i-1}^{n-1}\binom{2n-2}{p}\binom{2n-1}{n+p-i+1}\\ &= 4\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n-1}{n+p-i}p + 2(n-i+1)\binom{2n}{n}\binom{2n-1}{i-1}\end{aligned}$$

and

$$\frac{i(n+i)}{2n}\sum_{p=i}^{n}\binom{2n}{p}\binom{2n}{n-i+p} = 2i\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n}{n-i+p+1},$$

we have to check that

$$\begin{aligned}0 &= 2\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n-1}{n+p-i}p-(5n+i-2)\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n-1}{n-i+p}\\&\quad+3n\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n}{n-i+p+1}-i\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\binom{2n}{n-i+p+1}\\&=\sum_{p=i-1}^{n-1}\binom{2n-1}{p}\left(\binom{2n-1}{n+p-i}2p-(5n+i-2)\binom{2n-1}{n-i+p}\right.\\&\quad\left.+3n\binom{2n}{n-i+p+1}-i\binom{2n}{n-i+p+1}\right)\end{aligned}$$

which holds by Lemma 4. □

As a consequence of Theorem 5, we derive some new relationships between the Catalan numbers C_n and the numbers $B_{n,p}$.

Corollary 6. *Let $n, p \in \mathbb{N}$. The Catalan numbers C_n and the numbers $B_{n,p}$ satisfy the equalities*

$$\sum_{p=1}^{n} p(B_{n,p})^2 = \sum_{p=1}^{n} p\left(\sum_{i_1+i_2+\cdots+i_p=n} C_{i_1}C_{i_2}\cdots C_{i_p}\right)^2 = \frac{(n+1)n}{2}C_nC_{n-1}.$$

Proof. If $i = n$ in Theorem 5, then the identity

$$\sum_{p=1}^{n} p(B_{n,p})^2 = \frac{(n+1)n}{2}C_nC_{n-1}$$

holds and from (2) the proof is finished. □

We may join the results from Theorems 2, 3 and Corollary 6 to give the next corollary.

Corollary 7. *Take $n \in \mathbb{N}$ and $f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2$ a polynomial of second order. Then*

$$\sum_{p=1}^{n} f(p)(B_{n,p})^2 = \alpha_0 C_{2n-1} + \alpha_1 \frac{(n+1)n}{2}C_nC_{n-1} + \alpha_2(3n-2)C_{2n-2}.$$

3. Open questions, remarks and conclusions

Now we present three interesting open questions which are closer to our previous results.

(1) Is it possible to give a interpretation of $\sum_{p=1}^{n} p(B_{n,p})^2$ in terms of pairs of paths? If there exists this interpretation, it will be possible to give an alternative proof of Corollary 6

$$\sum_{p=1}^{n} p(B_{n,p})^2 = \frac{n(n+1)}{2}C_nC_{n-1}.$$

In addition, Theorem 5 could also be interpreted under this point of view.

(2) For any $k \in \mathbb{N}$, is it possible to give an expression of

$$\sum_{p=1}^{n} p^k(B_{n,p})^2$$

in terms of Catalan numbers? Notice that then this could be generalized to

$$\sum_{p=1}^{n} P(p)(B_{n,p})^2$$

where P a polynomial. Observe that for $k = 0, 1, 2$ this result has been proved (Corollary 7). For $k = 3$ we have numerically tested that

$$\sum_{p=1}^{n} p^3(B_{n,p})^2 = \frac{n^2(n+1)}{2} C_n C_{n-1}.$$

(3) Is it possible to give an expression of $\sum_{p=1}^{n}(B_{n,p})^3$ in terms of Catalan numbers? Notice that

$$\sum_{p=1}^{n} B_{n,p} = \frac{n+1}{2} C_n \quad \text{and} \quad \sum_{p=1}^{n}(B_{n,p})^2 = C_{2n-1},$$

see [10] and Theorem 2.

To finish this paper we present our conclusions. Firstly we have applied Vandermonde's convolution formula to show new identities in the Catalan triangle which (n, p) entry is denoted by $B_{n,p}$. The main identity is given in Theorem 5. This identity was conjectured in [9] working in the setting of one-point iterative processes to approximate roots of a complex quadratic polynomial. Other results proved in the paper give the value of

$$\sum_{p=1}^{n} p^k(B_{n,p})^2$$

in terms of Catalan numbers for any $n \in \mathbb{N}$ and $k = 0$ (Theorem 2); $k = 1$ (Corollary 6) and $k = 2$ (Theorem 3).

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