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Lie Superalgebras with Semisimple Even Part

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Finite-dimensional Lie superalgebras $G = G_{\bar{0}} \oplus G_{\bar{1}}$ over an algebraically closed field of characteristic zero, in which $G_{\bar{1}}$ is a completely reducible module for the Lie algebra $G_{\bar{0}}$, are described. In particular, the Lie superalgebras with semisimple even part are described.

1. INTRODUCTION

The finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero are divided into two classes: the classical and the nonclassical ones. The classical Lie superalgebras are the simple Lie superalgebras $G = G_{\bar{0}} \oplus G_{\bar{1}}$, in which the action of $G_{\bar{0}}$ in $G_{\bar{1}}$ is completely reducible. This allows the use of representation theory to determine such structures.

There appears the question of determining all the Lie superalgebras, not necessarily simple, satisfying the latter condition and, in particular, the Lie superalgebras with semisimple even part.

This problem has been treated (without proofs) in [6, Chap. II, Proposition 1.2] with the use of the description of the semisimple Lie superalgebras given in the seminal paper [5] (see also [3]). However, some Lie superalgebras are missing. The purpose of this paper is to fill in these missing superalgebras in [6].

It should be noted that, with some extra conditions, Lie superalgebras with semisimple even part have been dealt with in [2] and that, quite recently, the analogous problems for Malcev superalgebras have been reduced to the Lie superalgebra case in [1]. The exact class of Lie

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superalgebras that we will determine completely is the class formed by the Lie superalgebras $G = G_{\bar{0}} \oplus G_{\bar{1}}$ with reductive $G_{\bar{0}}$ and completely reducible action of $G_{\bar{0}}$ in $G_{\bar{1}}$.

The paper is structured as follows. The second section will deal with some previous results needed in the paper. The next one will be devoted, following the ideas in [6], to the classification of the semisimple Lie superalgebras $G = G_{\bar{0}} \oplus G_{\bar{1}}$ with a completely reducible action of $G_{\bar{0}}$ in $G_{\bar{1}}$. The semisimplicity restriction will be removed in the last section.

Throughout this paper, we shall work over an algebraically closed field of characteristic zero F and all the algebras and superalgebras considered will be assumed to have finite dimension over F. The undefined notations will always be taken from [5].

The overall idea of the paper is to show that if G is a semisimple Lie superalgebra with a completely reducible action of $G_{\bar{0}}$ in $G_{\bar{1}}$ (or just with semisimple $G_{\bar{0}}$) and if $\operatorname{socle}(G)$ denotes the sum, necessarily direct, of the minimal ideals of G, then there is a subalgebra L such that $G = \operatorname{socle}(G) \oplus L$ (Proposition 7). The elements of L act as derivations of $\operatorname{socle}(G)$ and it is important to determine the possibilities for the minimal ideals of G and for L. Clearly it is enough to deal with superalgebras G which are not a direct sum of proper ideals. This forces some homogeneity in the minimal ideals of G, but still G may have quite different minimal ideals "sharing" the subalgebra of derivations L. If the semisimplicity condition is dropped, then some extra work is needed to take account of the radical.

2. PRELIMINARY RESULTS

We shall give in this section some results needed to describe the Lie superalgebras in the title. First of all, we need the description of the Lie superalgebra of derivations of some simple superalgebras. For any simple Lie superalgebra G we identify G with the inner derivation superalgebra inder G, so we identify G with a subalgebra of der G.

For future reference, we explicitly state the next result:

PROPOSITION 1 (see [5, Proposition 5.1.2]). *Let G be a classical simple Lie superalgebra strictly contained in* der G; *then one of the following possibilities occurs*:

(1) *$G = G_{-1} \oplus G_0 \oplus G_1$ is a classical simple Lie superalgebra of type $\mathbf{A}(n,n)$ or $\mathbf{P}(n)$ with $n \geq 2$, $\mathbb{Z}$-graded as in* [5, *Proposition* 2.1.2], *and* der $G = G \oplus Fd$ *is a semidirect sum, where d is the even derivation determined by $[d, x] = kx$ for $x \in G_k$, $k = 0, 1, -1$.*

(2) *$G \cong \mathbf{Q}(n) = \mathbf{Q}(n)_{\bar{0}} \oplus \mathbf{Q}(n)_{\bar{1}}$ $(n \geq 3)$ and* der $G = G \oplus Fd$ *is a semidirect sum, where d is the (within proportionality unique) odd endomor-*

phism of $\mathbf{Q}(n)$ *for which* $d(\mathbf{Q}(n)_{\bar{0}}) = 0$, $d(\mathbf{Q}(n)_{\bar{1}}) = \mathbf{Q}(n)_{\bar{0}}$, *and* $d\colon \mathbf{Q}(n)_{\bar{1}} \to \mathbf{Q}(n)_{\bar{0}}$ *is an isomorphism of* $\mathbf{Q}(n)_{\bar{0}}$*-modules.*

(3) $G \cong \mathbf{A}(1,1) = G_{-1} \oplus G_0 \oplus G_1$, $\mathbb{Z}$*-graded as in* [5, *Proposition* 2.1.2], *and* $\operatorname{der} G = G \oplus P$ *is a semidirect sum, where* $P = P_{\bar{0}} = FD_{-1} + Fz + FD_{+1}$ *is the three-dimensional simple Lie algebra,* $[z, g] = kg$ *for* $g \in G_k$, $k = 0, 1, -1$, $D_{\pm 1}$ *are the* (*up to a constant factor unique*) *endomorphisms of* G *for which* $D_{\pm 1}(G_0) = 0$, $D_{\pm 1}(G_{\pm 1}) = 0$, $D_{\pm 1}(G_{\mp 1}) = G_{\pm 1}$, *and* $D_{\pm 1}\colon G_{\mp 1} \to G_{\pm 1}$ *are isomorphisms of* $G_{\bar{0}}$*-modules.*

In case (3) *of the proposition above, we will deal with Lie superalgebras* G *between* $\mathbf{A}(1,1)$ *and* $\operatorname{der}\mathbf{A}(1,1)$. *We will make use of the following*:

PROPOSITION 2. *Let* G *be the classical simple Lie superalgebra* $\mathbf{A}(1,1)$, *so that* $\operatorname{der} G = G \oplus P$ *as in Proposition* 1. *Then*:

(i) *If* φ *is an automorphism of* P, *then there is an automorphism of* $\operatorname{der} G$ (*which necessarily preserves* G) *extending* φ.

(ii) *If* d *is a nonzero element of* P *such that the action of* $G_{\bar{0}} \oplus Fd$ *in* $G_{\bar{1}}$ *is completely reducible, then* G *admits a consistent* $\mathbb{Z}$*-grading* $G = G_{-1} \oplus G_0 \oplus G_1$ *such that, up to a scalar multiple,* $[d, g] = kg$ *for any* $g \in G_k$, $k = 0, 1, -1$.

Proof. Using the notation in [7, p. 17], $\operatorname{der} G = \Gamma(1, -1, 0)$ (see [7, p. 235]), so $(\operatorname{der} G)_{\bar{0}} = sl(2) \oplus sl(2) \oplus sl(2)$ and $(\operatorname{der} G)_{\bar{1}} = W = V_1 \otimes V_2 \otimes V_3$, where V_i is the two-dimensional irreducible module for the ith copy of $sl(2)$, and with the product of two odd elements given by a map $S^2W \to sl(2) \oplus sl(2) \oplus sl(2)$ in [7, p. 17] with $\sigma_1 = 1 = -\sigma_2$ and $\sigma_3 = 0$. G is the subalgebra of $\operatorname{der} G$ generated by $(\operatorname{der} G)_{\bar{1}}$. That is, $G_{\bar{0}} = sl(2) \oplus sl(2) \oplus 0$, $G_{\bar{1}} = W$, and P is the third copy of $sl(2)$. Any automorphism φ of P is of the form $x \mapsto \alpha x \alpha^{-1}$, where $\alpha \in SL(2) = SL(V_3)$ and φ extends to the automorphism μ of $\operatorname{der} G$ given by $\mu|_{(\operatorname{der} G)_{\bar{0}}} = 1 \oplus 1 \oplus \varphi$, $\mu|_{(\operatorname{der} G)_{\bar{1}}} = 1 \otimes 1 \otimes \alpha$. This proves (i).

For (ii), notice that under the hypotheses, d acts diagonally on V_3, so there is an automorphism φ of P such that, up to a scalar multiple, $\varphi(d) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in P = sl(2)$. Using (i), φ extends to an automorphism of $\operatorname{der} G$ and this preserves its minimal ideal G. Hence, without loss of generality, we can assume that $d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the third copy of $sl(2)$, $V_3 = Fv_1 + Fv_{-1}$ and $[d, v_{\pm 1}] = \pm v_{\pm 1}$. We now take $G_0 = G_{\bar{0}}$, $G_{\pm 1} = V_1 \otimes V_2 \otimes Fv_{\pm 1}$. ∎

In order to study the Lie superalgebras mentioned in the title, it is enough to restrict ourselves to indecomposable ones. This means that they cannot be decomposed as a direct sum of two proper ideals. To deal with this situation, we require the next definition.

DEFINITION. Let $A_1, \dots, A_n$ be arbitrary nonassociative algebras, $A = A_1 \oplus \cdots \oplus A_n$ its direct sum, and B a subalgebra of A with full projections on each A_i. Then B will be called *splittable* if there is a partition of $\{1, \dots, n\}$ in two nonempty subsets I and J such that $B = (B \cap \bigoplus_{i \in I} A_i) \oplus (B \cap \bigoplus_{j \in J} A_j)$. Otherwise, B will be called *unsplittable*.

Recall that an algebra A is termed *solvable* if there is an n such that $A^{(n)} = 0$, where $A^{(0)} = A$ and $A^{(i+1)} = A^{(i)}A^{(i)}$.

PROPOSITION 3. *Let A_1 and A_2 be nonassociative algebras with $(A_1)^2 = A_1$ and A_2 solvable. Let B be a subalgebra of the direct sum $A = A_1 \oplus A_2$ with full projection on A_1. Then $B = (B \cap A_1) \oplus (B \cap A_2)$ (B is splittable).*

Proof. Assume $A_2^{(n)} = 0$ and let π_i denote the projection of A onto A_i, $i = 1, 2$. Then $A_1 = A_1^{(n)} = \pi_1(B^{(n)}) = B^{(n)} \subseteq B$ and $B = A_1 \oplus (B \cap A_2) = (B \cap A_1) \oplus (B \cap A_2)$, as required. ∎

This proposition will allow us to separate adequately the minimal ideals of some semisimple Lie superalgebras. The following will be also of interest:

PROPOSITION 4. *Let L be an unsplittable subalgebra of a direct sum $P_1 \oplus \cdots \oplus P_r$ of simple Lie algebras P_i. Then L is a "diagonal subalgebra." More precisely, for each $i \geq 2$ there is an isomorphism of Lie algebras $\varphi_i \colon P_1 \to P_i$ such that $L = \{\varphi_1(a) + \cdots + \varphi_r(a) : a \in P_1\}$, where φ_1 is the identity mapping of P_1.*

Proof. Let π_i denote the projection onto P_i. Since $\pi_i(L) = P_i$ for each i, it is clear that L is semisimple ($\pi_i(\operatorname{rad} L) = 0$ for each i). Let $N = \ker \pi_1|_L$, which is an ideal of L. By semisimplicity, L contains another ideal N_1 such that $L = N \oplus N_1$ and $N_1 \cong L/N \cong P_1$, which is simple. Therefore, N_1 is the only simple ideal of L with nonzero projection on P_1. Similarly, define N_i to be the unique simple ideal of L with nonzero projection onto P_i. L is the sum of the N_i's, which are not necessarily different. Let $I = \{i \in \{1, \dots, , r\} : \pi_i(N_1) \neq 0\} = \{i \in \{1, \dots, r\} : N_i = N_1\}$ and $J = \{1, \dots, r\} - I$. Then $L = N_1 + \cdots + N_r = N_1 \oplus \sum_{j \in J} N_j = (L \cap \bigoplus_{i \in I} P_i) \oplus (L \cap \bigoplus_{j \in J} P_j)$. By unsplittability, $I = \{1, \dots, r\}$ and $L = N_1$ is simple. Now, for any $a_1 \in P_1$, there are unique $a_2 \in P_2, \dots, a_r \in P_r$ such that $a_1 + \cdots + a_r \in L$, because $\ker \pi_1|_L = N = 0$, and we define $\varphi_i(a_1) = a_i$ for each i. The rest is straightforward. ∎

3. SEMISIMPLE LIE SUPERALGEBRAS WITH COMPLETELY REDUCIBLE ACTION OF $G_{\bar 0}$ IN $G_{\bar 1}$

Let $G = G_{\bar 0} \oplus G_{\bar 1}$ be a semisimple Lie superalgebra and denote by $\rho\colon G_{\bar 0} \to \operatorname{End}_F(G_{\bar 1})$ be the representation of $G_{\bar 0}$ in $G_{\bar 1}$ afforded by the multiplication in G. According to [5, Theorem 6] (see also [3, Corollary 6.1]), the minimal ideals of G are isomorphic to tensor products $S \otimes \Lambda(n)$, where S is a simple Lie superalgebra and $\Lambda(n)$ is the Grassmann superalgebra on an n-dimensional vector space. Therefore, the minimal ideals of G contained in $G_{\bar 0}$ are necessarily simple Lie algebras.

On the other hand, if S is a simple Lie algebra and a minimal ideal in the semisimple Lie superalgebra G, again by [5, Theorem 6], or [3, Proposition 7.2], there is another ideal T of G such that $G = S \oplus T$. Hence $G_{\bar 1} = T_{\bar 1}$ and $[S, G_{\bar 1}] = 0$. We proceed now with T. Then:

PROPOSITION 5. *Let $G = G_{\bar 0} \oplus G_{\bar 1}$ be a semisimple Lie superalgebra and $H = \{x \in G_{\bar 0} : [x, G_{\bar 1}] = 0\}$. Then, H is an ideal of G and there is another ideal T of G such that $G = H \oplus T$. Moreover, H is a semisimple Lie algebra* (*in particular, a direct sum of classical simple Lie superalgebras*).

Because of Proposition 5, we may restrict our attention to semisimple Lie superalgebras G with a completely reducible and faithful action of $G_{\bar 0}$ in $G_{\bar 1}$. We will do so and, as a consequence, $G_{\bar 0}$ will be assumed to be a reductive Lie algebra.

Now, by [5, Theorem 6] we know that, because G is semisimple, its socle (the sum, necessarily direct, of the minimal ideals) verifies

$$\operatorname{socle}(G) = \bigoplus_{i=1}^{r} S_i \otimes \Lambda(n_i), \tag{1}$$

with $S_1, \dots, S_r$ simple Lie superalgebras. But

$$\big(S_i \otimes \Lambda(n_i)\big)_{\bar 0} = (S_i)_{\bar 0} \otimes \Lambda(n_i)_{\bar 0} \bigoplus (S_i)_{\bar 1} \otimes \Lambda(n_i)_{\bar 1}.$$

If $n_i \geq 1$, $\Lambda(n_i) = F1 \oplus N_i$, where $N_i = (N_i)_{\bar 0} \oplus \Lambda(n_i)_{\bar 1}$ is the nilpotent radical of $\Lambda(n_i)$. Thus, $\Lambda(n_i)_{\bar 0} = F1 \oplus (N_i)_{\bar 0}$, with nilpotent $(N_i)_{\bar 0}$. Hence, $(S_i)_{\bar 0} \otimes (N_i)_{\bar 0} \bigoplus (S_i)_{\bar 1} \otimes \Lambda(n_i)_{\bar 1}$ is a nilpotent ideal of $(S_i \otimes \Lambda(n_i))_{\bar 0}$, and this is a reductive Lie algebra. Therefore,

$$\big((S_i)_{\bar 0} \otimes 1\big)\big((S_i)_{\bar 1} \otimes \Lambda(n_i)_{\bar 1}\big) = 0,$$

so $(S_i)_{\bar 0}(S_i)_{\bar 1} = 0$. Hence, by simplicity of S_i, $(S_i)_{\bar 1} = 0$ and $S_i = (S_i)_{\bar 0}$ is a simple Lie algebra. Moreover, $n_i = 1$ since $(S_i \otimes \Lambda(n_i))_{\bar 0}$ would not be reductive otherwise.

Proposition 6. *Let $G = G_{\bar 0} \oplus G_{\bar 1}$ be a semisimple Lie superalgebra with a completely reducible and faithful action of $G_{\bar 0}$ in $G_{\bar 1}$, and let* $\operatorname{socle}(G) = \bigoplus_{i=1}^r S_i \otimes \Lambda(n_i)$ *as in Eq.* (1). *Then, for any $i = 1, \dots, r$, $n_i = 0$ or 1. Moreover, if $n_i = 0$, then S_i is a classical simple Lie superalgebra, and if $n_i = 1$, then $(S_i)_{\bar 1} = 0$ and $S_i = (S_i)_{\bar 0}$ is a simple Lie algebra.*

Proof. The only thing which remains to be determined is what happens if $n_i = 0$. In this case $(S_i)_{\bar 0}$ is reductive and S_i is simple. This forces S_i to be a classical simple Lie superalgebra. ∎

As mentioned in the previous section, it is enough to consider indecomposable Lie superalgebras so, for the time being, $G = G_{\bar 0} \oplus G_{\bar 1}$ will be an indecomposable semisimple Lie superalgebra with a completely reducible and faithful action of $G_{\bar 0}$ in $G_{\bar 1}$. Again, by [5, Theorem 6],

$$\bigoplus_{i=1}^r S_i \otimes \Lambda(n_i) \subseteq G \subseteq \bigoplus_{i=1}^r \left(\operatorname{der} S_i \otimes \Lambda(n_i) \oplus 1 \otimes \operatorname{der} \Lambda(n_i)\right). \quad (2)$$

For each $i = 1, \dots, r$, let G_i be the projection of G into $\operatorname{der} S_i \otimes \Lambda(n_i) \oplus 1 \otimes \operatorname{der} \Lambda(n_i)$.

If $n_i = 1$, then S_i is a simple Lie algebra by Proposition 6, so $S_i = \operatorname{der} S_i$ and $G_i = (S_i \otimes \Lambda(n_i)) \oplus \hat{G}_i$, a semidirect sum, where $\hat{G}_i$ is the projection of G_i in $1 \otimes \operatorname{der} \Lambda(n_i) = 1 \otimes \operatorname{der} \Lambda(1)$. By [5, Theorem 6], $\Lambda(n_i)$ must be $\hat{G}_i$-simple, but $\Lambda(n_i) = \Lambda(1) = F1 + Fe_i$, with $e_i^2 = 0$, and $\operatorname{der} \Lambda(n_i) = Fd_{\bar 0}^i + Fd_{\bar 1}^i$, where $d_{\bar 0}^i(1) = 0 = d_{\bar 1}^i(1)$, $d_{\bar 0}^i(e_i) = e_i$, $d_{\bar 1}^i(e_i) = 1$, and $[d_{\bar 1}^i, d_{\bar 0}^i] = d_{\bar 1}^i$. Then, $\Lambda(1)$ is $Fd_{\bar 1}^i$-simple but not $Fd_{\bar 0}^i$-simple. Hence, either $\hat{G}_i = (\hat{G}_i)_{\bar 1} = Fd_{\bar 1}^i$, or $\hat{G}_i = \operatorname{der} \Lambda(1) = Fd_{\bar 0}^i + Fd_{\bar 1}^i$.

On the other hand, if $n_i = 0$ $(1 \le i \le r)$ in Eq. (2), then $S_i \subseteq G_i \subseteq \operatorname{der} S_i$, where S_i is a classical simple Lie superalgebra. Moreover, if $n_i = 0$ and any derivation of S_i is inner, then $S_i = G_i = \operatorname{der} S_i$ in (2). Hence

$$G = (S_i \otimes 1) \oplus \left(G \cap \bigoplus_{j \ne i} \left(\operatorname{der} S_j \otimes \Lambda(n_j) + 1 \otimes \operatorname{der} \Lambda(n_j)\right)\right).$$

By indecomposability, $r = 1$ and G is a simple classical Lie superalgebra. The same happens if $n_i = 0$ and $G_i = S_i \otimes 1$.

Otherwise, with G indecomposable in (2), $n_i = 0$, $S_i \subset G_i \subseteq \operatorname{der} S_i$, S_i classical simple, and $(G_i)_{\bar 0}$ reductive, Propositions 1, 2, 5, and 6 and the previous arguments immediately imply:

Proposition 7. *Let G be a semisimple indecomposable Lie superalgebra with a completely reducible action of $G_{\bar 0}$ and $G_{\bar 1}$. Then, either G is a classical simple Lie superalgebra, or if $T_1, \dots, T_r$ are the minimal ideals of G, we have*

$$\operatorname{socle}(G) = \bigoplus_{i=1}^r T_i \subset G \subseteq \bigoplus_{i=1}^r \left(T_i \oplus \hat{G}_i\right) \subseteq \bigoplus_{i=1}^r \operatorname{der} T_i, \quad (3)$$

where $T_i \oplus \hat{G}_i$ *is a semidirect sum, the projection of* G *on each* $\operatorname{der} T_i$ *is* $T_i \oplus \hat{G}_i$, *and for each* $i = 1, \dots, r$, *one of the following situations occurs*:

(1) $T_i = S_i \otimes \Lambda(1) = S_i \oplus e_i S_i$, *where* $(T_i)_{\bar{0}} = S_i$ *is a simple Lie algebra,* $(T_i)_{\bar{1}} = e_i S_i$, *and the multiplication is determined by the multiplication* $[x,y]$ *in* S_i, $[x, e_i y] = [e_i x, y] = e_i[x, y]$ *and* $[e_i x, e_i y] = 0$ *for any* $x, y \in S_i$. *Besides,* $\hat{G}_i = Fd^i_{\bar{0}} + Fd^i_{\bar{1}}$ *with* $[\hat{G}_i, (T_i)_{\bar{0}}] = 0$ *and* $[d^i_{\bar{0}}, e_i x] = e_i x$, $[d^i_{\bar{1}}, e_i x] = x$ *for any* $x \in S_i$, *so that* $[d^i_{\bar{1}}, d^i_{\bar{0}}] = d^i_{\bar{1}}$. *In this case,* $G_i = T_i \oplus \hat{G}_i$ *has a consistent* $\mathbb{Z}$*-grading* $G_i = (G_i)_{-1} \oplus (G_i)_0 \oplus (G_i)_1$ *with*

$$(G_i)_{-1} = e_i S_i, \qquad (G_i)_0 = S_i \oplus Fd^i_{\bar{0}}, \qquad (G_i)_1 = Fd^i_{\bar{1}}.$$

(2) T_i *is as in case* (1) *but with* $\hat{G}_i = Fd^i_{\bar{1}}$ *with* $d^i_{\bar{1}}$ *also as in* (1). *Again,* $G_i = T_i \oplus \hat{G}_i$ *has a consistent* $\mathbb{Z}$*-grading with the same* $(G_i)_{\pm 1}$, *but with* $(G_i)_0 = S_i$.

(3) $T_i = (T_i)_{-1} \oplus (T_i)_0 \oplus (T_i)_1$ *is a classical simple Lie superalgebra of type* $\mathbf{A}(n,n)$ $(n \geq 1)$ *or* $\mathbf{P}(n)$ $(n \geq 2)$, *with a consistent* $\mathbb{Z}$*-grading as in* [5, *Proposition* 2.1.2] *and* $\hat{G}_i = Fd^i_{\bar{0}}$ *with* $[d^i_{\bar{0}}, x] = kx$ *for any* $x \in (T_i)_k$, $k = 0, 1, -1$. *Thus,* $G_i = T_i \oplus \hat{G}_i$ *is consistently* $\mathbb{Z}$*-graded with* $(G_i)_{\pm 1} = (T_i)_{\pm 1}$ *and* $(G_i)_0 = (T_i)_0 \oplus Fd^i_{\bar{0}}$.

(4) T_i *is a classical simple Lie superalgebra of type* $\mathbf{Q}(n)$ $(n \geq 3)$ *and* $\hat{G}_i = Fd^i_{\bar{1}}$, *where* $d^i_{\bar{1}}$ *is the* (*within proportionality unique*) *odd endomorphism of* T_i *with* $d^i_{\bar{1}}((T_i)_{\bar{0}}) = 0$ *and* $d^i_{\bar{1}} : (T_i)_{\bar{1}} \to (T_i)_{\bar{0}}$ *an isomorphism of* $(T_i)_{\bar{0}}$*-modules. In this case* $G_i = T_i \oplus \hat{G}_i = (G_i)_{-1} \oplus (G_i)_0 \oplus (G_i)_1$, *with* $(G_i)_0 = (T_i)_{\bar{0}}$, $(G_i)_1 = (T_i)_{\bar{1}}$, *and* $(G_i)_{-1} = \hat{G}_i = Fd^i_{\bar{1}}$, *although this is not an algebra gradation.*

(5) $T_i = (T_i)_{-1} \oplus (T_i)_0 \oplus (T_i)_1$ *is a classical simple Lie superalgebra of type* $\mathbf{A}(1,1)$ *with consistent* $\mathbb{Z}$*-grading as in* [5, *Proposition* 2.1.2] *and* $\hat{G}_i = (\hat{G}_i)_{\bar{0}}$ *is the three-dimensional simple Lie algebra with basis* $\{D^i_{-1}, z^i, D^i_{+1}\}$ *and action in* T_i *given by* $[z^i, x] = kx$ *for any* $x \in (T_i)_k$, $k = 0, 1, -1$, $D_{\pm 1}((T_i)_{\pm 1}) = 0 = D_{\pm 1}((T_i)_0)$, $D_{\pm 1}((T_i)_{\mp 1}) = (T_i)_{\pm 1}$, *and* $D_{\pm 1} : (T_i)_{\mp 1} \to (T_i)_{\pm 1}$ *are isomorphisms of* $(T_i)_0$*-modules.*

Moreover, $G = \operatorname{socle}(G) \oplus L$, *a semidirect sum, with* L *an unsplittable subalgebra of* $\hat{G} = \hat{G}_1 \oplus \cdots \oplus \hat{G}_r$ *in* (3).

Notice that in the proposition above, L has full projections on each $\hat{G}_i$, because $T_i \oplus \hat{G}_i$ is the projection of G onto $\operatorname{der} T_i$ for each i and L is unsplittable because of the indecomposability of G.

Therefore, our problem in this section reduces to the easiest question of determining the unsplittable subalgebras L of Lie superalgebras $\hat{G} = \hat{G}_1 \oplus \cdots \oplus \hat{G}_r$, which are direct sums of ideals of four types:

(i)

$\hat{G}_i = Fd^i_{\bar{0}}$, an even one-dimensional superalgebra, for case (3) in Proposition 7.

(ii)

$\hat{G}_i = Fd^i_{\bar{1}}$, an odd one-dimensional superalgebra, for cases (2) and (4) in Proposition 7.

(iii)

$\hat{G}_i = Fd^i_{\bar{0}} + Fd^i_{\bar{1}}$, with $\left[d^i_{\bar{1}}, d^i_{\bar{0}}\right] = d^i_{\bar{1}}$, a two-dimensional superalgebra isomorphic to $\operatorname{der}\Lambda(1)$, for case (1).

(iv)

$\hat{G}_i = \left(\hat{G}_i\right)_{\bar{0}}$ isomorphic to $sl(2)$, for case (5). (4)

We will see first that if some $\hat{G}_i$ is of type (iv), then so are all the direct summands of $\hat{G}$. That is, type (iv) does not mix with the other types for unsplittable L. Then, we will see that the same happens for type (ii). However, types (i) and (iii) may stand together.

Hence, assume $\hat{G} = \hat{G}_1 \oplus \cdots \oplus \hat{G}_r$ with each ideal $\hat{G}_i$ as in types (i)–(iv) above, and let L be an unsplittable subalgebra of $\hat{G}$. Let $I = \{i \in \{1, \dots, r\} : \hat{G}_i \cong sl(2)\}$. Let π_I (respectively $\pi_{\bar{I}}$) be the projection of $\hat{G}$ onto $\bigoplus_{i \in I} \hat{G}_i$ (respectively, onto $\bigoplus_{i \notin I} \hat{G}_i$). Then, L is a subalgebra of $\pi_I(L) \oplus \pi_{\bar{I}}(L)$. Since the projection of L on each $\hat{G}_i$ is the whole $\hat{G}_i$, because L is unsplittable, the same happens with $\pi_I(L)$ on each $\hat{G}_i$, $i \in I$. Thus, $\pi_I(L)$ is a semisimple Lie algebra, because the projection of its radical onto each copy of $sl(2)$ is trivial. By Proposition 3, $L = \pi_I(L) \oplus \pi_{\bar{I}}(L)$ and, since L is unsplittable, either $I = \{1, \dots, r\}$ of $I = \emptyset$.

In the case $I = \{1, \dots, r\}$, Propositions 4 and 7 force G to be isomorphic to a superalgebra described in the next example:

EXAMPLE 1. Let T be a direct sum of r copies $T_1, \dots, T_r$ of the simple Lie superalgebra $\mathbf{A}(1,1)$, so that $\operatorname{der} T$ is the semidirect sum of the ideal $T_1 \oplus \cdots \oplus T_r$ and an even subalgebra which is a direct sum of r copies of $sl(2)$:

$$\operatorname{der} T = (T_1 \oplus \cdots \oplus T_r) \oplus \left(\hat{G}_1 \oplus \cdots \oplus \hat{G}_r\right).$$

Take a basis $\{D^i_{-1}, z^i, D^i_{+1}\}$ of each $\hat{G}_i$ as in Proposition 7, case (5), and consider the elements $D_{\pm 1} = \sum_{i=1}^r D^i_{\pm 1}$, $z = \sum_{i=1}^r z^i$, the diagonal subalgebra $P = FD_{-1} + Fz + FD_{+1}$ and the Lie superalgebra

$$G = \mathbf{G}(\mathbf{A}(1,1); r) = (T_1 \oplus \cdots \oplus T_r) \oplus P,$$

with the multiplication inherited from $\operatorname{der} T$. Notice that P is an ideal of $G_{\bar{0}}$ and that $\mathbf{G}(\mathbf{A}(1,1); 1) = \operatorname{der} \mathbf{A}(1,1)$.

Assume now that $\hat{G} = \hat{G}_1 \oplus \cdots \oplus \hat{G}_r$, with each $\hat{G}_i$ of type (i), (ii), or (iii) in (4), and that L is an unsplittable subalgebra of $\hat{G}$. Let $I = \{i \in \{1, \dots, r\} : \hat{G}_i$ is of type (i)$\}$, $J = \{j \in \{1, \dots, r\} : \hat{G}_j$ is of type (ii)$\}$ and $K = \{k \in \{1, \dots, r\} : \hat{G}_k$ of type (iii)$\}$. Also let $A = \bigoplus_{i \in I} \hat{G}_i$, $B = \bigoplus_{j \in J} \hat{G}_j$ and $C = \bigoplus_{k \in K} \hat{G}_k$. Then $\hat{G}_{\bar{0}} = A \oplus C_{\bar{0}}$ and $\hat{G}_{\bar{1}} = B \oplus C_{\bar{1}}$. Let π be the projection of $\hat{G}$ onto $C_{\bar{1}}$ and let $u \in \pi(L)$, $u \notin L$, $u = \sum_{k \in K} \alpha_k d_{\bar{1}}^k$, with a minimum positive number of nonzero α_k's. We may assume that $u = d_{\bar{1}}^h + \sum_{k \in K,\ k \neq h} \alpha_k d_{\bar{1}}^k$. Let $u_{\bar{0}} \in L_{\bar{0}}$ such that the projection of $u_{\bar{0}}$ on $\hat{G}_h$ is $d_{\bar{0}}^h$ (recall that $L = L_{\bar{0}} \oplus L_{\bar{1}}$ has full projections on each $\hat{G}_i$). Then, $[u, u_{\bar{0}}] \in L$ since $[A, L] = [B, L] = 0$, and $[u, u_{\bar{0}}] - u \in \pi(L)$, but with a smaller number of nonzero coefficients. Thus, $[u, u_{\bar{0}}] - u \in L$ and so $u \in L$ too, a contradiction. We conclude that $\pi(L) = L \cap C_{\bar{1}}$. Hence, $L_{\bar{1}} = (L \cap B) \oplus (L \cap C_{\bar{1}})$ and $L = (L \cap B) \oplus (L \cap (A \oplus C))$. Since L is unsplittable, this forces either $\hat{G} = B$ or $\hat{G} = A \oplus C$. In the case $\hat{G} = B$ $(J = \{1, \dots, r\})$, the superalgebra G in Proposition 7 is described by the next example (inspired by [6, p. 54]):

EXAMPLE 2. Let $S_1, \dots, S_m$ be simple Lie algebras and V an m-dimensional vector space with basis $\{e_1, \dots, e_m\}$ and let $\{d_1, \dots, d_m\}$ be the dual basis in V^*. We identify d_i with $d_{\bar{1}}^i$ in Proposition 7, case (2). Let us take also $T_{m+1} = \mathbf{Q}(n_1), \dots, T_{m+r} = \mathbf{Q}(n_r)$ $(n_i \geq 3)$ and the r-dimensional odd subspace $W = Fd_{m+1} + \cdots + Fd_{m+r}$ of $\operatorname{der}(T_{m+1} \oplus \cdots \oplus T_{m+r})$, where d_i is the $d_{\bar{1}}^i$ described in Proposition 7, case (4). We define the Lie superalgebra

$$G = \mathbf{G}_{\mathbf{Q}}^{m,r}(S_1, \dots, S_m; n_1, \dots, n_r; L) = G_{-1} \oplus G_0 \oplus G_1,$$

where

$$G_{\bar{0}} = G_0, \qquad G_{\bar{1}} = G_{-1} + G_1,$$

$$G_0 = S_1 \oplus \cdots \oplus S_m \oplus (T_{m+1})_{\bar{0}} \oplus \cdots \oplus (T_{m+r})_{\bar{0}},$$

$$G_1 = e_1 S_1 \oplus \cdots \oplus e_m S_m \oplus (T_{m+1})_{\bar{1}} \oplus \cdots \oplus (T_{m+r})_{\bar{1}},$$

$$G_{-1} = L, \text{ a subspace of } V^* \oplus W = Fd_1 + \cdots + Fd_{m+r} \text{ with full projections on each } Fd_i.$$

For each $i = 1, \dots, m$ (respectively $i = m+1, \dots, m+r$) we identify componentwise $Fd_i \oplus S_i \oplus e_i S_i$ (respectively, $Fd_i \oplus (T_i)_{\bar{0}} \oplus (T_i)_{\bar{1}}$) with $G_i = (G_i)_{-1} \oplus (G_i)_0 \oplus (G_i)_1$ in Proposition 7, case (2) (respectively, case (4)) and the multiplication in G is that inherited from these identifications as a subalgebra of $G_1 \oplus \cdots \oplus G_{m+r}$.

The Lie superalgebra G is semisimple, since its minimal ideals are the $(S_i \otimes \Lambda(1))$'s $(i = 1, \ldots, m)$ and the T_i's $(i = m+1, \ldots, m+r)$. Its even part $G_{\bar{0}}$ is semisimple and G is indecomposable if and only if L is unsplittable as a subalgebra of $Fd_1 \oplus \cdots \oplus Fd_{m+r}$.

Notice that we admit $m = 0$ or $r = 0$ and that for $m = 0$ and $r = 1$, G is just $\operatorname{der} \mathbf{Q}(n_1)$.

Finally, we are left with types (i) and (iii) in (4). Hence, after a suitable permutation,

$$\hat{G} = Fd_{\bar{0}}^{i} \oplus \cdots \oplus Fd_{\bar{0}}^{r} \oplus D_{r+1} \oplus \cdots \oplus D_{r+t}, \tag{5}$$

with $D_i = Fd_{\bar{0}}^{i} \oplus Fd_{\bar{1}}^{i}$ as in (4).

LEMMA 8. *Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a subalgebra of the Lie superalgebra $\hat{G}$ in* (5) *with full projections on each direct summand. Then, L is unsplittable if and only if $L_{\bar{0}}$ is.*

Proof. Obviously, any splitting of L gives one of $L_{\bar{0}}$, so that if $L_{\bar{0}}$ is unsplittable, so is L. Let us assume now that $L_{\bar{0}}$ is splittable and L is not. Then, $\{1, \ldots, r\} = I_1 \cup I_2$, $\{r+1, \ldots, r+t\} = J_1 \cup J_2$ for disjoint sets I_1 and I_2 and J_1 and J_2, $J_1 \neq \emptyset \neq J_2$, so that $L_{\bar{0}} = L_{\bar{0}}^{1} \oplus L_{\bar{0}}^{2}$, where

$$L_{\bar{0}}^{1} = L_{\bar{0}} \cap \left(\left(\bigoplus_{i \in I_1} Fd_{\bar{0}}^{i} \right) \oplus \left(\bigoplus_{j \in J_1} D_j \right) \right)$$

$$L_{\bar{0}}^{2} = L_{\bar{0}} \cap \left(\left(\bigoplus_{i \in I_2} Fd_{\bar{0}}^{i} \right) \oplus \left(\bigoplus_{j \in J_2} D_j \right) \right).$$

Since L is unsplittable, there is an element $u \in L_{\bar{1}}$ such that $u = u^{J_1} + u^{J_2}$, where $u^{J_k} \in \bigoplus_{j \in J_k} (D_j)_{\bar{1}}$ and $u^{J_k} \notin L_{\bar{1}}$, $k = 1, 2$. We take such a u with the smallest possible number of nonzero components in the basis $\{d_{\bar{1}}^{1}, \ldots, d_{\bar{1}}^{r+t}\}$. As in a previous argument, we may take $u_{\bar{0}}^{J_k} \in L_{\bar{0}}^{k}$ such that $[u^{J_k}, u_{\bar{0}}^{J_k}] - u^{J_k}$ has a smaller number of nonzero components, $k = 1, 2$. Then $u_{\bar{0}} = u_{\bar{0}}^{J_1} + u_{\bar{0}}^{J_k} \in L_{\bar{0}}$ and $[u, u_{\bar{0}}] - u = ([u^{J_1}, u_{\bar{0}}^{J_1}] - u^{J_1}) + ([u^{J_2}, u_{\bar{0}}^{J_2}] - u^{J_2}) \in L$. By minimality $[u^{J_1}, u_{\bar{0}}^{J_1}] - u^{J_1} \in L$ and, since $[u^{J_1}, u_{\bar{0}}^{J_1}] = [u, u_{\bar{0}}^{J_1}] \in L$, we obtain $u^{J_1} \in L$ and, similarly, $u^{J_2} \in L$; a contradiction which proves the lemma. ∎

Therefore, if we take any unsplittable subalgebra S of $\hat{G}_{\bar{0}}$ in Eq. (5) and any subspace T of $\hat{G}_{\bar{1}}$ with full projections on each $(\hat{G}_i)_{\bar{1}}$ and such that $[S, T] \subseteq T$, then $L = L_{\bar{0}} \oplus L_{\bar{1}}$, with $L_{\bar{0}} = S$ and $L_{\bar{1}} = T$, gives an unsplittable subalgebra of $\hat{G}$ and all unsplittable subalgebras of $\hat{G}$ are obtained in this way.

With this in mind, the next example covers the nonsimple Lie superalgebras G in Proposition 7 with socle consisting of ideals in cases (1) and (3):

EXAMPLE 3. Let $S_1, \ldots, S_m$ be simple Lie algebras and for each $i = 1, \ldots, m$ consider $G_i = \operatorname{der}(S_i \otimes \Lambda(1)) = (G_i)_{-1} \oplus (G_i)_0 \oplus (G_i)_1$ as in Proposition 7, case (1), with $(G_i)_{-1} = Fd^i_{\bar{1}}$ and $(G_i)_{\bar{0}} = S_i \oplus Fd^i_{\bar{0}}$.

Now, let us consider classical simple Lie superalgebras $T_{m+i} = \mathbf{A}(p_i, p_i)$ $(p_i \geq 1)$, $i = 1, \ldots, r$, and $T_{m+r+j} = \mathbf{P}(q_j)$ $(q_j \geq 2)$, $j = 1, \ldots, s$, and $G_i = (G_i)_{-1} \oplus (G_i)_0 \oplus (G_i)_1$ as in Proposition 7, case (3), for each $i = m + 1, \ldots, m + r + s$, so that $(G_i)_0 = (T_i)_{\bar{0}} \oplus Fd^i_{\bar{0}}$.

Take a subalgebra L of the superalgebra $(\bigoplus_{i=1}^{m}(Fd^i_{\bar{0}} + Fd^i_{\bar{1}})) \oplus (\bigoplus_{i=m+1}^{m+r+s} Fd^i_{\bar{0}})$ with full projections on each direct summand. We define the superalgebra

$$G = \mathbf{G}^{m,r,s}_{\mathbf{A},\mathbf{P}}(S_1, \ldots, S_m; p_1, \ldots, p_r; q_1, \ldots, q_s; L)$$

as the subalgebra of $\bigoplus_{i=1}^{m+r+s} G_i$ which is the semidirect sum of the socle

$$\left(\bigoplus_{i=1}^{m} (e_i S_i \oplus S_i)\right) \oplus \left(\bigoplus_{j=1}^{r+s} T_{m+j}\right)$$

and the subalgebra L.

The Lie superalgebra G is indecomposable if and only if $L_{\bar{0}}$ is an unsplittable subalgebra of $\bigoplus_{i=1}^{m+r+s} Fd^i_{\bar{0}}$. Notice that for $m = 1$, $r = s = 0$, $G = \operatorname{der}(S_1 \otimes \Lambda(1))$; for $m = 0 = s$, $r = 1$ and $p_1 \geq 2$, $G = \operatorname{der}\mathbf{A}(p_1, p_1)$; and for $m = r = 0$ and $s = 1$, $G = \operatorname{der}\mathbf{P}(q_1)$.

All the work in this section is summarized in the following:

THEOREM A. *Let G be a semisimple Lie superalgebra. Then, the action of $G_{\bar{0}}$ in $G_{\bar{1}}$ is completely reducible if and only if G is a direct sum of ideals which are either classical simple Lie superalgebras or superalgebras described in Examples* 1, 2, *and* 3.

4. LIE SUPERALGEBRAS $L = L_{\bar{0}} \oplus L_{\bar{1}}$ WITH $L_{\bar{0}}$ REDUCTIVE AND COMPLETELY REDUCIBLE ACTION OF $L_{\bar{0}}$ IN $L_{\bar{1}}$

The goal of this section is to describe the superalgebras in its title. It will be shown that this problem reduces to the description given in Theorem A by means of some extensions of the following type:

DEFINITION. (1) Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra with $G_{\bar{0}} = A \oplus [G_{\bar{1}}, G_{\bar{1}}]$, for some ideal A of $G_{\bar{0}}$, and let V be a Lie module for the Lie algebra A. Consider the superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$, with $L_{\bar{0}} = G_{\bar{0}}$, $L_{\bar{1}} =$

$G_{\bar 1} \oplus V$ and multiplication given by $[L_{\bar 1}, V] = [[G_{\bar 1}, G_{\bar 1}], V] = 0$, $[a, v]$ defined by the A-module structure of V for any $a \in A$ and $v \in V$, and the given multiplication in G. L is easily seen to be a Lie superalgebra, which will be called an *elementary odd extension* of G by V.

(2) Let $G = G_{\bar 0} \oplus G_{\bar 1}$ be a Lie superalgebra, A a vector space, and μ: $G_{\bar 1} \times G_{\bar 1} \to A$ a symmetric bilinear mapping satisfying $\mu([g, x], y) + \mu(x, [g, y]) = 0$ for any $g \in G_{\bar 0}$, $x, y \in G_{\bar 1}$. Consider the superalgebra $L = L_{\bar 0} \oplus L_{\bar 1}$ with $L_{\bar 0} = G_{\bar 0} \oplus A$, a direct sum of ideals, $L_{\bar 1} = G_{\bar 1}$, and multiplication $[x, y]^{\sim}$ determined by the multiplication $[u, v]$ in G if at least one of u, v is even, $[A, L]^{\sim} = 0$, and $[u, v]^{\sim} = [u, v] + \mu(u, v)$, for $u,v \in G_{\bar 1}$. Then, L is also easily seen to be a Lie superalgebra, which will be called an *elementary even extension* of G by (A, μ).

The elementary odd extensions were called just elementary extensions in [6, Chap. II].

In case (2) of the above definition, if $\{a_1, \dots, a_n\}$ is a basis of A, let μ_i: $G_{\bar 1} \times G_{\bar 1} \to F$ be given by

$$\mu(x, y) = \sum_{i=1}^{n} \mu_i(x, y) a_i. \tag{6}$$

The μ_i's are symmetric bilinear and $G_{\bar 0}$-invariant forms. Conversely, given symmetric invariant bilinear forms μ_i: $G_{\bar 1} \times G_{\bar 1} \to F$, Eq. (6) gives a μ satisfying the conditions in (2) above.

Moreover, if $G_{\bar 1}$ is a $G_{\bar 0}$-module completely reducible, $G_{\bar 1} = \bigoplus_{i=1}^{n} W_i$, with the W_i's $G_{\bar 0}$-irreducible modules, each μ_i is determined by $\mu_i^{j,k}$: $W_j \times W_k \to F$, which must satisfy $\mu_i^{j,k}(w_j, w_k) = \mu_i^{k,j}(w_k, w_j)$ for any j, k and any $w_j \in W_j$, $w_k \in W_k$. By irreducibility of the W_i's and the $G_{\bar 0}$-invariance of the μ_i's, $\mu_i^{j,k}$ can only be nonzero if W_j and W_k are dual $G_{\bar 0}$-modules. Besides, if $j \neq k$ and W_j and W_k are dual modules for $G_{\bar 0}$, such a nonzero $\mu_i^{j,k}$ is unique up to a scalar multiple. If $j = k$ and W_j is a self-dual $G_{\bar 0}$-module, either there is no nonzero $\mu_i^{j,j}$ or there is a unique one up to scalar multiples.

One more kind of extension will be needed:

PROPOSITION 9. *Let $G = G_{\bar 0} \oplus G_{\bar 1}$ be a Lie superalgebra with $G_{\bar 0} = A \oplus [G_{\bar 1}, G_{\bar 1}]$ (direct sum of ideals) and let $U = U_{\bar 0} \oplus U_{\bar 1}$ be another Lie superalgebra with abelian $U_{\bar 0}, [U_{\bar 1}, U_{\bar 1}] = 0$ and completely reducible action of $U_{\bar 0}$ in $U_{\bar 1}$, so that $U_{\bar 1} = \bigoplus_{\lambda \in \Lambda} U^{\lambda}$, $\Lambda \subseteq (U_{\bar 0})^*$, and $U^{\lambda} = \{u \in U_{\bar 1} : [x, u] = \lambda(x) u$ for any $x \in U_{\bar 0}\}$. Let each U^{λ} be equipped with an A-module structure and consider the superalgebra $L = L_{\bar 0} \oplus L_{\bar 1}$, with $L_{\bar 0} = U_{\bar 0} \oplus G_{\bar 0}$, $L_{\bar 1} = U_{\bar 1} \oplus G_{\bar 1}$, and multiplication defined in such a way that G is a subalgebra of L, U an ideal, $[G_{\bar 1}, U] = 0 = [[G_{\bar 1}, G_{\bar 1}], U] = [U_{\bar 0}, G_{\bar 0}]$, and the product $[a, u]$ for $a \in A$ and $u \in U^{\lambda}$ is given by the corresponding A-module structure in U^{λ}. Then, L is a Lie superalgebra.*

Proof. Notice that $\hat{G} = G \oplus U_{\bar{0}}$ is a direct sum of ideals, so it is a Lie superalgebra, and $\hat{G}_{\bar{0}} = [\hat{G}_{\bar{1}}, \hat{G}_{\bar{1}}] \oplus (A \oplus U_{\bar{0}})$. Now, L is an elementary odd extension of $\hat{G}$ by $U_{\bar{1}}$. ∎

DEFINITION. Let G, A, U, and L be as in the previous proposition. If each U^λ is a completely reducible A-module, then L will be called a *nice extension* of G by U.

LEMMA 10. *Let $U = U_{\bar{0}} \oplus U_{\bar{1}}$ be a Lie superalgebra with abelian $U_{\bar{0}}$ and completely reducible action of $U_{\bar{0}}$ in $U_{\bar{1}}$. Then,* $[[U_{\bar{1}}, U_{\bar{1}}], U_{\bar{1}}] = 0$.

Proof. Let $\rho\colon U_{\bar{0}} \to \operatorname{End}_F(U_{\bar{1}})$ be the representation of $U_{\bar{0}}$ in $U_{\bar{1}}$ afforded by the multiplication in U. Then, $\ker\rho$ is an ideal of U. By complete reducibility, $U_{\bar{1}} = \bigoplus_{\lambda\in\Lambda} U^\lambda$ with $\Lambda \subseteq (U_{\bar{0}})^*$ and $U^\lambda = \{u \in U_{\bar{1}} : [x, u] = \lambda(x)u$ for any $x \in U_{\bar{0}}\}$. If $\mu, \lambda \in \Lambda$ and $\mu \neq -\lambda$, $[U^\lambda, U^\mu] \subseteq \{a \in U_{\bar{0}} : [x, a] = (\lambda + \mu)(x)a$ for any $x \in U_{\bar{0}}\} = 0$ since $U_{\bar{0}}$ is abelian. Now, if $0 \neq \lambda \in \Lambda$, $0 \neq x \in U^\lambda$, and $0 \neq y \in U^{-\lambda}$, $0 = [[x, x], y] = 2[x, [x, y]]$. Hence, $\lambda([U^\lambda, U^{-\lambda}]) = 0$ and $[[U^\lambda, U^{-\lambda}], U^{\pm\lambda}] = 0$. Besides, if $\mu \neq \pm\lambda$, $[[U^\lambda, U^{-\lambda}], U^\mu] \subseteq [[U^\lambda, U^\mu], U^{-\lambda}] + [U^\lambda, [U^{-\lambda}, U^\mu]] = 0$. Hence, for any $0 \neq \lambda \in \Lambda$, $[U^\lambda, U^{-\lambda}] \subseteq \ker\rho$, and also $[U^0, U^0] \subseteq \ker\rho$. Thus, $[U_{\bar{1}}, U_{\bar{1}}] \subseteq \ker\rho$ and $[[U_{\bar{1}}, U_{\bar{1}}], U_{\bar{1}}] = 0$. ∎

Now, let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra with reductive $L_{\bar{0}}$ and completely reducible action of $L_{\bar{0}}$ in $L_{\bar{1}}$, and let $\rho\colon L_{\bar{0}} \to \operatorname{End}_F(L_{\bar{1}})$ be the corresponding representation. Let $\operatorname{rad} L$ be the solvable radical of L (incidentally, this is the same considering L either as a superalgebra or as a (ungraded) nonassociative algebra [4]). Since $L_{\bar{0}}$ is reductive, $(\operatorname{rad} L)_{\bar{0}}$ is contained in the center of $L_{\bar{0}}$. By complete reducibility, $L_{\bar{1}} = (\operatorname{rad} L)_{\bar{1}} \oplus G_{\bar{1}}$. Finally, let $K = \ker\rho \cap (\operatorname{rad} L)_{\bar{0}}$, which is easily seen to be equal to $Z(L)_{\bar{0}}$, where $Z(L) = \{x \in L : [x, L] = 0\}$. Then, $L_{\bar{0}} = K \oplus \tilde{L}_{\bar{0}}$ for some ideal $\tilde{L}_{\bar{0}}$ of $L_{\bar{0}}$. For $x, y \in L_{\bar{1}}$, $[x, y] = \mu(x, y) + [x, y]^\sim$, with $\mu(x, y) \in K$ and $[x, y]^\sim \in \tilde{L}_{\bar{0}}$, and L is easily seen to be an elementary even extension of $\tilde{L} = \tilde{L}_{\bar{0}} \oplus L_{\bar{1}}$ by (K, μ).

Hence, in what follows, we assume that $K = 0$ (or equivalently, we pass from L to $\tilde{L}$). Because the representation ρ is completely reducible, its restriction

$$\rho|_{(\operatorname{rad} L)_{\bar{0}}}\colon (\operatorname{rad} L)_{\bar{0}} \to \operatorname{End}_F\big((\operatorname{rad} L)_{\bar{1}}\big)$$

is also completely reducible, since the elements of $Z(L_{\bar{0}})$ ($\supseteq (\operatorname{rad} L)_{\bar{0}}$) act simultaneously diagonalizably on $L_{\bar{1}}$. Moreover, if $a \in (\operatorname{rad} L)_{\bar{0}}$ and $[a, (\operatorname{rad} L)_{\bar{1}}] = 0$, then $[a, G_{\bar{1}}] \subseteq G_{\bar{1}} \cap (\operatorname{rad} L)_{\bar{1}} = 0$, so $[a, L_{\bar{1}}] = 0$ and $a \in \ker\rho \cap (\operatorname{rad} L)_{\bar{0}} = 0$. Hence, $\rho|_{(\operatorname{rad} L)_{\bar{0}}}$ is completely reducible and faithful. By Lemma 10, $[(\operatorname{rad} L)_{\bar{1}}, (\operatorname{rad} L)_{\bar{1}}] \subseteq K = 0$. Thus, $[[G_{\bar{1}}, G_{\bar{1}}], (\operatorname{rad} L)_{\bar{1}}] \subseteq [G_{\bar{1}}, [G_{\bar{1}}, (\operatorname{rad} L)_{\bar{1}}]] \subseteq [G_{\bar{1}}, (\operatorname{rad} L)_{\bar{0}}] = 0$.

Now, $(\text{rad } L)_{\bar{1}} = \bigoplus_{\lambda \in \Lambda} V^{\lambda}$, with $\Lambda \subseteq ((\text{rad } L)_{\bar{0}})^*$ and $V^{\lambda} = \{x \in (\text{rad } L)_{\bar{1}} : [a, x] = \lambda(a)x$ for any $a \in (\text{rad } L)_{\bar{0}}\}$. For any $0 \neq \lambda \in \Lambda$,

$$\begin{aligned}\left[L_{\bar{1}}, V^{\lambda}\right] = \left[G_{\bar{1}}, V^{\lambda}\right] &\subseteq \{x \in L_{\bar{0}} : [a, x] = \lambda(a)x \text{ for any } a \in (\text{rad } L)_{\bar{0}}\}\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\left[\left[L_{\bar{1}}, V^0\right], V^{\lambda}\right] = \left[\left[G_{\bar{1}}, V^0\right], V^{\lambda}\right] &\subseteq \left[\left[G_{\bar{1}}, V^{\lambda}\right], V^0\right] + \left[G_{\bar{1}}, [V^0, V^{\lambda}]\right]\\ &\subseteq 0 + \left[G_{\bar{1}}, \left[L_{\bar{1}}, V^{\lambda}\right]\right] = 0.\end{aligned}$$

Therefore, $[L_{\bar{1}}, V^0] \subseteq \ker \rho \cap (\text{rad } L)_{\bar{0}} = 0$, so $[L_{\bar{1}}, (\text{rad } L)_{\bar{1}}] = 0$ and, as a consequence, $[G_{\bar{1}}, \text{rad } L] = 0$.

Also, $[[L_{\bar{1}}, L_{\bar{1}}] \cap (\text{rad } L)_{\bar{0}}, (\text{rad } L)_{\bar{1}}] = 0$, so $[L_{\bar{1}}, L_{\bar{1}}] \cap (\text{rad } L)_{\bar{0}} = 0$ because $\rho|_{(\text{rad } L)_{\bar{0}}}$ is faithful. Hence, since $L_{\bar{0}}$ is reductive, there is an ideal $G_{\bar{0}}$ of $L_{\bar{0}}$ such that $L_{\bar{0}} = (\text{rad } L)_{\bar{0}} \oplus G_{\bar{0}}$ and $[L_{\bar{1}}, L_{\bar{1}}] = [G_{\bar{1}}, G_{\bar{1}}] \subseteq G_{\bar{0}}$. Therefore, $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is a subalgebra of L and $L = \text{rad } L \oplus G$, a semidirect sum, so G is semisimple. Moreover, the action of $G_{\bar{0}}$ in $G_{\bar{1}}$ is completely reducible and $G_{\bar{0}}$ is reductive, so there is an ideal A of $G_{\bar{0}}$ with $G_{\bar{0}} = A \oplus [G_{\bar{1}}, G_{\bar{1}}]$ and, since $[[G_{\bar{1}}, G_{\bar{1}}], (\text{rad } L)_{\bar{1}}] = 0$, A acts completely reducibly on each V^{λ}. With $U = \text{rad } L$ and $U^{\lambda} = V^{\lambda}$ for all λ, we have obtained that L is a nice extension of the semisimple Lie superalgebra G.

Summarizing these arguments, we arrive at the main result of the paper:

THEOREM B. *Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra with reductive $L_{\bar{0}}$ and completely reducible action of $L_{\bar{0}}$ in $L_{\bar{1}}$. Then, L is an elementary even extension of a nice extension of a semisimple Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ with a completely reducible action of $G_{\bar{0}}$ in $G_{\bar{1}}$.*

The converse is clear, that is, any Lie superalgebra L constructed as an elementary even extension of a nice extension of a semisimple Lie superalgebra $G_{\bar{0}} \oplus G_{\bar{1}}$ with a completely reducible action of $G_{\bar{0}}$ in $G_{\bar{1}}$ verifies that $L_{\bar{0}}$ is reductive and that the action of $L_{\bar{0}}$ in $L_{\bar{1}}$ is completely reducible.

Let us remark that if G is a semisimple Lie superalgebra with a completely reducible action of $G_{\bar{0}}$ in $G_{\bar{1}}$, then by Theorem A and the known classification of the classical simple Lie superalgebras, it is easy to show explicitly an ideal A such that $G_{\bar{0}} = A \oplus [G_{\bar{1}}, G_{\bar{1}}]$, which plays such an important role in Proposition 9. Simply notice that for G in Example 1, $A = P \cong sl(2)$, for G in Example 2, $A = 0$, and for G in Example 3, A is the even part $L_{\bar{0}}$ of the subalgebra L that appears there; A is abelian in this case.

We finish the paper with the classification of the Lie superalgebras in the title:

COROLLARY. *Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra with semisimple even part. Then, L is an elementary odd extension of a direct sum of the following Lie superalgebras: simple Lie algebras,* $\mathbf{A}(n, n)$, $\mathbf{B}(m, n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1, \alpha)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{P}(n)$, $\mathbf{Q}(n)$, $\mathbf{G}(\mathbf{A}(1, 1); r)$, $\mathbf{G}_{\mathbf{Q}}^{m, r}(S_1, \ldots, S_m; n_1, \ldots, n_r; L)$. *Moreover, the only simple ideals of $L_{\bar{0}}$ which act nontrivially in the (odd) solvable radical are the simple ideals of L which are simple Lie algebras and the ideals P of $L_{\bar{0}}$ isomorphic to $sl(2)$ which appear in each* $\mathbf{G}(\mathbf{A}(1, 1); r)$.

Proof. If $L_{\bar{0}}$ is semisimple, then $\operatorname{rad} L = (\operatorname{rad} L)_{\bar{1}}$, $K = 0$, and the nice extension in Theorem B reduces to the elementary odd extension of $G = L_{\bar{0}} \oplus G_{\bar{1}}$ by $(\operatorname{rad} L)_{\bar{1}}$. Now Theorem A, the known classification of the classical simple Lie superalgebras, and the fact that $L_{\bar{0}}$ is semisimple complete the proof. ∎

The superalgebras $\mathbf{G}(\mathbf{A}(1, 1); r)$ and $\mathbf{G}_{\mathbf{Q}}^{m, r}(S_1, \ldots, S_m; n_1, \ldots, n_r; L)$ with $m, r > 0$ or $m = 0$, $r \geq 2$, are the ones missing in [6, Chap. II, Proposition 1.2].

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