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Perturbation of Orthogonal Fourier Expansions*

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In this paper, a generalized Jacobi measure on $[-1, 1]$ is perturbed by exponentials of functions b of bounded mean oscillation. If we consider the Fourier series in orthogonal polynomials associated to each modification, then certain estimates (uniform in $n \in \mathbb{N}$ and b belonging to some neighbourhood of the origin) are obtained. As a consequence, the partial sum operators depend analytically on the functional parameter b. The case of the Bessel series is also considered.

Key Words: Fourier–Jacobi series; Bessel series; BMO; A_p weights.

0. INTRODUCTION

Let μ be a positive Borel measure on the real line and assume that $(P_n)_{n \geqslant 0}$ is a sequence of orthonormal polynomials in $L^2(\mu)$, obtained from the sequence $(1, x, x^2, ...)$ by the Gram–Schmidt orthogonalization process. The system $(P_n)_{n \geqslant 0}$ satisfies a three-term recurrence relation of the form

$$\begin{gathered} xP_n = a_{n+1} P_{n+1} + b_n P_n + a_n P_{n-1}, \qquad n = 0, 1, 2, ... \\ P_{-1} = 0, \qquad P_0 = 1 \end{gathered} \tag{1}$$

with $a_n > 0$, $b_n \in \mathbb{R}$. Conversely, any such recurrence relation (with the initial conditions $P_{-1} = 0$, $P_0 = 1$) determines a sequence $(P_n)_{n \geqslant 0}$ which is orthonormal with respect to some positive measure μ on the real line.

The recurrence relation (1) can be reformulated as $xP = JP$, where $P = (P_0, P_1, ...)$ and J is an infinite tridiagonal matrix, i.e., $J = (a_{ij})_{i, j \geqslant 0}$

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with $a_{ij}=0$ if $|i-j|>1$. The matrix J is called the Jacobi matrix of the polynomial system.

These matrices appear frequently to be attached to physical problems. For example, the motion of a number of particles interacting on the real line can be described, in the context of Toda lattices, by nonlinear matrix differential equations (see [4, 1]). The solutions are a flow of infinite tri-diagonal matrices, so that the spectral analysis of the system can be transformed into the problem of determining the polynomials associated to a three-term recurrence relation.

A perturbation of the physical system corresponds to a perturbation of the three-term recurrence relation. For a study of this type of perturbation (in the compact case) and further references, see [11, 8].

Following Coifman and Murray [3], another way of considering perturbed orthogonal polynomials consists of modifying the corresponding spectral measure. More precisely, let μ be a nonnegative measure on the real line and let $(P_n)_{n\geqslant 0}$ be an orthonormal system in $L^2(\mu)$. Often, but not necessarily, this is a polynomial system. Consider a perturbed space $L^2(u^2\,d\mu)$, where $u(x)=e^{b(x)}$ is a suitable function such that $(P_n)_{n\geqslant 0}\subseteq L^2(\mu)$. When b is close to zero (in a sense to be determined later), the new measure $u^2 d\mu$ is in some sense close to μ. Now the Gram–Schmidt orthogonalization process can be applied to $(P_n)_{n\geqslant 0}$ so as to get a perturbed orthonormal system in $L^2(u^2\,d\mu)$.

Let $S_n(b)$ stand for the nth partial sum operator of the Fourier expansion relative to the perturbed system in $L^2(u^2 d\mu)$. We consider the mapping $b\to S_n(b)$ depending on the functional parameter b. This is not a convenient setting, since $S_n(b)$ is a bounded operator on $L^2(e^{2b}\,d\mu)$, which varies with b. Instead, we take

$$b\to V_n(b)=e^b S_n(b)\,e^{-b} \tag{2}$$

so that each operator $V_n(b)$ acts on $L^2(\mu)$, the $L^2(\mu)$-boundedness of $V_n(b)$ is equivalent to the $L^2(u^2 d\mu)$-boundedness of $S_n(b)$ and the operator norms are equal.

The mapping (2) can now be seen in the context of calculus on Banach spaces where notions like continuity, differentiability, and analyticity are well defined. We are interested in the uniform analyticity of the sequence $(V_n)_{n\geqslant 0}$.

In specific examples it is difficult to deal with the operators $S_n(b)$ and $V_n(b)$, and it is much more convenient to work with the family of operators

$$T_n(b)=e^b S_n(0)\,e^{-b}.$$

The formula

$$V = T(I + (T - T^*))^{-1}$$

of Kerzman and Stein makes it possible. Here, V is a self-adjoint projection of a Hilbert space H onto a closed subspace K and T is a bounded oblique projection (non self-adjoint) from H onto K. We can take $V = V_n(b)$ and $T = T_n(b)$, so that $T^* = T_n(-b)$ and the uniform analyticity of $(T_n)_{n \geqslant 0}$ implies, via the Kerzman–Stein formula, the uniform analyticity of $(V_n)_{n \geqslant 0}$.

Let B be a real Banach space, $\bar{B}$ the complexification of B, and Y another complex Banach space. A sequence of operators $F_n: \bar{B} \to Y$ $(n \geqslant 0)$ is uniformly analytic in a neighbourhood U of $0 \in \bar{B}$ if and only if there exists a constant $C > 0$ such that for every n, and all $b \in U$, we have that

(a) F_n is Gâteaux differentiable

(b) $\|F_n(b)\| \leqslant C$, where $\|\cdot\|$ denotes the operator norm.

Consider

$$S_n(0) f(x) = S_n f(x) = \int_{-1}^{1} K_n(x, y) f(y) \, d\mu(y)$$

where $K_n(x, y)$ is the corresponding nth kernel of the orthonormal polynomials in $L^2(\mu)$. Then

$$T_n(b) f(x) = \int_{-1}^{1} \exp[b(x) - b(y)] \, K_n(x, y) f(y) \, d\mu(y).$$

For this particular kind of projection it is enough to prove (b), to obtain uniform analyticity for $\{T_n(b)\}_{n \geqslant 0}$, acting on $L^2(\mu)$, and so the problem can be reduced to obtaining a uniform weighted norm inequality for the operators S_n.

For Jacobi polynomials, $d\mu(x) = (1-x)^{\alpha} (1+x)^{\beta} \, dx$, with $\alpha, \beta \geqslant -1/2$, and $B = \mathrm{BMO}$ (bounded mean oscillation), Coifman and Murray proved that the operators $T_n(b)$ are bounded from $L^2(d\mu)$ into itself when $\|b\|_*$ (the norm of b in BMO) is small enough. This implies that the T_n are uniformly analytic in a neighbourhood U of 0 and so are the V_n.

The aim of this paper is to deal with a more general class of measures and to study the uniform boundedness of the operators $T_n(b)$ in $L^p(d\mu)$ where p belongs to an interval including $p = 2$. This will be possible because there exists enough information in a more general context about uniform weighted norm inequalities for the partial sum operators in L^p spaces. As a consequence we obtain the uniform analyticity of the operators T_n and V_n.

The paper is organized as follows: in Section 1 we present the notation and the main results (Theorems 1 and 2), which will be proved in Sections 3 and 4, respectively. In Section 2 we state and prove some auxiliary results about the A_p class of weights.

1. NOTATION AND RESULTS

Throughout this paper, C will denote a universal constant which may be different from line to line. If $1<p<\infty$, we use the notation $q=p/(p-1)$, i.e., $1/p+1/q=1$.

Let $1<p<\infty$ and $-\infty\leqslant a<b\leqslant\infty$. The class $A_p(a, b)$ consists of those weights u such that

$$\left(\frac{1}{|I|}\int_I u(x)\,dx\right)\left(\frac{1}{|I|}\int_I u(x)^{-1/(p-1)}\,dx\right)^{p-1}\leqslant C$$

where I ranges over all the finite intervals $I\subseteq(a, b)$ and $|I|$ stands for the length of the interval I. The least constant C will be referred to as the A_p constant of u and will be denoted $A_p(u)$. The Hilbert transform is bounded on $L^p(u)$ if $u\in A_p$ (see [7]).

If b is an integrable function on $[-1, 1]$, the mean of b on an interval I is

$$b_I=\frac{1}{|I|}\int_I b(x)\,dx.$$

The function b is said to have bounded mean oscillation on $[-1, 1]$ if

$$\|b\|_*=\sup_I\frac{1}{|I|}\int_I|b(x)-b_I|\,dx$$

is finite, where the supremum is taken over all the intervals $I\subseteq[-1, 1]$. The space BMO of real-valued functions (modulo constants) having bounded mean oscillation on $[-1, 1]$ is a Banach space with $\|\cdot\|_*$ as its norm.

Consider a generalized Jacobi weight

$$w(x)=h(x)(1-x)^\alpha\,(1+x)^\beta\prod_{i=1}^{N}|x-t_i|^{\gamma_i},\qquad x\in[-1, 1]\tag{3}$$

where

(a) $\alpha, \beta > -1$, $\gamma_i \geqslant 0$, $t_i \in (-1, 1)$, $t_i \neq t_j$ $\forall i \neq j$;

(b) h is a positive, continuous function on $[-1, 1]$ and $\omega(h, \delta)\, \delta^{-1} \in L^1(0, 1)$, $\omega(h, \delta)$ being the modulus of continuity of h.

Let $(P_n)_{n \geqslant 0}$ be the orthonormal polynomials with respect to $w(x)\, dx$. Badkov [2] proved that there exists a constant C such that for every $x \in [-1, 1]$ and $n \in \mathbb{N}$

$$|P_n(x)| \leqslant C(1 - x + n^{-2})^{-(2\alpha+1)/4} (1 + x + n^{-2})^{-(2\beta+1)/4} \prod_{i=1}^{N} |x - t_i|^{-\gamma_i/2}. \tag{4}$$

Also, if $(Q_n)_{n \geqslant 0}$ denotes the orthonormal polynomials with respect to the measure $(1 - x^2)\, w(x)\, dx$, we have

$$|Q_n(x)| \leqslant C(1 - x)^{-(2\alpha+3)/4} (1 + x)^{-(2\beta+3)/4} \prod_{i=1}^{N} (|x - t_i|)^{-\gamma_i/2}.$$

For $f \in L^1(w)$, let $S_n f$ denote the nth partial sum of the orthonormal Fourier expansion of f in $(P_n)_{n \geqslant 0}$, i.e.

$$S_n f(x) = \int_{-1}^{1} f(y)\, K_n(x, y)\, w(y)\, dy, \qquad K_n(x, y) = \sum_{k=0}^{n} P_k(x)\, P_k(y).$$

For a suitable function $b \in \mathrm{BMO}$, consider the perturbed measure $e^{2b} w\, dx$. The classical Gram–Schmidt procedure applied to the orthonormal system $(P_n)_{n \geqslant 0}$ leads to a perturbed orthonormal system related to the measure $e^{2b} w\, dx$. We then have the perturbed Fourier expansion operators $(S_n(b))_{n \geqslant 0}$. Thus, the sequences of operators $(V_n)_{n \geqslant 0}$ and $(T_n)_{n \geqslant 0}$ are given by

$$V_n(b) = e^b S_n(b)\, e^{-b}$$

and

$$\begin{aligned} T_n(b)(f)(x) &= e^{b(x)} S_n(e^{-b} f)(x) \\ &= \int_{-1}^{1} \exp[b(x) - b(y)]\, K_n(x, y)\, f(y)\, w(y)\, dy. \end{aligned}$$

Theorem 1. *Let $1 < p < \infty$, w be as in* (3) *and assume*

(a) $w^{1-p/2}(1 - x^2)^{-p/4} \in L^1$,

(b) $w^{1-q/2}(1 - x^2)^{-q/4} \in L^1$.

Then, there exist some constants $C, \delta > 0$ *such that for all* $b \in \mathrm{BMO}$ *with* $\|b\|_* < \delta$,

$$\sup_n \|T_n(b)\|_{L^p(w) \to L^p(w)} \leqslant C.$$

Remark. Conditions (a) and (b) are also necessary for the uniform boundedness of $T_n(b)$ in a neighbourhood of $0 \in \mathrm{BMO}$. More precisely, the uniform boundedness of $T_n(0) = S_n$ implies (a) and (b) [9, Theorem 1].

A real number p satisfies conditions (a) and (b) if and only if the following inequalities hold:

$$\left|(\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right)\right| < \min\left\{\frac{1}{4}, \frac{\alpha+1}{2}\right\}$$

$$\left|(\beta+1)\left(\frac{1}{p}-\frac{1}{2}\right)\right| < \min\left\{\frac{1}{4}, \frac{\beta+1}{2}\right\}$$

$$\left|(\gamma_i+1)\left(\frac{1}{p}-\frac{1}{2}\right)\right| < \min\left\{\frac{1}{2}, \frac{\gamma_i+1}{2}\right\}, \qquad i = 1, ..., N.$$

It is clear that these conditions determine an interval containing $p = 2$.

The uniform boundedness in Theorem 1, together with the general arguments pointed out in the Introduction, leads to the following corollary.

COROLLARY. *Let* w *be as in* (3). *Then, the sequences of operators* $\{T_n(b)\}$ *and* $\{V_n(b)\}$, *acting on* $L^2(w)$, *are uniformly analytic in a neighbourhood of* 0 *in the complexification of* BMO.

Now, let J_α be the Bessel function or order $\alpha > -1$, and put

$$j_n(x) = 2^{1/2} |J_\alpha(\alpha_n)|^{-1} J_\alpha(\alpha_n x),$$

where $(\alpha_n)_{n \geqslant 1}$ is the increasing sequence of the zeros of J_α. The functions $(j_n)_{n \geqslant 1}$ constitute an complete orthonormal system in $L^2((0, 1); x\,dx)$, called the Bessel system of order α ([13]).

For the sake of simplicity we will write $L^p(x\,dx)$ for $L^p((0, 1); x\,dx)$.

For $f \in L^1(x\,dx)$, let $s_n f$ denote the nth partial sum of the orthonormal Fourier–Bessel of f in $(j_n)_{n \geqslant 1}$, i.e.

$$s_n f(x) = \int_0^1 f(y)\, \mathscr{K}_n(x, y)\, y\, dy, \qquad \mathscr{K}_n(x, y) = \sum_{k=1}^n j_k(x)\, j_k(y),$$

and in a similar way define

$$\mathscr{V}_n(b) = e^b s_n(b)\, e^{-b}$$

and

$$\mathcal{T}_n(b)(f)(x) = e^{b(x)} s_n(e^{-b} f)(x) = \int_0^1 \exp[b(x) - b(y)]\, \mathcal{K}_n(x, y)\, f(y)\, y\, dy.$$

THEOREM 2. *Let* $\alpha > -1$ *and assume*

(a) $\frac{4}{3} < p < 4$, *if* $-\frac{1}{2} \leqslant \alpha$;

(b) $\frac{2}{\alpha+2} < p < \frac{2}{-\alpha}$, *if* $-1 < \alpha \leqslant -\frac{1}{2}$.

Then, there exist some constants $C, \delta > 0$ *such that for all* $b \in \mathrm{BMO}$ *with* $\|b\|_* < \delta$

$$\sup_n \|\mathcal{T}_n(b)\|_{L^p(x\,dx) \to L^p(x\,dx)} \leqslant C.$$

Remark. Also, conditions (a) and (b) are necessary for the uniform boundedness of $\mathcal{T}_n(0) = s_n$ [6, Theorem 2].

The uniform analyticity follows as a consequence of Theorem 2.

COROLLARY. *The sequences of operators* $\{\mathcal{T}_n(b)\}$ *and* $\{\mathcal{V}_n(b)\}$, *acting on* $L^2(x\,dx)$, *are uniformly analytic in a neighbourhood of* 0 *in the complexification of* BMO.

2. AUXILIARY RESULTS ABOUT THE A_p CLASS

In this section, u, v, w, u_n, w_n will denote weights on some subset $Q \subseteq \mathbb{R}^m$ which we may think of as $(-1, 1) \subseteq \mathbb{R}$. A sequence $(u_n)_{n \geqslant 1}$ is said to belong to A_p uniformly if the corresponding sequence of A_p constants is bounded or, equivalently, if some constant C, independent of n, satisfies the A_p condition for every weight u_n.

With $u \sim v$ we mean $C_1 \leqslant u/v \leqslant C_2$ for some positive constants C_1, C_2. With $u_n \overset{\text{unif}}{\sim} v_n$ we mean that the respective constants are independent of n.

LEMMA 1. (a) $w \in A_p,\ \lambda \in (0, +\infty) \Rightarrow \lambda w \in A_p,\ A_p(\lambda w) = A_p(w)$.

(b) $u \sim v,\ v \in A_p \Rightarrow u \in A_p,\ \ A_p(u) \sim A_p(v)$ (*with constants depending only on the ratio constants in* $u \sim v$).

(c) $u, v \in A_p \Rightarrow u + v \in A_p,\ A_p(u+v) \leqslant A_p(u) + A_p(v)$.

(d) $u, v \in A_p,\ w^{-1} = u^{-1} + v^{-1} \Rightarrow w \in A_p,\ A_p(w) \leqslant 2^p[A_p(u) + A_p(v)]$.

Proof. (a) and (b) are immediate; (d) is analogous to (c).

(c) We have

$$\left(\frac{1}{|I|}\int_I (u+v)^{-p'/p}\right)^{p/p'} \leqslant \min\left\{\left(\frac{1}{|I|}\int_I u^{-p'/p}\right)^{p/p'}, \left(\frac{1}{|I|}\int_I v^{-p'/p}\right)^{p/p'}\right\}.$$

Therefore,

$$\begin{aligned}&\frac{1}{|I|}\int_I (u+v)\left(\frac{1}{|I|}\int_I (u+v)^{-p'/p}\right)^{p/p'}\\ &\quad\leqslant \frac{1}{|I|}\int_I u\left(\frac{1}{|I|}\int_I u^{-p'/p}\right)^{p/p'} + \frac{1}{|I|}\int_I v\left(\frac{1}{|I|}\int_I v^{-p'/p}\right)^{p/p'}\\ &\quad\leqslant A_p(u)+A_p(v). \quad \blacksquare\end{aligned}$$

The properties above easily yield the following result.

LEMMA 2.

(a) $u_n \overset{\text{unif}}{\sim} w+w_n$, $w\in A_p$, $w_n\in A_p$ *uniformly* $\Rightarrow u_n\in A_p$ *uniformly*.

(b) $u_n^{-1} \overset{\text{unif}}{\sim} w^{-1}+w_n^{-1}$, $w\in A_p$, $w_n\in A_p$ *uniformly* $\Rightarrow u_n\in A_p$ *uniformly*.

Also, as a consequence of Hölder's inequality we have:

LEMMA 3. *Let u and v be two weights, $1<p<\infty$, $1<\delta<\infty$, $1/\delta+1/\delta'=1$. If u^δ, $v^{\delta'}\in A_p$, then $uv\in A_p$ and the A_p constants verify*

$$A_p(uv)\leqslant A_p(u^\delta)^{1/\delta}\, A_p(v^{\delta'})^{1/\delta'}.$$

COROLLARY. *Let $r, R\in\mathbb{R}$, $u_n(x)=(1-x)^r\,(1-x+n^{-2})^R$. Then, $u_n\in A_p(-1,1)$ uniformly* $\Leftrightarrow -1<r<p-1,\ -1<r+R<p-1$.

Remark. Analogous results hold when $1-x$ is replaced by $|x-t|$, for some $t\in(-1,1)$.

Proof. $(\Rightarrow)$ From the inequality

$$\tfrac{1}{2}\int_{-1}^{1} u_n\left(\tfrac{1}{2}\int_{-1}^{1} u_n^{-p'/p}\right)^{p/p'}\leqslant C$$

it follows that $-1<r<p-1$, and taking the limit in n yields $-1<r+R<p-1$, by integrability.

$(\Leftarrow)$ It is known that $(1-x)^r\in A_p(-1,1)\Leftrightarrow -1<r<p-1$.

Case $R \geqslant 0$. We have

$$\tfrac{1}{2}[(1-x)^R + n^{-2R}] \leqslant (1-x+n^{-2})^R \leqslant 2^R[(1-x)^R + n^{-2R}],$$

so that $(1-x)^r (1-x+n^{-2})^R \overset{\text{unif}}{\sim} (1-x)^{r+R} + n^{-2R}(1-x)^r$. Parts (a) of Lemmas 1 and 2 show that $u_n \in A_p(-1, 1)$ uniformly.

Case $R < 0$. Now,

$$\begin{aligned} u_n(x)^{-1} = (1-x)^{-r}(1-x+n^{-2})^{-R} &\overset{\text{unif}}{\sim} [(1-x)^{r+R}]^{-1} \\ &+ [n^{-2R}(1-x)^r]^{-1} \end{aligned}$$

and we can use part (a) of Lemma 1 together with part (b) of Lemma 2. ∎

3. PROOF OF THEOREM 1

The boundedness of

$$T_n(b) \colon L^p(w) \to L^p(w)$$

is equivalent to the boundedness of

$$S_n \colon L^p(e^{pb}w) \to L^p(e^{pb}w).$$

By using Pollard's decomposition of the kernels $K_n(x, y)$ (see [12, 10])

$$K_n(x, y) = r_n A_{1,n}(x, y) + s_n A_{2,n}(x, y) + s_n A_{3,n}(x, y),$$

where

$$A_{1,n}(x, y) = P_{n+1}(x)\, P_{n+1}(y),$$

$$A_{2,n}(x, y) = (1-y^2)\, \frac{P_{n+1}(x)\, Q_n(y)}{x-y},$$

$$A_{3,n}(x, y) = (1-x^2)\, \frac{P_{n+1}(y)\, Q_n(x)}{y-x},$$

and $(r_n)_{n \geqslant 0}$, $(s_n)_{n \geqslant 0}$ are bounded sequences, the uniform boundedness of S_n can be reduced to that of $W_{1,n}$, $W_{2,n}$, $W_{3,n}$, where

$$W_{1,n} f(x) = P_{n+1}(x) \int_{-1}^{1} P_{n+1} fw \, dy,$$

$$W_{2,n} f(x) = P_{n+1}(x) \, H((1-y^2) \, Q_n \, fw, x),$$

$$W_{3,n} f(x) = (1-x^2) \, Q_n(x) \, H(P_{n+1} \, fw, x),$$

and H is the Hilbert transform on the interval $[-1, 1]$. For $W_{1,n}$, by duality, it is enough to show

$$\|P_n e^b\|_{L^p(w)} \leqslant C, \qquad \|P_n e^{-b}\|_{L^q(w)} \leqslant C.$$

Now, by the estimates (4) for P_n and the dominated convergence theorem, it is enough to prove

$$\|(1-x^2)^{-1/4} \, w^{-1/2} e^b\|_{L^p(w)} \leqslant C, \tag{5}$$

$$\|(1-x^2)^{-1/4} \, w^{-1/2} e^{-b}\|_{L^q(w)} \leqslant C. \tag{6}$$

By the definition of w and the hypothesis $w^{1-p/2}(1-x^2)^{-p/4} \in L^1$, there exists some $\varepsilon > 0$ such that $w^{1-p/2}(1-x^2)^{-p/4} \in L^{1+\varepsilon}$. On the other hand, there exists some $\gamma > 0$ such that if $\|b\|_* < \gamma$ then $e^{pb} \in L^{(1+\varepsilon)/\varepsilon}$ (see [5, p. 409]).

Now, inequality (5) follows easily from Hölder's inequality. In a similar way (6) follows.

The uniform boundedness of $W_{2,n}$ and $W_{3,n}$ is equivalent to that of the Hilbert transform with pairs of weights

$$(e^{pb} \, |P_{n+1}|^p \, w, \, e^{pb} \, |Q_n|^{-p} \, (1-x^2)^{-p} \, w^{1-p})$$

and

$$(e^{pb} \, |Q_n|^p \, (1-x^2)^p \, w, \, e^{pb} \, |P_{n+1}|^{-p} \, w^{1-p}).$$

Then the proof of Theorem 1 will be finished if we prove the following lemmas:

Lemma 4. *With the above notation, there exist two constants $C_1, C_2 > 0$ and a sequence $(\phi_n)_{n \geqslant 1}$ uniformly in $A_p(-1, 1)$ such that*

$$C_1 \, |P_{n+1}|^p \, w \leqslant \phi_n \leqslant C_2 \, |Q_n|^{-p} \, (1-x^2)^{-p} \, w^{1-p}.$$

Lemma 5. *With the above notation, there exist two constants $C_1, C_2 > 0$ and a sequence $(\phi_n)_{n \geqslant 1}$ uniformly in $A_p(-1, 1)$ such that*

$$C_1 \, |Q_n|^p \, (1-x^2)^p \, w \leqslant \phi_n \leqslant C_2 \, |P_{n+1}|^{-p} \, w^{1-p}.$$

LEMMA 6. *Let $1 < p < \infty$. For each $\phi \in A_p$, there exists some $\gamma > 0$ such that $e^{pb}\phi \in A_p$ whenever $b \in BMO$ with $\|b\|_* < \gamma$. Moreover, γ and the A_p constant of $e^{pb}\phi$ depend only on the A_p constant of ϕ.*

Proof of Lemma 4. *Let* $w_1 = \prod_{i=1}^{N} |x - t_i|^{\gamma_i}$. From the estimates for P_n and Q_n we have

$$|P_{n+1}|^p\, w \leqslant C w_1^{1-(p/2)} (1-x)^\alpha\, (1+x)^\beta\, (1-x+n^{-2})^{-p(\alpha/2+1/4)}$$
$$\times (1+x+n^{-2})^{-p(\beta/2+1/4)},$$
$$|Q_n|^{-p}\, (1-x^2)^{-p}\, w^{1-p} \geqslant C w_1^{1-p/2} (1-x)^{-p+\alpha(1-p)}\, (1+x)^{-p+\beta(1-p)}$$
$$\times (1-x+n^{-2})^{p(\alpha/2+3/4)}\, (1+x+n^{-2})^{p(\beta/2+3/4)}.$$

It is not difficult to see, from the hypothesis, that we can take a real number R such that

$$-p + \alpha(1-p) \leqslant R \leqslant \alpha,$$
$$-1 < R < p - 1$$

and choose S such that

$$-p + \alpha(1-p) + p(\alpha/2 + 3/4) \leqslant R + S \leqslant \alpha - p(\alpha/2 + 1/4),$$
$$-1 < R + S < p - 1.$$

Now, it is a straightforward calculation to verify that

$$C(1-x)^\alpha\, (1-x+n^{-2})^{-p(\alpha/2+1/4)}$$
$$\leqslant (1-x)^R\, (1-x+n^{-2})^S$$
$$\leqslant C(1-x)^{-p+\alpha(1-p)}\, (1-x+n^{-2})^{p(\alpha/2+3/4)}.$$

We can also take $\tilde{R}$ and $\tilde{S}$ such that

$$-p + \beta(1-p) \leqslant \tilde{R} \leqslant \beta,$$
$$-1 < \tilde{R} < p - 1,$$
$$-p + \beta(1-p) + p(\beta/2 + 3/4) \leqslant \tilde{R} + \tilde{S} \leqslant \beta - p(\beta/2 + 1/4),$$
$$-1 < \tilde{R} + \tilde{S} < p - 1,$$

so that

$$
\begin{aligned}
C(1+x)^{\beta} & (1+x+n^{-2})^{-p(\beta/2+1/4)} \\
& \leqslant (1+x)^{\tilde{R}} (1+x+n^{-2})^{\tilde{S}} \\
& \leqslant C(1+x)^{-p+\beta(1-p)} (1+x+n^{-2})^{p(\beta/2+3/4)}.
\end{aligned}
$$

If we write

$$
\begin{aligned}
u_n(x) &= (1-x)^R (1-x+n^{-2})^S, \\
v_n(x) &= (1+x)^{\tilde{R}} (1+x+n^{-2})^{\tilde{S}}
\end{aligned}
$$

then

$$C_1 |P_{n+1}|^p w \leqslant w_1^{1-p/2} u_n v_n \leqslant C_2 |Q_n|^{-p} (1-x^2)^{-p} w^{1-p}.$$

Now, from the corollary in the previous section we have

$$
\begin{aligned}
u_n &\in A_p \qquad \text{uniformly}, \\
v_n &\in A_p \qquad \text{uniformly}, \\
w_1^{1-p/2} &\in A_p.
\end{aligned}
$$

Then, splitting in pieces the integrals appearing in the A_p condition it can be shown that

$$\phi_n = w_1^{1-p/2} u_n v_n \in A_p \qquad \text{uniformly.} \quad \blacksquare$$

The proof of Lemma 5 is entirely similar, so we omit it.

Proof of Lemma 6. If $\phi \in A_p$, there exists some $\varepsilon > 1$ such that $\phi^{\varepsilon} \in A_p$. Moreover, ε and the A_p constant of ϕ^{ε} depend only on the A_p constant of ϕ [5, Theorem IV.2.7, p. 399]. Take now $1/\varepsilon + 1/\varepsilon' = 1$. There exists some $\delta > 0$ such that

$$\|b\|_* < \delta \Rightarrow e^{p\varepsilon' b} \in A_p$$

and this in turn implies $e^{pb}\phi \in A_p$, by Lemma 3. Also, δ and the A_p constant of $e^{p\varepsilon' b}$ depend only on ε' [5, p. 409]. This proves the lemma. $\blacksquare$

4. PROOF OF THEOREM 2

The following result can be seen in [6].

PROPOSITION 1. *Let* $\alpha \geqslant -1/2$, $1 < p < \infty$, *and let* u *be a weight. If* $x^{1-p/2}u^p \in A_p(0, 1)$, *then there exists some constant* $C > 0$ *such that*

$$\|us_n(u^{-1}f)\|_{L^p(x\,dx)} \leqslant C\,\|f\|_{L^p(x\,dx)}$$

for every $n \geqslant 0$, $f \in L^p(x\,dx)$.

For the case $-1 < \alpha < -1/2$ we have

PROPOSITION 2. *Let* $-1 < \alpha < -1/2$, $1 < p < \infty$, *and let* u *be a weight. If*

$$\begin{aligned} x^{1-p/2}u^p &\in A_p(0, 1) \\ x^{1+\alpha p}u^p &\in A_p(0, 1) \\ x^{1-(\alpha+1)p}u^p &\in A_p(0, 1) \end{aligned} \tag{7}$$

then there exists some constant $C > 0$ *such that*

$$\|us_n(u^{-1}f)\|_{L^p(x\,dx)} \leqslant C\|f\|_{L^p(x\,dx)}$$

for every $n \geqslant 0$, $f \in L^p(x\,dx)$.

Proof. As shown in [6] it is enough to have

$$x^{1+\alpha p}u^p(M_n^{-1} + x)^{-p(\alpha+1/2)} \in A_p(0, 1)$$

$$x^{1-(\alpha+1)p}u^p(M_n^{-1} + x)^{p(\alpha+1/2)} \in A_p(0, 1)$$

where M_n are certain positive constants with $M_n \to +\infty$. With the help of Lemmas 1 and 2 this turns out to be equivalent to (7). ∎

Remark. In both cases, the constant C depends only on the A_p constants.

Proof of Theorem 2. Assume first that $-1/2 \leqslant \alpha$. According to Proposition 1, we only need to show that there exist some constants $C, \delta > 0$ such that

$$\|b\|_* < \delta \Rightarrow x^{1-p/2}e^{b(x)} \in A_p$$

with a constant C. Since $x^{1-p/2} \in A_p$, there exists some $\varepsilon > 1$ such that $x^{\varepsilon(1-p/2)} \in A_p$ [5, Theorem IV.2.7, p. 399]. Take $1/\varepsilon + 1/\varepsilon' = 1$; there also exist some constants $C, \delta > 0$ such that

$$\|b\|_* < \delta \Rightarrow e^{\varepsilon' b} \in A_p$$

with an A_p constant C (see [5, p. 409). Finally, apply Lemma 3. The case $-1 < \alpha < -1/2$ is analogous. ∎

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